FUNCTIONS 11

Nelson

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Lead Author
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Barbara Alldred • Andrew Dmytriw • Shawn Godin
Angelo Lillo • David Pilmer • Susanne Trew • Noel Walker

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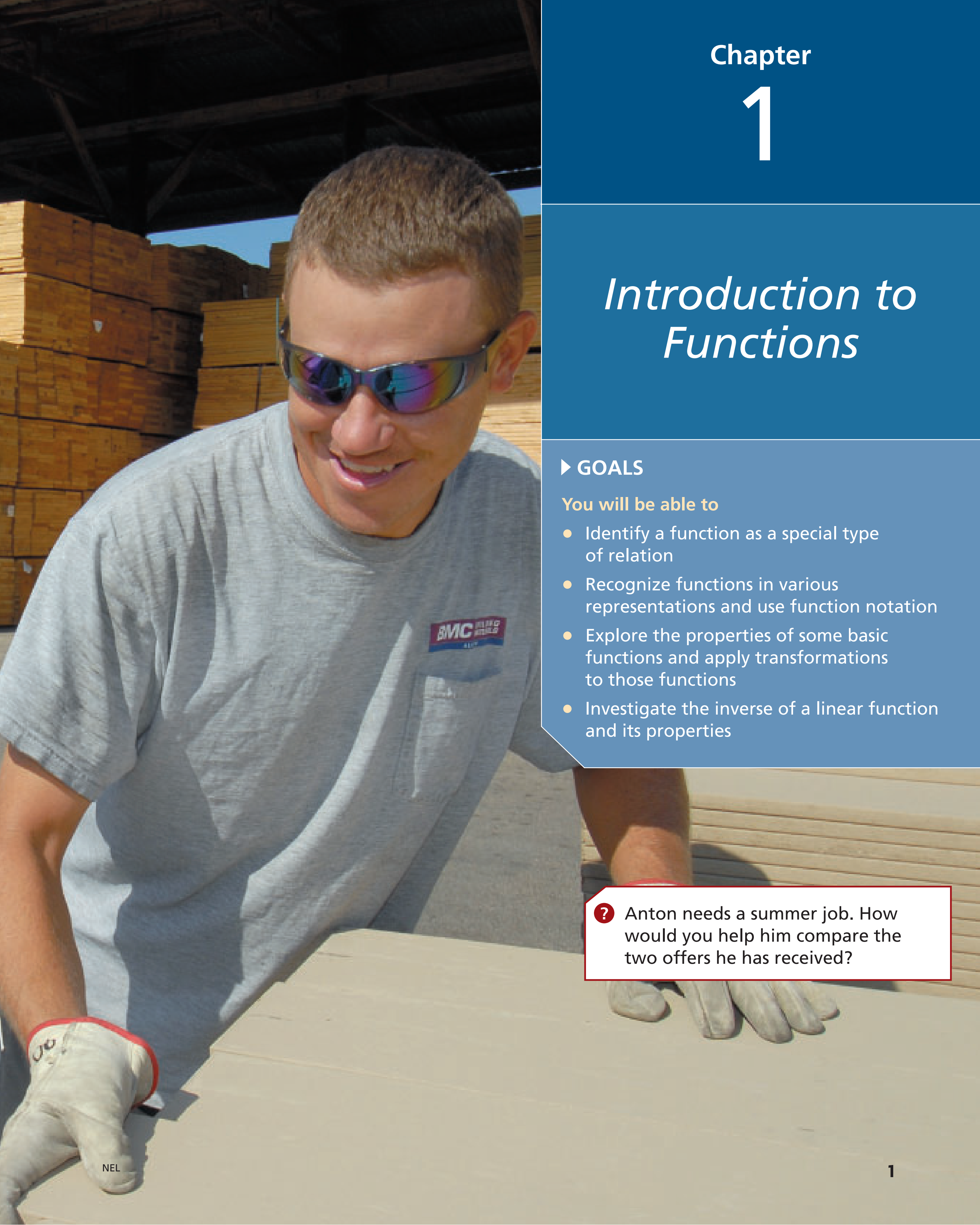
Help wanted:
Shipper at lumber store
$13.50/h
Potlatch
Help wanted:
Student-operated
house painting business
is hiring students now.
$650 per house



Chapter 1

Introduction to Functions

GOALS

You will be able to

- Identify a function as a special type of relation
- Recognize functions in various representations and use function notation
- Explore the properties of some basic functions and apply transformations to those functions
- Investigate the inverse of a linear function and its properties

? Anton needs a summer job. How would you help him compare the two offers he has received?

1 Getting Started

SKILLS AND CONCEPTS You Need

Study *Aid*

For help, see the Essential Skills Appendix.

Question	Appendix
1	A-8
2	A-7
4	A-5
5	A-15
6	A-12
7	A-14
8	A-9, A-10

1. Simplify each expression.

a) $3(x + y) - 5(x - y)$

b) $(4x - y)(4x + y)$

c) $\frac{1}{2}(x^2 + 1) - \frac{3}{2}(x^2 - 1)$

d) $4x(x + 2) - 2x(x - 4)$

2. Evaluate each expression in question 1 when $x = 3$ and $y = -5$.

3. Solve each linear equation.

a) $5x - 8 = 7$

b) $-2(x - 3) = 2(1 - 2x)$

c) $\frac{5}{6}y - \frac{3}{4}y = -3$

d) $\frac{x - 2}{4} = \frac{2x + 1}{3}$

4. Graph each **linear relation**.

a) $y = 2x - 3$

b) $3x + 4y = 12$

5. Graph each circle.

a) $x^2 + y^2 = 9$

b) $3x^2 + 3y^2 = 12$

6. Graph each **parabola**, labelling the **vertex** and the **axis of symmetry**.

a) $y = x^2 - 6$

b) $y = (x - 2)^2 - 1$

c) $y = -3(x + 4)^2 + 2$

d) $y = -x^2 + 6x$

7. For each **quadratic relation**, list the **transformations** you need to apply to $y = x^2$ to graph the relation. Then sketch the graph.

a) $y = x^2 - 2$

b) $y = -4x^2 + 3$

c) $y = \frac{1}{2}(x - 1)^2 - 4$

d) $y = -2(x + 3)^2 + 5$

8. Solve each quadratic equation.

a) $x^2 - 5x + 6 = 0$

b) $3x^2 - 5 = 70$

9. Compare the properties of linear relations, circles, and quadratic relations. Begin by completing a table like the one shown. Then list similarities and differences among the three types of relations.

Property	Linear Relations	Circles	Quadratic Relations
Equation(s)			
Shape of graph			
Number of quadrants graph enters			
Descriptive features of graph			
Types of problems modelled by the relation			

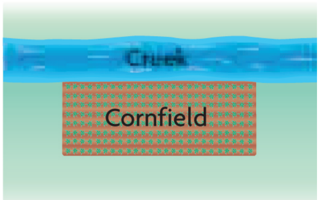
APPLYING What You Know

Fencing a Cornfield

Rebecca has 600 m of fencing for her cornfield. The creek that goes through her farmland will form one side of the rectangular boundary. Rebecca considers different widths to maximize the area enclosed.

YOU WILL NEED

- graph paper



? How are the length and area of the field related to its width?

A. What are the minimum and maximum values of the width of the field?

B. What equations describe each?
i) the relationship between the length and width of the field
ii) the relationship between the area and width of the field

C. Copy and complete this table of values for widths that go from the least to the greatest possible values in intervals of 50 m.

Width (m)	Length (m)	Area (m^2)

D. Graph the data you wrote in the first two columns. Use width as the **independent variable**. Describe the graph. What type of relationship is this?

E. Now graph the data you wrote in the first and third columns. Use width as the independent variable again. Describe the graph. What type of relationship is this?

F. How could you have used the table of values to determine the types of relationships you reported in parts D and E?

G. How could you have used the equations from part B to determine the types of relationships you reported in parts D and E?

1.1 Relations and Functions

YOU WILL NEED

- graphing calculator or graph paper

GOAL

Recognize functions in various representations.

INVESTIGATE the Math

Ang recorded the heights and shoe sizes of students in his class.

Shoe Size	Height (cm)
10	158
11.5	175
10	173
9	164
9	167
10	170
11	172
8	160
8	174
11	175
8	166
7.5	153
10	171
11	181
11	171
10	170

Shoe Size	Height (cm)
8	156
7.5	161
12	179
11	178
10.5	173
8.5	177
8	165
12	182
13	177
13	192
7.5	157
8.5	163
12	183
10	168
11	180

? Can you predict a person's height from his or her shoe size?

A. Plot the data, using shoe size as the **independent variable**. Describe the relationship shown in the scatter plot.

B. Use your plot to predict the height of a person with each shoe size.
i) 8 **ii)** 10 **iii)** 13

C. Use your plot to predict what shoe size corresponds to each height.
i) 153 cm **ii)** 173 cm **iii)** 177 cm

D. Draw a **line of good fit** on your plot. Write the equation of your line, and use it to determine the heights corresponding to the shoe sizes in part B. How are your results different from those in part B?

Tech *Support*

For help drawing a line of best fit on a graphing calculator, see Technical Appendix, B-11.

E. Describe the domain and range of the relationship between shoe size and height in Ang's class.

F. Explain why the relation plotted in part A is not a function.

G. Is the relation drawn in part D a function? Explain.

H. Which of the relations in parts A and D could be used to predict a single height for a given shoe size? Explain.

domain

the set of all values of the independent variable of a relation

range

the set of all values of the dependent variable of a relation

relation

a set of ordered pairs; values of the independent variable are paired with values of the dependent variable

function

a relation where each value of the independent variable corresponds with only one value of the dependent variable

Reflecting

I. How did the numbers in the table of values show that the relation was not a function?

J. How did the graph of the linear function you drew in part D differ from the graph of the relation you plotted in part A?

K. Explain why it is easier to use the linear function than the scatter plot of the actual data to predict height.

APPLY the Math

EXAMPLE 1 Representing functions in different ways

The ages and soccer practice days of four students are listed.

Student	Age	Soccer Practice Day
Craig	15	Tuesday
Magda	16	Tuesday
Stefani	15	Thursday
Amit	17	Saturday

Communication *Tip*

Use braces to list the values, or elements, in a set.

For example, the set of the first five even numbers is {2, 4, 6, 8, 10}.

For each of the given relations, state the domain and range and then determine whether or not the relations are functions.

a) students and the day for soccer practice

b) ages and the day for soccer practice

Jenny's Solution: Using Set Notation

a) {(Craig, Tuesday),
(Magda, Tuesday),
(Stefani, Thursday),
(Amit, Saturday)}

Domain = {Craig, Magda, Stefanie, Amit}

Range = {Tuesday, Thursday, Saturday}

I wrote the relation as a set of ordered pairs, (student's name, day for practice). I wrote the domain by listing the students' names—the independent variable, or first elements, in each ordered pair.

I listed the day for practice—the dependent variable, or second elements—to write the range.

Each element of the domain corresponds with only one element in the range, so the relation between students and their soccer practice day is a function. The first elements appear only once in the list of ordered pairs. No name is repeated.

Each student has only one practice day, so the relation is a function. In this case, if I know the student's name, I can predict his or her practice day.

b) {(15, Tuesday), (16, Tuesday), (15, Thursday), (17, Saturday)}

I noticed that one 15-year-old practiced on Tuesday, but another practiced on Thursday, so I can't predict a practice day just by knowing the age. This is not a function.

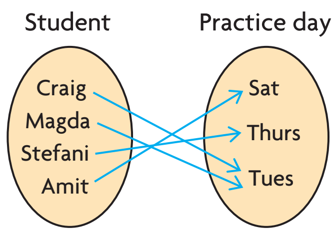
Domain = {15, 16, 17}

Range = {Tuesday, Thursday, Saturday}

15 in the domain corresponds with two different days in the range, so this relation is not a function.

Olivier's Solution: Using a Mapping Diagram

a)



I drew a diagram of the relation between students and soccer practice days by listing the student names in an oval and the days in another oval. Then I drew arrows to match the students with their practice days. The diagram is called a mapping diagram, since it *maps* the elements of the domain onto the elements of the range.

Domain = {Craig, Magda, Stefanie, Amit}

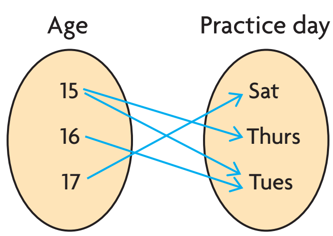
Range = {Tuesday, Thursday, Saturday}

The elements in the left oval are the values of the independent variable and make up the domain. The elements in the right oval are the values of the dependent variable and make up the range. I wrote the domain and range by listing what was in each oval.

Each element of the domain has only one corresponding element in the range, so the relation is a function.

The relation is a function because each student name has only one arrow leaving it.

b)



I drew another mapping diagram for the age and practice day relation. I matched the ages to the practice days.

Domain = {15, 16, 17}

Range = {Tuesday, Thursday, Saturday}

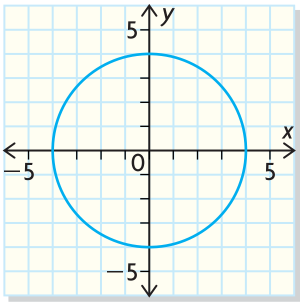
The value 15 of the independent variable, age, maps to two different values of the dependent variable, days. This relation is not a function.

Two arrows go from 15 to two different days. This cannot be a function. An element of the domain can't map to two elements in the range.

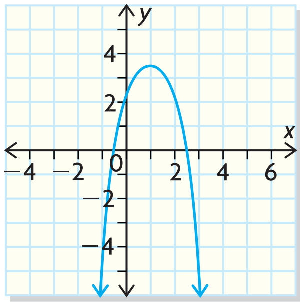
EXAMPLE 2 Selecting a strategy to recognize functions in graphs

Determine which of the following graphs are functions.

a)

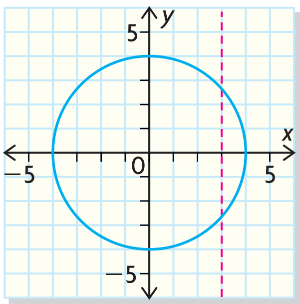


b)



Ken's Solution

a)



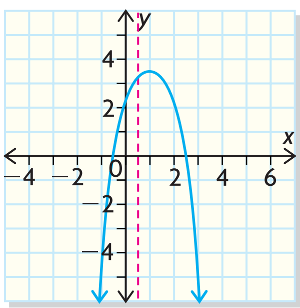
I used the **vertical-line test** to see how many points on the graph there were for each value of *x*.

An easy way to do this was to use a ruler to represent a vertical line and move it across the graph.

The ruler crossed the graph in two places everywhere except at the leftmost and rightmost ends of the circle. This showed that there are *x*-values in the domain of this relation that correspond to two *y*-values in the range.

At least one vertical line drawn on the graph intersects the graph at two points. This is not the graph of a function.

b)



I used the vertical-line test again. Wherever I placed my ruler, the vertical line intersected the graph in only one place. This showed that each *x*-value in the domain corresponds with only one *y*-value in the range.

Any vertical line drawn on the graph intersects the graph at only one point. This is the graph of a function.

vertical-line test

if any vertical line intersects the graph of a relation more than once, then the relation is not a function

EXAMPLE 3 Using reasoning to recognize a function from an equation

Determine which equations represent functions.

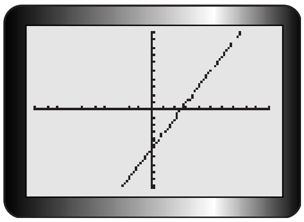
a) $y = 2x - 5$ **b)** $x^2 + y^2 = 9$ **c)** $y = 2x^2 - 3x + 1$

Keith's Solution: Using the Graph Defined by its Equation

a) This equation defines the graph of a linear function with a positive slope. Its graph is a straight line that increases from left to right.

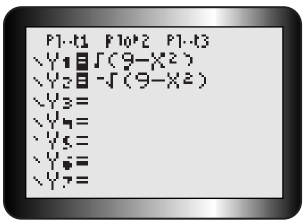
I used my graphing calculator and entered $y = 2x - 5$. I graphed the function and checked it with the vertical-line test.

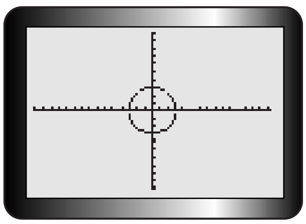
This graph passes the vertical-line test, showing that for each *x*-value in the domain there is only one *y*-value in the range. This is the graph of a function.



$y = 2x - 5$ is a function.

b) This equation defines the graph of a circle centred at (0, 0) with a radius of 3.





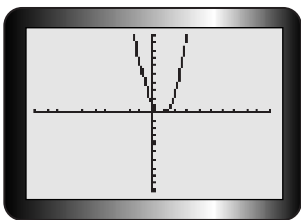
I used my graphing calculator and entered the upper half of the circle in Y1 and the lower half in Y2. Then I applied the vertical-line test to check.

This graph fails the vertical-line test, showing that there are *x*-values in the domain of this relation that correspond to two *y*-values in the range. This is not the graph of a function.

$x^2 + y^2 = 9$ is not a function.

c) This equation defines the graph of a parabola that opens upward.

I used my graphing calculator to enter $y = 2x^2 - 3x + 1$ and applied the vertical-line test to check.



This graph passes the vertical-line test, showing that for each *x*-value in the domain there is only one *y*-value in the range. This is the graph of a function.

$y = 2x^2 - 3x + 1$ is a function.

Mayda's Solution: Substituting Values

a) For any value of x, the equation $y = 2x - 5$ produces only one value of y. For example,

$y = 2(1) - 5 = -3$

This equation defines a function.

I substituted numbers for x in the equation.

No matter what number I substituted for x, I got only one answer for y when I doubled the number for x and then subtracted 5.

b) Substitute 0 for x in the equation

$$x^2 + y^2 = 9.$$
$$(0)^2 + y^2 = 9$$
$$y = 3 \text{ or } -3$$

I substituted 0 for x in the equation and solved for y. I used 0 because it's an easy value to calculate with. I got two values for y with $x = 0$.

There are two values for y when $x = 0$, so the equation defines a relation, but not a function.

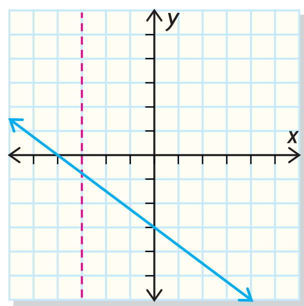
c) Every value of x gives only one value of y in the equation $y = 2x^2 - 3x + 1$.
This equation represents a function.

No matter what number I choose for x, I get only one number for y that satisfies the equation.

In Summary

Key Ideas

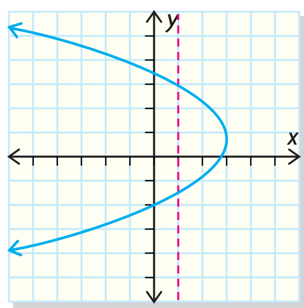
- A function is a relation in which each value of the independent variable corresponds with only one value of the dependent variable.
- Functions can be represented in various ways: in words, a table of values, a set of ordered pairs, a mapping diagram, a graph, or an equation.



A relation that is a function

Need To Know

- The domain of a relation or function is the set of all values of the independent variable. This is usually represented by the x-values on a coordinate grid.
- The range of a relation or function is the set of all values of the dependent variable. This is usually represented by the y-values on a coordinate grid.
- You can use the vertical-line test to check whether a graph represents a function. A graph represents a function if every vertical line intersects the graph in at most one point. This shows that there is only one element in the range for each element of the domain.
- You can recognize whether a relation is a function from its equation. If you can find even one value of x that gives more than one value of y when you substitute x into the equation, the relation is *not* a function. Linear relations, which have the general forms $y = mx + b$ or $Ax + By = C$ and whose graphs are straight lines, are all functions. Vertical lines are not functions but horizontal lines are. Quadratic relations, which have the general forms $y = ax^2 + bx + c$ or $y = a(x - h)^2 + k$ and whose graphs are parabolas, are also functions.



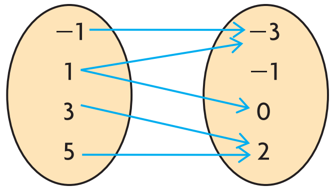
A relation that is not a function

CHECK Your Understanding

1. State which relations are functions. Explain.

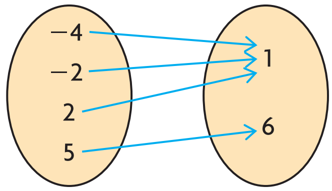
a) $\{(-5, 1), (-3, 2), (-1, 3), (1, 2)\}$

b)



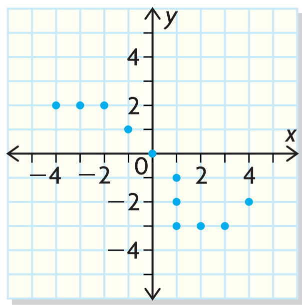
c) $\{(0, 4), (3, 5), (5, -2), (0, 1)\}$

d)



2. Use a ruler and the vertical-line test to determine which graphs are functions.

a)



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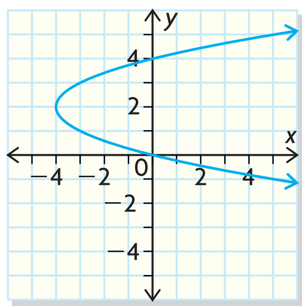
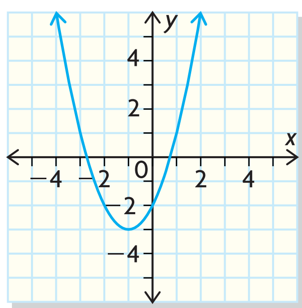


c)



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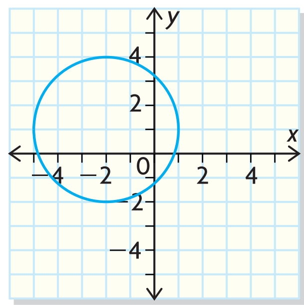

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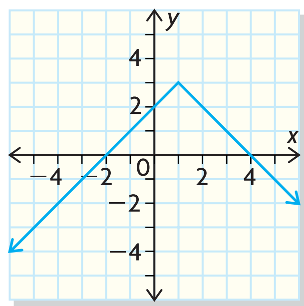

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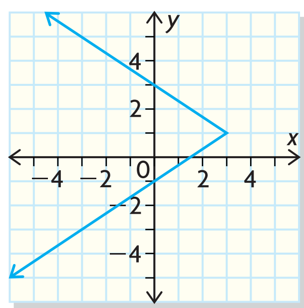


3. Substitute -6 for x in each equation and solve for y. Use your results to explain why $y = x^2 - 5x$ is a function but $x = y^2 - 5y$ is not.

PRACTISING

4. The grades and numbers of credits for students are listed.
K

Student	Grade	Number of Credits
Barbara	10	8
Pierre	12	25
Kateri	11	15
Mandeep	11	18
Elly	10	16

a) Write a list of ordered pairs and create a mapping diagram for the relation between
- students and grades
- grades and numbers of credits
- students and numbers of credits

b) State the domain and range of each relation in part (a).
c) Which relations in part (a) are functions? Explain.

5. Graph the relations in question 4. Then use the vertical-line test to confirm your answers to part (c).

6. Describe the graphs of the relations $y = 3$ and $x = 3$. Are these relations functions? Explain.

7. Identify each type of relation and predict whether it is a function. Then graph each function and use the vertical-line test to determine whether your prediction was correct.

a) $y = 5 - 2x$
b) $y = 2x^2 - 3$
c) $y = -\frac{3}{4}(x + 3)^2 + 1$
d) $x^2 + y^2 = 25$

8. a) Substitute $x = 0$ into each equation and solve for y. Repeat for $x = -2$.
i) $3x + 4y = 5$
ii) $x^2 + y^2 = 4$
iii) $x^2 + y = 2$
iv) $x + y^2 = 0$

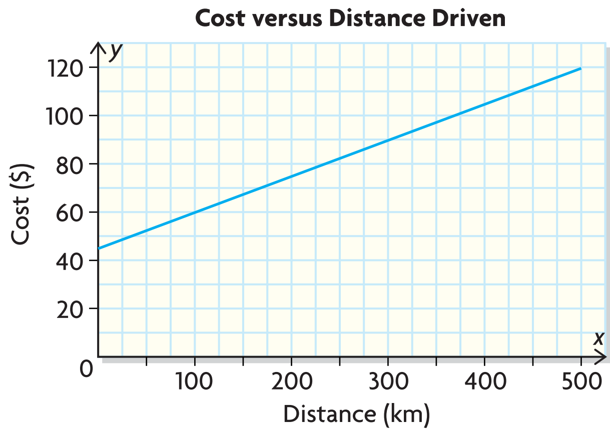
b) Which relations in part (a) appear to be functions?
c) How could you verify your answer to part (b)?

9. Determine which relations are functions.
a) $y = \sqrt{x + 2}$
b) $y = 2 - x$
c) $3x^2 - 4y^2 = 12$
d) $y = -3(x + 2)^2 - 4$

10. Use numeric and graphical representations to investigate whether the relation $x - y^2 = 2$ is a function. Explain your reasoning.

11. Determine which of the following relations are functions.
A
a) The relation between earnings and sales if Olwen earns \$400 per week plus 5% commission on sales
b) The relation between distance and time if Bran walks at 5 km/h
c) The relation between students' ages and the number of credits earned

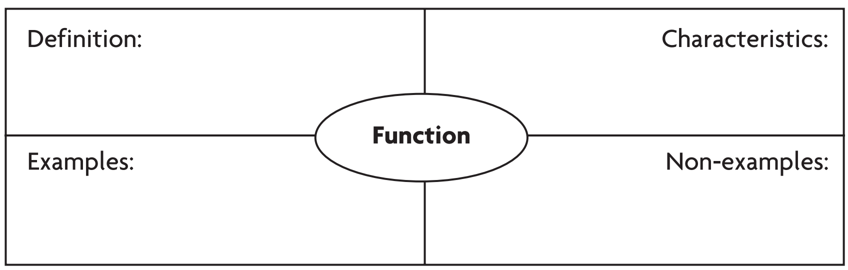
12. The cost of renting a car depends on the daily rental charge and the number of kilometres driven. A graph of cost versus the distance driven over a one-day period is shown.



a) What are the domain and range of this relation?
b) Explain why the domain and range have a lower limit.
c) Is the relation a function? Explain.

13. T **a)** Sketch a graph of a function that has the set of integers as its domain and all integers less than 5 as its range.
b) Sketch a graph of a relation that is not a function and that has the set of real numbers less than or equal to 10 as its domain and all real numbers greater than -5 as its range.

14. C Use a chart like the following to summarize what you have learned about functions.



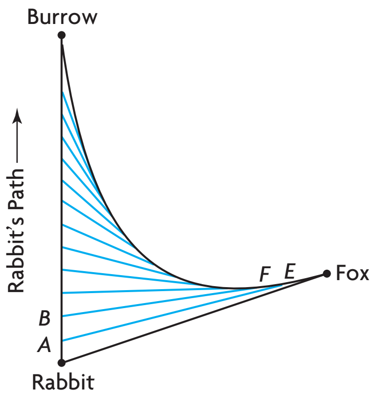
Extending

15. A freight delivery company charges \$4/kg for any order less than 100 kg and \$3.50/kg for any order of at least 100 kg.
a) Why must this relation be a function?
b) What is the domain of this function? What is its range?
c) Graph the function.
d) What suggestions can you offer to the company for a better pricing structure? Support your answer.

Curious | *Math*

Curves of Pursuit

A fox sees a rabbit sitting in the middle of a field and begins to run toward the rabbit. The rabbit sees the fox and runs in a straight line to its burrow. The fox continuously adjusts its direction so that it is always running directly toward the rabbit.



If the fox and rabbit are running at the same speed, the fox reaches point E when the rabbit reaches point A. The fox then changes direction to run along line EA. When the fox reaches point F, the rabbit is at B, so the fox begins to run along FB, and so on. The resulting curve is called a *curve of pursuit*.

1. If the original position of the rabbit represents the origin and the rabbit's path is along the positive y-axis, is the fox's path the graph of a function? Explain.

2. **a)** Investigate what happens by drawing a curve of pursuit if
  i) the burrow is farther away from the rabbit than it is in the first example
  ii) the burrow is closer to the rabbit than it is in the first example
b) Where does the fox finish in each case? How does the location of the burrow relative to the rabbit affect the fox's path?
c) Will the path of the fox always be a function, regardless of where the rabbit is relative to its burrow? Explain.

3. **a)** Draw a curve of pursuit in which
  i) the rabbit runs faster than the fox
  ii) the fox runs faster than the rabbit
b) Are these relations also functions? How do they differ from the one in question 1?

1.2 Function Notation

YOU WILL NEED

- graphing calculator

GOAL

Use function notation to represent linear and quadratic functions.

LEARN ABOUT the Math

The deepest mine in the world, East Rand mine in South Africa, reaches 3585 m into Earth's crust. Another South African mine, Western Deep, is being deepened to 4100 m. Suppose the temperature at the top of the mine shaft is 11 °C and that it increases at a rate of 0.015 °C/m as you descend.

? What is the temperature at the bottom of each mine?

function notation

notation, such as $f(x)$, used to represent the value of the dependent variable—the output—for a given value of the independent variable, x—the input

EXAMPLE 1 Representing a situation with a function and using it to solve a problem

a) Represent the temperature in a mine shaft with a function. Explain why your representation is a function, and write it in function notation.

b) Use your function to determine the temperature at the bottom of East Rand and Western Deep mines.

Lucy's Solution: Using an Equation

a) An equation for temperature is $T = 11 + 0.015d$, where T represents the temperature in degrees Celsius at a depth of d metres.

I wrote a linear equation for the problem.

I used the fact that T starts at 11 °C and increases at a steady rate of 0.015 °C/m.

The equation represents a function. Temperature is a function of depth.

Since this equation represents a linear relationship between temperature and depth, it is a function.

In function notation, $T(d) = 11 + 0.015d$.

I wrote the equation again. $T(d)$ makes it clearer that T is a function of d.

Communication *Tip*

The notations y and $f(x)$ are interchangeable in the equation or graph of a function, so y is equal to $f(x)$. The notation $f(x)$ is read "f at x" or "f of x." The symbols $f(x)$, $g(x)$, and $h(x)$ are often used to name the outputs of functions, but other letters are also used, such as $v(t)$ for velocity as a function of time.

b) $T(3585) = 11 + 0.015(3585)$
$= 11 + 53.775$
$= 64.775$

I found the temperature at the bottom of East Rand mine by calculating the temperature at a depth of 3585 m. I substituted 3585 for d in the equation.

$T(4100) = 11 + 0.015(4100)$
$= 11 + 61.5$
$= 72.5$

For the new mine, I wanted the temperature when $d = 4100$, so I calculated $T(4100)$.

The temperatures at the bottom of East Rand mine and Western Deep mine are about 65 °C and 73 °C, respectively.

Stuart's Solution: Using a Graph

a) $T(d) = 11 + 0.015d$

This is a function because it is a linear relationship.

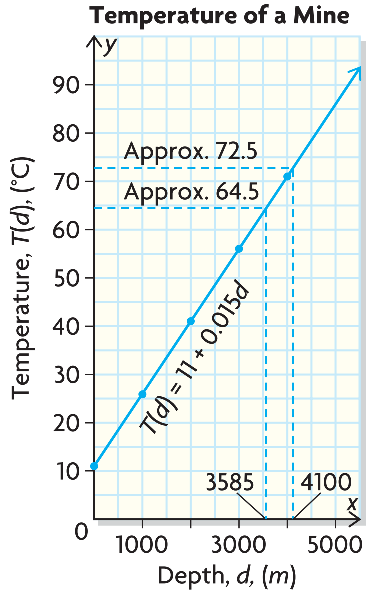
I wrote an equation to show how the temperature changes as you go down the mine.
I knew that the relationship was linear because the temperature increases at a steady rate. I used d for depth and called the function $T(d)$ for temperature.

b)

***d* (m)**	***T*(*d*)(°C)**
0	$T(0) = 11 + 0.015(0) = 11$
1000	26
2000	41
3000	56
4000	71

I made a table of values for the function.

I substituted the d-values into the function equation to get the $T(d)$-values.



I plotted the points (0, 11), (1000, 26), (2000, 41), (3000, 56), and (4000, 71). Then I joined them with a straight line.

East Rand mine is 3585 m deep. The temperature at the bottom is $T(3585)$.

I interpolated to read $T(3585)$ from the graph. It was approximately 65.

The other mine is 4100 m deep.

By extrapolating, I found that $T(4100)$ was about 73.

The temperature at the bottom of East Rand mine is about 65 °C. The temperature at the bottom of Western Deep mine is about 73 °C.

Tech *Support*

For help using a graphing calculator to graph and evaluate functions, see Technical Appendix, B-2 and B-3.

Eli's Solution: Using a Graphing Calculator

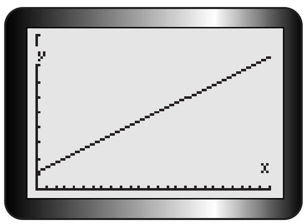
a) Let $T(d)$ represent the temperature in degrees Celsius at a depth of d metres.

I used function notation to write the equation.

$$T(d) = 11 + 0.015d$$

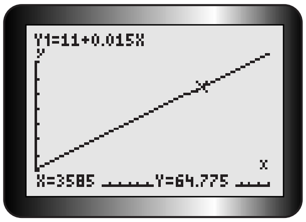
Temperature increases at a steady rate, so it is a function of depth.

b)

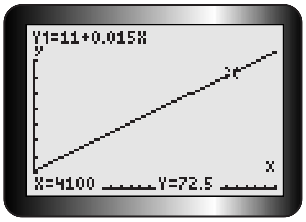


I graphed the function by entering Y1 = 11 + 0.015X into the equation editor.

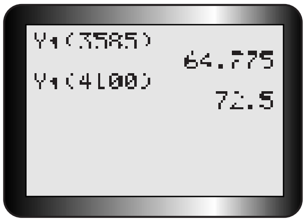
I used **WINDOW** settings of $0 \leq X \leq 5000$, Xscl 200, and $0 \leq Y \leq 100$, Yscl 10.



I used the value operation to find the temperature at the bottom of East Rand mine. This told me that $T(3585) = 64.775$.



Then I used the value operation again to find the temperature at the bottom of the other mine. I found that $T(4100) = 72.5$.



As a check, I called up the function on my calculator home screen, using VARS and function notation to display both answers.

The temperature at the bottom of the East Rand mine is about 65 °C. The temperature at the bottom of Western Deep mine is about 73 °C.

Reflecting

A. How did Lucy, Stuart, and Eli know that the relationship between temperature and depth is a function?

B. How did Lucy use the function equation to determine the two temperatures?

C. What does $T(3585)$ mean? How did Stuart use the graph to determine the value of $T(3585)$?

APPLY the Math

EXAMPLE 2 Representing a situation with a function model

A family played a game to decide who got to eat the last piece of pizza. Each person had to think of a number, double it, and subtract the result from 12. Finally, they each multiplied the resulting difference by the number they first thought of. The person with the highest final number won the pizza slice.

a) Use function notation to express the final answer in terms of the original number.

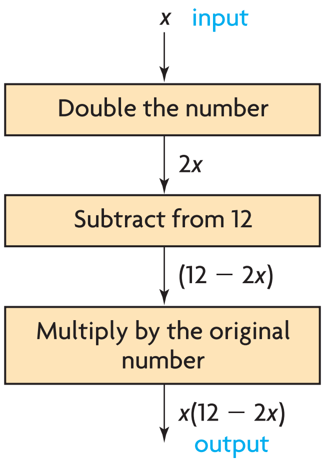
b) The original numbers chosen by the family members are shown. Who won the pizza slice?

c) What would be the best number to choose? Why?

Tim	5
Rhea	-2
Sara	7
Andy	10

Barbara's Solution

a)



I used a flow chart to show what happens to the original number in the game.

I chose x to be the original number, or input.

$$f(x) = x(12 - 2x)$$
$$= 12x - 2x^2$$

The expression for the final answer is quadratic, so the final result must be a function of the original number.

I chose $f(x)$ as the name for the final answer, or output.

b) Tim: $f(5) = 12(5) - 2(5)^2$
$= 60 - 2(25)$
$= 60 - 50$
$= 10$

I found the values of $f(5)$, $f(-2)$, $f(7)$, and $f(10)$ to see who had the highest answer.

Tim's answer was 10.

Rhea: $f(-2) = 12(-2) - 2(-2)^2$
$= -24 - 2(4)$
$= -24 - 8$
$= -32$

Rhea's answer was -32.

Sara: $f(7) = 12(7) - 2(7)^2$
$= 84 - 2(49)$
$= 84 - 98 = -14$

Sara's answer was -14.

Andy: $f(10) = 12(10) - 2(10)^2$
$= 120 - 2(100)$
$= 120 - 200 = -80$

Andy's answer was -80.

Tim won the pizza slice.

Tim's answer was the highest.

c) $f(x) = 12x - 2x^2$

I recognized that the equation was quadratic and that its graph would be a parabola that opened down, since the coefficient of x^2 was negative.

This meant that this quadratic function had a maximum value at its vertex.

$f(x) = -2x(x - 6)$

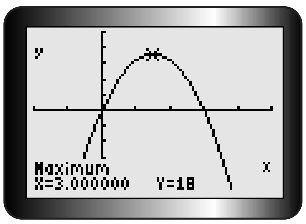
The x-intercepts are $x = 0$ and $x = 6$.

Vertex: $x = (0 + 6) \div 2$
$x = 3$

The best number to choose is 3.

I put the equation back in factored form by dividing out the common factor, $-2x$.

I remembered that the x-coordinate of the vertex is halfway between the two x-intercepts.



I checked my answer by graphing.

EXAMPLE 3 Connecting function notation to a graph

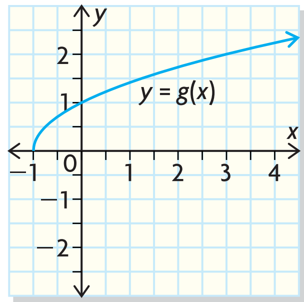
For the function shown in the graph, determine each value.

a) $g(3)$

b) $g(-1)$

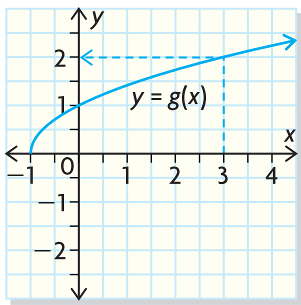
c) x if $g(x) = 1$

d) the domain and range of $g(x)$



Ernesto's Solution

a)



I looked at the graph to find the *y*-coordinate when $x = 3$.

I drew a line up to the graph from the *x*-axis at $x = 3$ and then a line across from that point of intersection to the *y*-axis.

When $x = 3, y = 2.$

The *y*-value was 2, so, in function notation, $g(3) = 2$.

$g(3) = 2$

b) When $x = -1, y = 0.$

I saw that $y = 0$ when $x = -1$, so -1 is the *x*-intercept and $g(-1) = 0$.

$g(-1) = 0$

c) $g(x) = 1$ when $x = 0$

I saw that the graph crosses the *y*-axis at $y = 1$. The *x*-value is 0 at this point.

d) The graph begins at the point $(-1, 0)$ and continues upward. The graph exists only for $x \geq -1$ and $y \geq 0$.

I saw that there was no graph to the left of the point $(-1, 0)$ or below that point.

So the only possible *x*-values are $x \geq -1$, and the only possible *y*-values are $y \geq 0$.

The domain is all real numbers greater than or equal to -1.

The range is all real numbers greater than or equal to 0.

EXAMPLE 4 Using algebraic expressions in functions

Consider the functions $f(x) = x^2 - 3x$ and $g(x) = 1 - 2x$.

a) Show that $f(2) > g(2)$, and explain what that means about their graphs.

b) Determine $g(3b)$.

c) Determine $f(c + 2) - g(c + 2)$.

Jamilla's Solution

a)
$$f(x) = x^2 - 3x$$
$$\begin{aligned} f(2) &= (2)^2 - 3(2) \\ &= 4 - 6 \\ &= -2 \end{aligned}$$
$$g(x) = 1 - 2x$$
$$\begin{aligned} g(2) &= 1 - 2(2) \\ &= 1 - 4 \\ &= -3 \end{aligned}$$

I substituted 2 for x in both functions.

$-2 > -3$, so $f(2) > g(2)$

That means that the point on the graph of $f(x)$ is above the point on the graph of $g(x)$ when $x = 2$.

b)
$$\begin{aligned} g(3b) &= 1 - 2(3b) \\ &= 1 - 6b \end{aligned}$$

I substituted $3b$ for x.

I simplified the equation.

c)
$$\begin{aligned} f(c + 2) - g(c + 2) &= [(c + 2)^2 - 3(c + 2)] - [1 - 2(c + 2)] \\ &= [(c^2 + 4c + 4 - 3c - 6)] - [1 - 2c - 4] \\ &= [c^2 + c - 2] - [-3 - 2c] \\ &= c^2 + c - 2 + 3 + 2c \\ &= c^2 + 3c + 1 \end{aligned}$$

I substituted $c + 2$ for x in both functions.

I used square brackets to keep the functions separate until I had simplified each one.

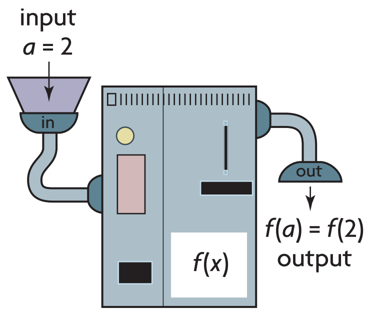
In Summary

Key Idea

- Symbols such as $f(x)$ are called function notation, which is used to represent the value of the dependent variable y for a given value of the independent variable x. For this reason, y and $f(x)$ are interchangeable in the equation of a function, so $y = f(x)$.

Need to Know

- $f(x)$ is read "f at x" or "f of x."
- $f(a)$ represents the value or output of the function when the input is $x = a$. The output depends on the equation of the function. To evaluate $f(a)$, substitute a for x in the equation for $f(x)$.
- $f(a)$ is the y-coordinate of the point on the graph of f with x-coordinate a. For example, if $f(x)$ takes the value 3 at $x = 2$, then $f(2) = 3$ and the point (2, 3) lies on the graph of f.



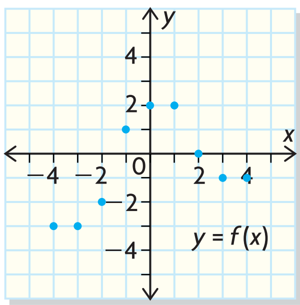
CHECK Your Understanding

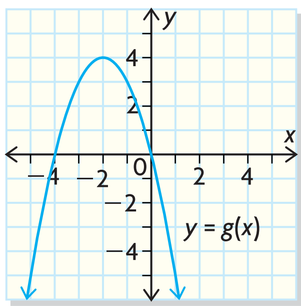
1. Evaluate, where $f(x) = 2 - 3x$.

a) $f(2)$ **c)** $f(-4)$ **e)** $f(a)$

b) $f(0)$ **d)** $f\left(\frac{1}{2}\right)$ **f)** $f(3b)$

2. The graphs of $y = f(x)$ and $y = g(x)$ are shown.





Using the graphs, evaluate

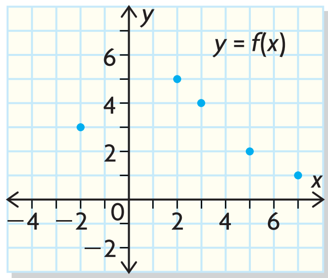
a) $f(1)$ **c)** $f(4) - g(-2)$

b) $g(-2)$ **d)** x when $f(x) = -3$

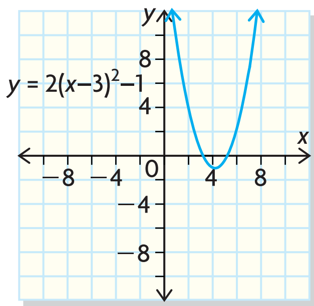
3. Milk is leaking from a carton at a rate of 3 mL/min. There is 1.2 L of milk in the carton at 11:00 a.m.
   a) Use function notation to write an equation for this situation.
   b) How much will be left in the carton at 1:00 p.m.?
   c) At what time will 450 mL of milk be left in the carton?

PRACTISING

4. Evaluate $f(-1)$, $f(3)$, and $f(1.5)$ for
   a) $f(x) = (x - 2)^2 - 1$
   b) $f(x) = 2 + 3x - 4x^2$
5. For $f(x) = \frac{1}{2x}$, determine
   a) $f(-3)$
   b) $f(0)$
   c) $f(1) - f(3)$
   d) $f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)$
6. The graph of $y = f(x)$ is shown at the right.
   a) State the domain and range of f.
   b) Evaluate.
      i) $f(3)$
      ii) $f(5)$
      iii) $f(5 - 3)$
      iv) $f(5) - f(3)$



7. For $h(x) = 2x - 5$, determine
   a) $h(a)$
   b) $h(b + 1)$
   c) $h(3c - 1)$
   d) $h(2 - 5x)$
8. Consider the function $g(t) = 3t + 5$.
   a) Create a table of values and graph the function.
   b) Determine each value.
      i) $g(0)$
      ii) $g(3)$
      iii) $g(1) - g(0)$
      iv) $g(2) - g(1)$
      v) $g(1001) - g(1000)$
      vi) $g(a + 1) - g(a)$
9. Consider the function $f(s) = s^2 - 6s + 9$.
   a) Create a table of values for the function.
   b) Determine each value.
      i) $f(0)$
      ii) $f(1)$
      iii) $f(2)$
      iv) $f(3)$
      v) $[f(2) - f(1)] - [f(1) - f(0)]$
      vi) $[f(3) - f(2)] - [f(2) - f(1)]$
   c) In part (b), what do you notice about the answers to parts (v) and (vi)? Explain why this happens.
10. The graph at the right shows $f(x) = 2(x - 3)^2 - 1$.
   K
   a) Evaluate $f(-2)$.
   b) What does $f(-2)$ represent on the graph of f?
   c) State the domain and range of the relation.
   d) How do you know that f is a function from its graph?



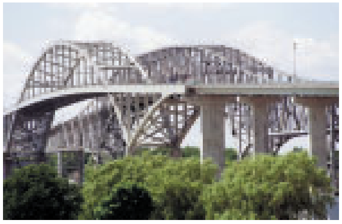
11. For $g(x) = 4 - 5x$, determine the input for x when the output of $g(x)$ is
   a) -6
   b) 2
   c) 0
   d) $\frac{3}{5}$

12. A company rents cars for \$50 per day plus \$0.15/km.

- **a)** Express the daily rental cost as a function of the number of kilometres travelled.
- **b)** Determine the rental cost if you drive 472 km in one day.
- **c)** Determine how far you can drive in a day for \$80.

13. **A** As a mental arithmetic exercise, a teacher asked her students to think of a number, triple it, and subtract the resulting number from 24. Finally, they were asked to multiply the resulting difference by the number they first thought of.

- **a)** Use function notation to express the final answer in terms of the original number.
- **b)** Determine the result of choosing numbers 3, -5, and 10.
- **c)** Determine the maximum result possible.



14. **T** The second span of the Bluewater Bridge in Sarnia, Ontario, is supported by two parabolic arches. Each arch is set in concrete foundations that are on opposite sides of the St. Clair River. The arches are 281 m apart. The top of each arch rises 71 m above the river. Write a function to model the arch.

15. **a)** Graph the function $f(x) = 3(x - 1)^2 - 4$.

- **b)** What does $f(-1)$ represent on the graph? Indicate on the graph how you would find $f(-1)$.
- **c)** Use the equation to determine
  i) $f(2) - f(1)$ **ii)** $2f(3) - 7$ **iii)** $f(1 - x)$

16. Let $f(x) = x^2 + 2x - 15$. Determine the values of x for which

a) $f(x) = 0$ **b)** $f(x) = -12$ **c)** $f(x) = -16$

17. Let $f(x) = 3x + 1$ and $g(x) = 2 - x$. Determine values for a such that

a) $f(a) = g(a)$ **b)** $f(a^2) = g(2a)$

18. **C** Explain, with examples, what function notation is and how it relates to the graph of a function. Include a discussion of the advantages of using function notation.

Extending

19. The highest and lowest marks awarded on an examination were 285 and 75. All the marks must be reduced so that the highest and lowest marks become 200 and 60.

- **a)** Determine a linear function that will convert 285 to 200 and 75 to 60.
- **b)** Use the function to determine the new marks that correspond to original marks of 95, 175, 215, and 255.

20. A function $f(x)$ has these properties:

- The domain of f is the set of natural numbers.
- $f(1) = 1$
- $f(x + 1) = f(x) + 3x(x + 1) + 1$

- **a)** Determine $f(2), f(3), f(4), f(5)$, and $f(6)$.
- **b)** Describe the function.

1.3 Exploring Properties of Parent Functions

GOAL

Explore and compare the graphs and equations of five basic functions.

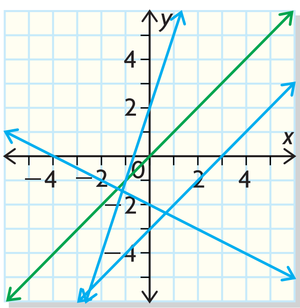
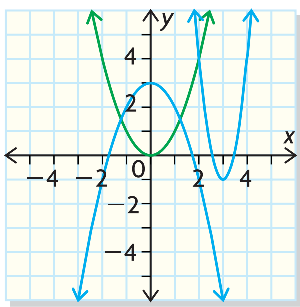
YOU WILL NEED

- graphing calculator or graphing technology
- graph paper

EXPLORE the Math

As a child, you learned how to recognize different animal families.

In mathematics, every function can be classified as a member of a **family.** Each member of a family of functions is related to the simplest, or most basic, function sharing the same characteristics. This function is called the **parent function.** Here are some members of the linear and quadratic families. The parent functions are in green.

Linear Functions	Quadratic Functions
 Parent function: $f(x) = x$	 Parent function: $f(x) = x^2$
Family members: $f(x) = mx + b$	Family members: $f(x) = a(x - h)^2 + k$
Examples: $f(x) = 3x + 2$, $f(x) = -\frac{1}{2}x - 2$	Examples: $f(x) = 5(x - 3)^2 - 1$, $f(x) = -x^2 + 3$

Communication Tip

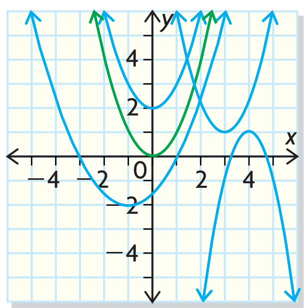
$\sqrt{x}$ always means the positive square root of x.

family

a collection of functions (or lines or curves) sharing common characteristics

parent function

the simplest, or base, function in a family



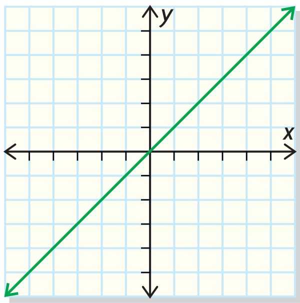
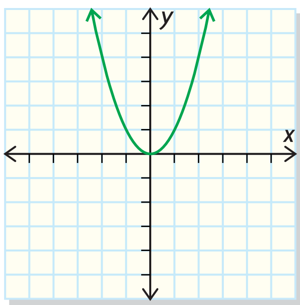
absolute value

written as $|x|$; describes the distance of x from 0; equals x when $x \geq 0$ or $-x$ when $x < 0$; for example, $|3| = 3$ and $|-3| = -(-3) = 3$

Three more parent functions are the square root function $f(x) = \sqrt{x}$, the reciprocal function $f(x) = \frac{1}{x}$, and the **absolute value** function $f(x) = |x|$.

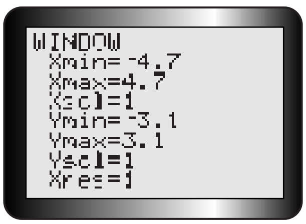
? What are the characteristics of these parent functions that distinguish them from each other?

A. Make a table like the one shown.

Equation of Function	Name of Function	Sketch of Graph	Special Features/ Symmetry	Domain	Range
$f(x) = x$	linear function		• straight line that goes through the origin • slope is 1 • divides the plane exactly in half diagonally • graph only in quadrants 1 and 3		
$f(x) = x^2$	quadratic function		• parabola that opens up • vertex at the origin • y has a minimum value • y-axis is axis of symmetry • graph only in quadrants 1 and 2		
$f(x) = \sqrt{x}$	square root function				
$f(x) = \frac{1}{x}$	reciprocal function				
$f(x) = \|x\|$	absolute value function				

Tech *Support*

Use the following **WINDOW** settings to graph the functions:



You can change to these settings by pressing ZOOM 4.

B. Use your graphing calculator to check the sketches shown for $f(x) = x$ and $f(x) = x^2$ and add anything you think is missing from the descriptions. Explain how you know that these equations are both functions.

C. In your table, record the domain and range of each of $f(x) = x$ and $f(x) = x^2$.

D. Clear all equations from the equation editor. Graph the square root function, $f(x) = \sqrt{x}$. In your table, sketch the graph and describe its shape. Is it a function? Explain. How is it different from the graphs of linear and quadratic functions?

E. Go to the table of values and scroll up and down the table. Does ERR: appear in the Y column? Explain why this happens.

F. Using the table of values and the graph, determine and record the domain and range of the function.

G. Repeat parts D through F for the reciprocal function $f(x) = \frac{1}{x}$. Use the table of values to see what happens to y when x is close to 0 and when x is far from 0. Explain why the graph is in two parts with a break in the middle.

H. Where are the **asymptotes** of this graph?

I. Repeat parts D through F for the absolute value function $f(x) = |x|$. Which of the other functions is the resulting graph most like? Explain. When you have finished, make sure that your table contains enough information for you to recognize each of the five parent functions.

Reflecting

J. Explain how each of the following helped you determine the domain and range.
a) the table of values
b) the graph
c) the function's equation

K. Which graphs lie in the listed quadrants?
a) the first and second quadrants
b) the first and third quadrants

L. Which graph has asymptotes? Why?

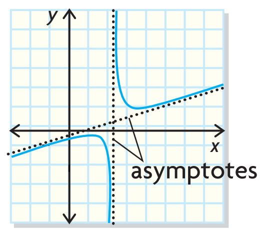
M. You have used the slope and y-intercept to sketch lines, vertices, and directions of opening to sketch parabolas. What characteristics of the new parent functions $f(x) = \sqrt{x}$, $f(x) = \frac{1}{x}$, and $f(x) = |x|$ could you use to sketch their graphs?

Tech Support

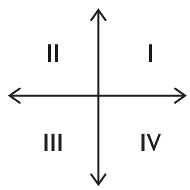
For help with the TABLE function of the graphing calculator, see Technical Appendix, B-6.
To graph $f(x) = |x|$, press MATH ▶ 1 X,T,Θ,n).

asymptote

a line that the graph of a relation or function gets closer and closer to, but never meets, on some portion of its domain



Quadrants



In Summary

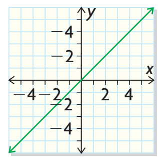
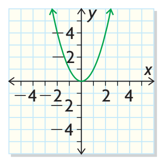
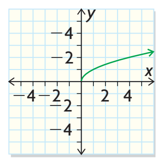
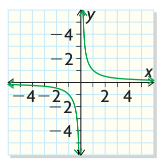
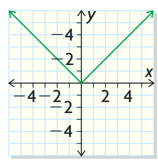
Key Idea

- Certain basic functions, called parent functions, form the building blocks for families of more complicated functions. Parent functions include, but are not limited to, $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = \frac{1}{x}$, and $f(x) = |x|$.

(continued)

Need to Know

- Each of the parent functions $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = \frac{1}{x}$, and $f(x) = |x|$ has unique characteristics that define the shape of its graph.

Equation of Function	Name of Function	Sketch of Graph
$f(x) = x$	linear function	
$f(x) = x^2$	quadratic function	
$f(x) = \sqrt{x}$	square root function	
$f(x) = \frac{1}{x}$	reciprocal function	
$f(x) = \|x\|$	absolute value function	

FURTHER Your Understanding

1. Sketch the graphs of $f(x) = x$ and $g(x) = \frac{1}{x}$ on the same axes. What do the graphs have in common? What is different about the graphs? Write equations of the asymptotes for the reciprocal function.
2. Sketch the graphs of $f(x) = x^2$ and $g(x) = |x|$ on the same axes. Describe how these graphs are alike and how they are different.
3. Sketch the graph of $f(x) = x^2$ for values of $x \geq 0$. On the same axes, sketch the graphs of $g(x) = \sqrt{x}$ and $h(x) = x$. Describe how the three graphs are related.

1.4 Determining the Domain and Range of a Function

GOAL

Use tables, graphs, and equations to find domains and ranges of functions.

YOU WILL NEED

- graph paper
- graphing calculator

LEARN ABOUT the Math

The CN Tower in Toronto has a lookout level that is 346 m above the ground.

A gull landing on the guardrail causes a pebble to fall off the edge.

The speed of the pebble as it falls to the ground is a **function** of how far it has fallen. The equation for this function is $v(d) = \sqrt{2gd}$, where

- d is the distance, in metres, the pebble has fallen
- $v(d)$ is the speed of the pebble, in metres per second (m/s)
- g is the acceleration due to gravity—about 9.8 metres per second squared (m/s^2)

? How can you determine the domain and range of the function $v(d)$?

EXAMPLE 1 Selecting a strategy to determine the domain and range

Determine the domain and range of $v(d)$, the pebble's speed.

Sally's Solution: Using a Graph

The pebble falls a total distance of 346 m. So the domain is $0 \leq d \leq 346$.

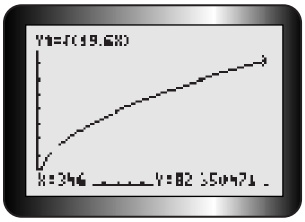
The distance d is 0 m when the pebble first falls off the edge and 346 m when the pebble lands on the ground. So d can take only values that lie in between 0 and 346. This gave me the domain for the function.

$2 \times 9.8 = 19.6$, so the function is

$v(d) = \sqrt{19.6d}$ for $0 \leq d \leq 346$

I entered Y= $\sqrt{19.6x}$ into my graphing calculator. I used $0 \leq X \leq 346$, Xscl 20 and $0 \leq Y \leq 100$, Yscl 10 for WINDOW settings.

Range:



$v(346) \doteq 82.4$

So the range is $0 \leq v(d) \leq 82.4$.

I saw that the graph started at the origin.

The pebble starts with no velocity. So 0 is the minimum value of the range.

The graph showed me that as the pebble's distance increases, so does its velocity. The pebble must be travelling the fastest when it hits the ground. This happens when $d = 346$. The maximum value of the range is $v(346)$. I evaluated this using the value operation.

Domain $= \{d \in \mathbf{R} \mid 0 \leq h \leq 346\}$

Range $= \{v(d) \in \mathbf{R} \mid 0 \leq v(d) \leq 82.4\}$

I used set notation to write the domain and range. I defined them as sets of **real numbers.**

real numbers

numbers that are either rational or irrational; these include positive and negative integers, zero, fractions, and irrational numbers such as $\sqrt{2}$ and π

Communication *Tip*

Set notation can be used to describe domains and ranges. For example, $\{x \in \mathbf{R} \mid 0 \leq x < 50\}$ is read "the set of all values x that belong to the set of real numbers, such that x is greater than or equal to 0 and less than 50." The symbol "|" stands for "such that."

David's Solution: Using the Function Equation

$d = 0$ when the pebble begins to fall, and $d = 346$ when it lands.

So the domain is $0 \leq d \leq 346$.

I found the domain by thinking about all the values that d could have. d is 0 m when the pebble first falls off the edge and 346 m when the pebble lands on the ground. So d must take values between 0 and 346.

The pebble starts with speed 0 m/s.

The pebble *fell* off the edge, so the speed was zero at the start.

$v(0) = \sqrt{19.6(0)} = 0$

I used the equation as a check.

As the pebble falls, its speed increases.

I knew that the pebble would gain speed until it hit the ground.

When the pebble lands, $d = 346$.

$$v(346) = \sqrt{19.6(346)}$$
$$= 82.4\text{, to one decimal place}$$

I used the function equation to find how fast the pebble was falling when it landed.

The domain is $\{d \in \mathbf{R} \mid 0 \leq d \leq 346\}$ and the range is $\{v(d) \in \mathbf{R} \mid 0 \leq v(d) \leq 82.4\}$.

I used set notation to write the domain and range.

Reflecting

A. Why did Sally need to think about the possible values for distance fallen before she graphed the function?

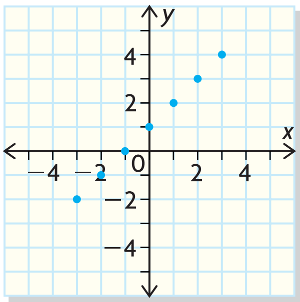
B. What properties of the square root function helped David use the given equation to find the domain and range?

APPLY the Math

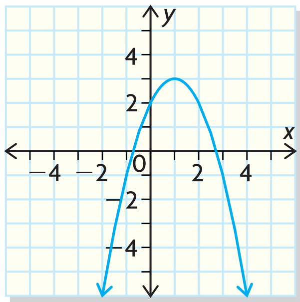
EXAMPLE 2 Determining domain and range from graphs

For each relation, state the domain and range and whether the relation is a function.

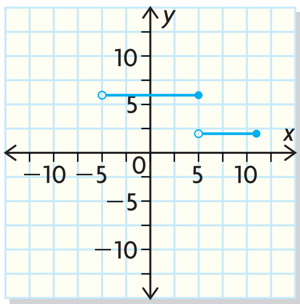
a)



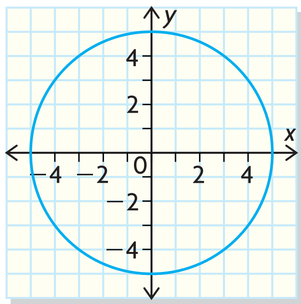
c)



b)



d)



Melanie's Solution

a) Domain $= \{x \in \mathbf{I} \mid -3 \le x \le 3\}$, or $\{-3, -2, -1, 0, 1, 2, 3\}$

Range $= \{y \in \mathbf{I} \mid -2 \le y \le 4\}$, or $\{-2, -1, 0, 1, 2, 3, 4\}$

I noticed that the *x*-coordinates were all the integers from -3 to 3 and the *y*-coordinates were all the integers from -2 to 4.

The graph is that of a function.

The graph passes the vertical-line test.

b) Domain $= \{x \in \mathbf{R} \mid -5 < x \leq 11\}$

An open circle on the graph shows that the endpoint of the line is not included in the graph. A closed circle means that the endpoint is included. So, x cannot be -5, but it can be 11.

Range $= \{2, 6\}$

There are only two y-values.

This is a function.

The graph passes the vertical-line test.

c) Domain $= \{x \in \mathbf{R}\}$

Range $= \{y \in \mathbf{R} \mid y \leq 3)\}$

The graph is a parabola with a maximum value at the vertex, which is the point (1, 3).

Therefore, x can be any real number, but y cannot be greater than 3.

This is a function.

The graph passes the vertical-line test.

d) Domain $= \{x \in \mathbf{R} \mid -5 \leq x \leq 5\}$

Range $= \{y \in \mathbf{R} \mid -5 \leq y \leq 5\}$

This is not a function.

The graph is a circle with centre (0, 0) and radius of 5. The graph fails the vertical-line test. There are many vertical lines that cross the graph in two places

EXAMPLE 3 Determining domain and range from the function equation

Determine the domain and range of each function.

a) $f(x) = 2x - 3$ **b)** $g(x) = -3(x + 1)^2 + 6$ **c)** $h(x) = \sqrt{2 - x}$

Jeff's Solution

a) $f(x) = 2x - 3$

This is the equation of a straight line that goes on forever in both directions.

This is a linear function, so x and y can be any value.

x and $f(x)$ can be any numbers.

Domain $= \{x \in \mathbf{R}\}$

Range $= \{y \in \mathbf{R}\}$

I used y instead of $f(x)$ to describe the range.

b) $g(x) = -3(x + 1)^2 + 6$

This is the equation of a parabola that opens down, so y can never be more than its value at the vertex.

This is a quadratic equation in vertex form. The function has a maximum value at the vertex $(-1, 6)$. x can be any value.

Any value of x will work in the equation, so x can be any number.

Domain $= \{x \in \mathbf{R}\}$

Range $= \{y \in \mathbf{R} \mid y \leq 6\}$

c) $h(x) = \sqrt{2 - x}$

$2 - x \geq 0$

$2 - x \geq 0$ as long as $x \leq 2$

Domain $= \{x \in \mathbf{R} \mid x \leq 2\}$

$\sqrt{2 - x}$ means the positive square root, so y is never negative.

Range $= \{y \in \mathbf{R} \mid y \geq 0\}$

You cannot take the square root of a negative number, so $2 - x$ must be positive or zero.

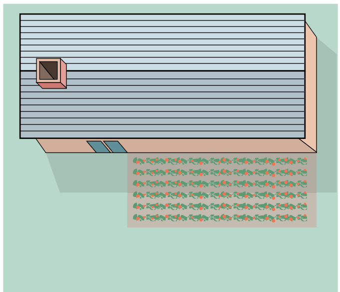
I thought about different possible values of x. 2 is okay, since $2 - 2 = 0$, but 4 is not, since $2 - 4$ is negative. I realized I had to use values less than or equal to 2.

EXAMPLE 4 Determining domain and range of an area function

Vitaly and Sherry have 24 m of fencing to enclose a rectangular garden at the back of their house.

a) Express the area of the garden as a function of its width.

b) Determine the domain and range of the area function.

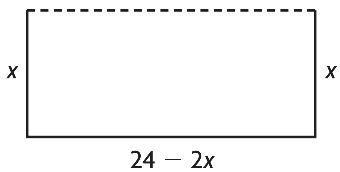


Jenny's Solution

a) Let the width of the garden be x m. Then the length is $(24 - 2x)$ m.

They need fencing on only three sides of the garden because the house forms the last side.

To find the length, I subtracted the two widths from 24.



Let the area be $A(x)$.

$$A(x) = x(24 - 2x)$$

$$A(x) = -2x(x - 12)$$

Area = width × length

I factored out -2 from $24 - 2x$ to write the function in factored form.

b) The smallest the width can approach is 0 m. The largest the width can approach is 12 m.

Domain $= \{x \in \mathbf{R} \mid 0 < x < 12\}$

This is a quadratic function that opens down. It has two zeros, at 0 and 12. The vertex lies halfway in between the zeros, above the x-axis, so the numbers in the domain have to be between 0 and 12. Any number ≤ 0 or ≥ 12 will result in a zero or negative area, which doesn't make sense.

$$x = (0 + 12) \div 2$$
$$x = 6$$

The vertex is halfway between $x = 0$ and $x = 12$. The x-coordinate of the vertex is 6.

$$A(6) = -2(6)(6 - 12)$$
$$= 72$$

The area ranges from 0 to 72 m^2.

Range
$= \{A(x) \in \mathbf{R} \mid 0 < A(x) \leq 72\}$

I substituted $x = 6$ into the area function to find the y-coordinate of the vertex. Since area must be a positive quantity, all the output values of the function must lie between 0 and 72.

In Summary

Key Ideas

- The domain of a function is the set of values of the independent variable for which the function is defined. The range of a function depends on the equation of the function. The graph depends on the domain and range.
- The domain and range of a function can be determined from its graph, from a table of values, or from the function equation. They are usually easier to determine from a graph or a table of values.

Need to Know

- All linear functions include all the real numbers in their domains. Linear functions of the form $f(x) = mx + b$, where $m \neq 0$, have range $\{y \in \mathbf{R}\}$. Constant functions $f(x) = b$ have range $\{b\}$.
- All quadratic functions have domain $\{x \in \mathbf{R}\}$. The range of a quadratic function depends on the maximum or minimum value and the direction of opening.
- The domains of square root functions are restricted because the square root of a negative number is not a real number. The ranges are restricted because the square root sign refers to the positive square root. For example,
  - The function $f(x) = \sqrt{x}$ has domain $= \{x \in \mathbf{R} \mid x \geq 0\}$ and range $= \{y \in \mathbf{R} \mid y \geq 0\}$.
  - The function $g(x) = \sqrt{x - 3}$ has domain $= \{x \in \mathbf{R} \mid x \geq 3\}$ and range $= \{y \in \mathbf{R} \mid y \geq 0\}$.
- When working with functions that model real-world situations, consider whether there are any restrictions on the variables. For example, negative values often have no meaning in a real context, so the domain or range must be restricted to nonnegative values.

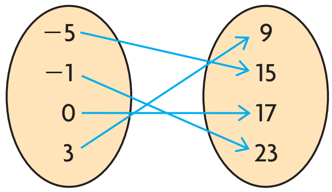
CHECK Your Understanding

1. State the domain and range of each relation.

a)

Year of Birth	Life Expectancy (years)
1900	47.3
1920	54.1
1940	62.9
1960	69.7
1980	73.7
2000	77.0

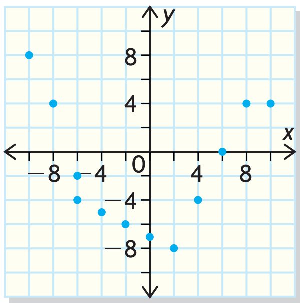
b)



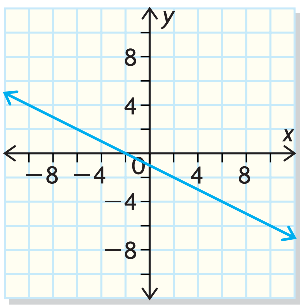
c) $\{(-4, 7), (0, 5), (0, 3), (3, 0), (5, -1)\}$

2. State the domain and range of each relation.

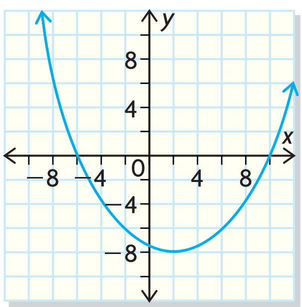
a)



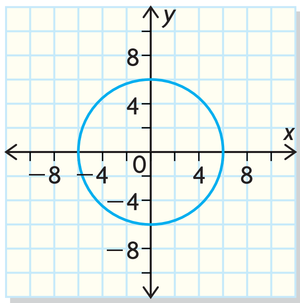
b)



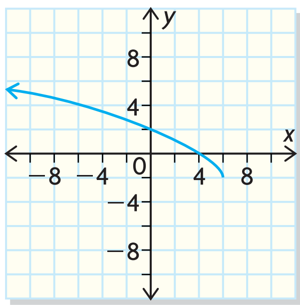
c)



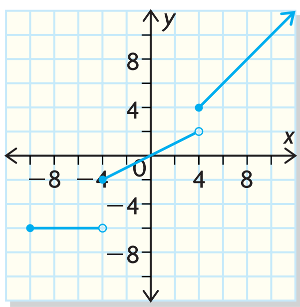
d)



e)



f)

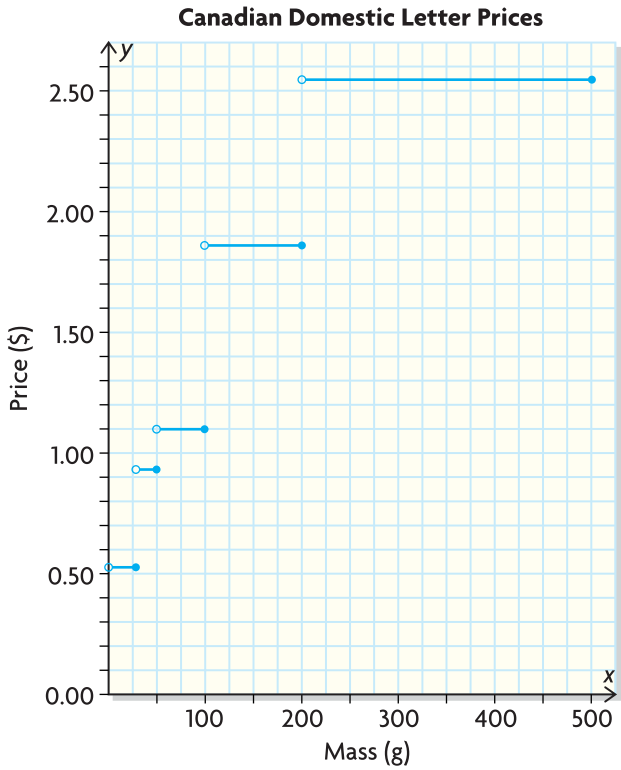


3. Identify which of the relations in questions 1 and 2 are functions.

4. Determine the domain and range of the function $f(x) = 2(x - 1)^2 - 3$ by sketching its graph.

PRACTISING

5. The graph shows how 2007 prices for mailing letters in Canada vary with mass.



a) Explain why this relation is a function. Why is it important for this to be so?

b) State the domain and range of the function.

6. The route for a marathon is 15 km long. Participants may walk, jog, run, or cycle. Copy and complete the table to show times for completing the marathon at different speeds.

Speed (km/h)	1	2	3	4	5	6	8	10	15	20
Time (h)	15.0	7.5								

Graph the relation in the table and explain how you know that it is a function. State the domain and range of the function.

7. A relation is defined by $x^2 + y^2 = 36$.

K a) Graph the relation.

b) State the domain and range of the relation.

c) Is the relation a function? Explain.

8. Write a function to describe coffee dripping into a 10-cup carafe at a rate of 1 mL/s. State the domain and range of the function (1 cup = 250 mL).

9. Determine the domain and range of each function.

a) $f(x) = -3x + 8$

b) $g(x) = -0.5(x + 3)^2 + 4$

c) $h(x) = \sqrt{x - 1}$

d) $p(x) = \frac{2}{3}(x - 2)^2 - 5$

e) $q(x) = 11 - \frac{5}{2}x$

f) $r(x) = \sqrt{5 - x}$

10. **A** A ball is thrown upward from the roof of a 25 m building. The ball reaches a height of 45 m above the ground after 2 s and hits the ground 5 s after being thrown.

a) Sketch a graph that shows the height of the ball as a function of time.

b) State the domain and range of the function.

c) Determine an equation for the function.

11. Write the domain and range of each function in set notation.

a) $f(x) = 4x + 1$

b) $f(x) = \sqrt{x - 2}$

c) $f(x) = 3(x + 1)^2 - 4$

d) $f(x) = -2x^2 - 5$

12. Use a graphing calculator to graph each function and determine the domain and range.

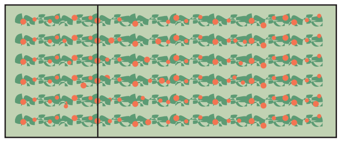
a) $f(x) = \sqrt{3 - x} + 2$

b) $g(x) = x^2 - 3x$

c) $h(x) = \frac{1}{x^2}$

d) $p(x) = \sqrt{x^2 - 5}$

13. **T** A farmer has 450 m of fencing to enclose a rectangular area and divide it into two sections as shown.



a) Write an equation to express the total area enclosed as a function of the width.

b) Determine the domain and range of this area function.

c) Determine the dimensions that give the maximum area.

14. Determine the range of each function if the domain is $\{-3, -1, 0, 2.5, 6\}$.

a) $f(x) = 4 - 3x$

b) $f(x) = 2x^2 - 3x + 1$

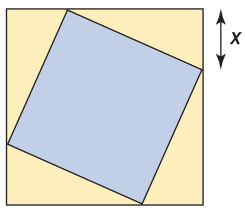
15. **C** Explain the terms "domain" and "range" as they apply to relations and functions. Describe, with examples, how the domain and range are determined from a table of values, a graph, and an equation.

Extending

16. **a)** Sketch the graph of a function whose domain is $\{x \in \mathbf{R}\}$ and range is $\{y \in \mathbf{R} \mid y \leq 2)\}$.

b) Sketch the graph of a relation that is not a function and whose domain is $\{x \in \mathbf{R} \mid x \geq -4\}$ and range is $\{y \in \mathbf{R}\}$.

17. You can draw a square inside another square by placing each vertex of the inner square on one side of the outer square. The large square in the diagram has side length 10 units.



a) Determine the area of the inscribed square as a function of x.

b) Determine the domain and range of this area function.

c) Determine the perimeter of the inscribed square as a function of x.

d) Determine the domain and range of this perimeter function.

1 Mid-Chapter Review

FREQUENTLY ASKED Questions

Study *Aid*

- See Lesson 1.1, Examples 1, 2, and 3.
- Try Mid-Chapter Review Questions 1 and 2.

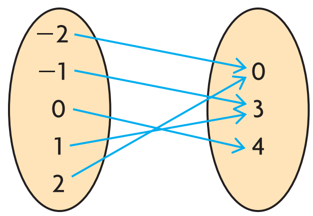
Q: How can you determine whether a relation is a function?

A1: For a relation to be a function, there must be only one value of the dependent variable for each value of the independent variable.

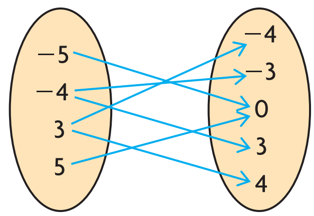
If the relation is described by a list of ordered pairs, you can see if any first elements appear more than once. If they do, the relation is not a function. For example, the relation $\{(-2, 0), (-1, 3), (0, 4), (1, 3), (2, 0)\}$ is a function; but the relation $\{(-5, 0), (-4, 3), (-4, -3), (3, -4), (3, 4), (5, 0)\}$ is not, because -4 and 3 each appear more than once as first elements.

A2: If the relation is shown in a mapping diagram, you can look at the arrows. If more than one arrow goes from an element of the domain (on the left) to an element of the range (on the right), then the relation is not a function. For example, diagram A shows a function but diagram B does not.

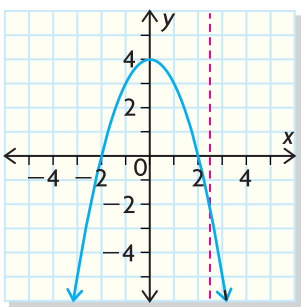
A



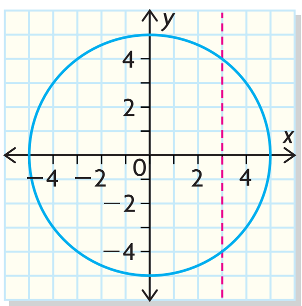
B



A



B



A3: If you have the graph of the relation, you can use the vertical-line test. If you can draw a vertical line that crosses the graph in more than one place, then an element in the domain corresponds to two elements in the range, so the relation is not a function. For example, graph A shows a function but graph B does not.

A4: If you have the equation of the relation, you can substitute numbers for x to see how many y-values correspond to each x-value. If a single x-value produces more than one corresponding y-value, the equation does not represent a function. For example, the equation $y = 4 - x^2$ is the equation of a function because you would get only one answer for y by putting a number in for x. The equation $x^2 + y^2 = 25$ does *not* represent a function because there are two values for y when x is any number between -5 and 5.

Study *Aid*

- See Lesson 1.2, Examples 1, 3, and 4.
- Try Mid-Chapter Review Questions 3 and 4.

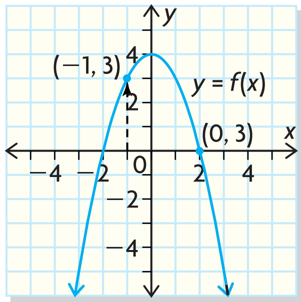
Q: What does function notation mean and why is it useful?

A: When a relation is a function, you can use function notation to write the equation. For example, you can write the equation $y = 4 - x^2$ in function notation as $f(x) = 4 - x^2$. f is a name for the function and $f(a)$ is the value of y or output when the input is $x = a$. The equation $f(-1) = 3$ means "When $x = -1$, $y = 3$;" in other words, the point $(-1, 3)$ belongs to the function.

To evaluate $f(-1)$, substitute -1 for x in the function equation:

$$\begin{aligned} f(-1) &= 4 - (-1)^2 \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

Or you can read the value from a graph.



Function notation is useful because writing $f(x) = 3$ gives more information about the function—you know that the independent variable is x—than writing $y = 3$. Also, you can work with more than one function at a time by giving each function a different name. You can choose meaningful names, such as $v(t)$ to describe velocity as a function of time, t, or $C(n)$ to describe the cost of producing n items.

Q: How can you determine the domain and range of a function?

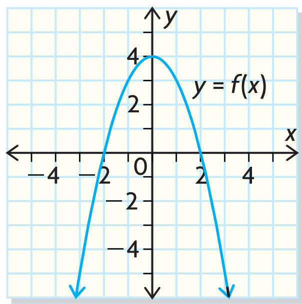
A: The domain of a function is the set of input values for which the function is defined. The range is the set of output values that correspond to the input values. Set notation can be used to describe the domain and range of a function.

Study *Aid*

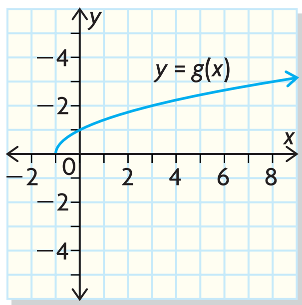
- See Lesson 1.4, Examples 2 and 3.
- Try Mid-Chapter Review Questions 6, 7, and 8.

If you have the graph of a function, you can see the domain and range, as in the following examples:

A



B



Because graph A goes on forever in both the positive and negative x direction, x can be any real number.

Because this function has a maximum value at the vertex, y cannot have a value greater than this maximum value.

You can express these facts in set notation:

$$\text{Domain} = \{x \in \mathbf{R}\};\ \ \text{Range} = \{y \in \mathbf{R} \mid y \leq 4\}$$

Graph B starts at the point $(-1, 0)$ and continues forever in the positive x direction and positive y direction. So x can be any real number greater than or equal to -1 and y can be any real number greater than or equal to 0.

$$\text{Domain} = \{x \in \mathbf{R} \mid x \geq -1\};\ \ \text{Range} = \{y \in \mathbf{R} \mid y \geq 0\}$$

You can also determine the domain and range from the equation of a function. For example, if $f(x) = 4 - x^2$, then any value of x will work in this equation, so $x \in \mathbf{R}$. Also, because x^2 is always positive or zero, $f(x)$ is always less than or equal to 4.

$$\text{Domain} = \{x \in \mathbf{R}\};\ \ \text{Range} = \{y \in \mathbf{R} \mid y \leq 4\}$$

If $g(x) = \sqrt{x + 1}$, then x cannot be less than -1, or the number inside the square root sign would be negative. Also, the square root sign refers to the positive square root, so $g(x)$ is always positive or zero.

$$\text{Domain} = \{x \in \mathbf{R} \mid x \geq -1\};\ \ \text{Range} = \{y \in \mathbf{R} \mid y \geq 0\}$$

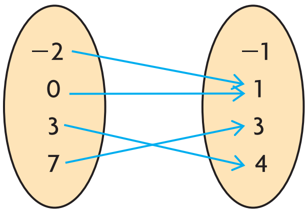
PRACTICE Questions

Lesson 1.1

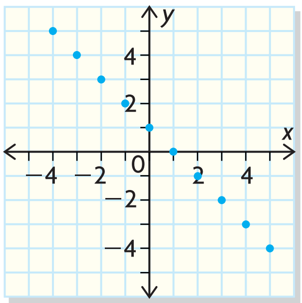
1. Determine which relations are functions. For those which are, explain why.

a) $\{(1, 2), (2, 3), (2, 4), (4, 5)\}$

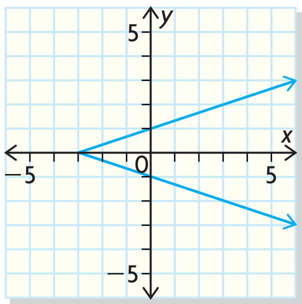
b)



c)



d)



e) $y = -(x - 3)^2 + 5$

f) $y = \sqrt{x - 4}$

2. Use numeric and graphical representations to show that $x^2 + y = 4$ is a function but $x^2 + y^2 = 4$ is not a function.

Lesson 1.2

3. **a)** Graph the function $f(x) = -2(x + 1)^2 + 3$.

b) Evaluate $f(-3)$.

c) What does $f(-3)$ represent on the graph of f?

d) Use the equation to determine **i)** $f(1) - f(0)$, **ii)** $3f(2) - 5$, and **iii)** $f(2 - x)$.

4. A teacher asked her students to think of a number, multiply it by 5, and subtract the product from 20. Then she asked them to multiply the resulting difference by the number they first thought of.

a) Use function notation to express the final answer in terms of the original number.

b) Determine the outputs for the input numbers 1, -1, and 7.

c) Determine the maximum result possible.

Lesson 1.3

5. Graph each function and state its domain and range.

a) $f(x) = x^2$

b) $f(x) = \dfrac{1}{x}$

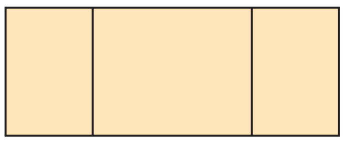
c) $f(x) = \sqrt{x}$

d) $f(x) = |x|$

Lesson 1.4

6. Determine the domain and range of each relation in question 1.

7. A farmer has 600 m of fencing to enclose a rectangular area and divide it into three sections as shown.



a) Write an equation to express the total area enclosed as a function of the width.

b) Determine the domain and range of this area function.

c) Determine the dimensions that give the maximum area.

8. Determine the domain and range for each.

a) A parabola has a vertex at $(-2, 5)$, and $y = 5$ is its maximum value.

b) A parabola has a vertex at $(3, 4)$, and $y = 4$ is its minimum value.

c) A circle has a centre at $(0, 0)$ and a radius of 7.

d) A circle has a centre at $(2, 5)$ and a radius of 4.

1.5 The Inverse Function and Its Properties

YOU WILL NEED

- graph paper
- Mira™ (transparent mirror) (optional)

GOAL

Determine inverses of linear functions and investigate their properties.

INVESTIGATE the Math

The Backyard Paving Company charges \$10/sq ft for installing interlocking paving stones, plus a \$50 delivery fee. The company calculates the cost to the customer as a function of the area to be paved. Tom wants to express area in terms of cost to see how much of his yard he can pave for different budget amounts.

A

x Area (sq ft)	y Cost (\$)
40	450
80	
120	
160	
200	

? What relation can Tom use, and how is it related to the function used by the company?

A. Copy and complete table A, using the company's prices. What is the independent variable in table A? the dependent variable?

B. Is the relation in table A a function? Explain.

C. Write the equation for $f(x)$ that describes the cost as a function of area.

D. Graph $f(x)$. Use the same scale of -100 to 2100 on each axis.

E. Tom needs to do the reverse of what the company's function does. Copy and complete table E for Tom. What is the independent variable? the dependent variable? How does this table compare with table A?

E

Cost (\$)	Area (sq ft)
450	40
850	
1250	
1650	
2050	

F. The relationship in part E is the inverse of the cost function. Graph this inverse relation on the same axes as those in part D. Is this relation a function? Explain.

inverse of a function
the reverse of the original function; undoes what the original function has done

G. Draw the line $y = x$ on your graph. Place a Mira along the line $y = x$, or fold your graph paper along that line. What do you notice about the two graphs? Where do they intersect?

H. Compare the coordinates of points that lie on the graph of the cost function with those which lie on the graph of its inverse. What do you notice?

I. Write the slopes and y-intercepts of the two lines.
i) How are the slopes related?
ii) How are the y-intercepts related?
iii) Use the slope and y-intercept to write an equation for the inverse.

J. Use inverse operations on the cost function, f, to solve for x. Compare this equation with the equation of the inverse you found in part I.

K. Make a list of all the connections you have observed between the Backyard Paving Company's cost function and the one Tom will use.

Reflecting

L. How would a table of values for a linear function help you determine the inverse of that function?

M. **i)** How can you determine the coordinates of a point on the graph of the inverse function if you know a point on the graph of the original function?
ii) How could you use this relationship to graph the inverse?

N. How are the domain and range of the inverse related to the domain and range of a linear function?

O. How could you use inverse operations to determine the equation of the inverse of a linear function from the equation of the function?

APPLY the Math

EXAMPLE 1 | Representing the equation of the inverse of a linear function

Find the inverse of the function defined by $f(x) = 2 - 5x$. Is the inverse a function? Explain.

Jamie's Solution: Reversing the Operations

In the equation $f(x) = 2 - 5x$, the operations on x are as follows: Multiply by -5 and then add 2.

I wrote down the operations on x in the order they were applied.

To reverse these operations, subtract 2 and then divide the result by -5.

Then I worked backward and wrote the inverse operations.

$$f^{-1}(x) = \frac{x-2}{-5} \quad \text{or}$$

$$f^{-1}(x) = -\frac{1}{5}x + \frac{2}{5}$$

I used these inverse operations to write the equation of the inverse.

The inverse is linear, so it must be a function, since all linear relations except vertical lines are functions.

I knew the inverse was a line.

Communication *Tip*

The function f^{-1} is the inverse of the function f. This use of -1 is different from raising values to the power -1.

Lynette's Solution: Interchanging the Variables

$$f(x) = 2 - 5x$$
$$y = -5x + 2$$

I wrote the function in $y = mx + b$ form by putting y in place of $f(x)$.

$$x = -5y + 2$$

I knew that if (x, y) is on the graph of $f(x)$, then (y, x) is on the inverse graph, so I switched x and y in the equation.

$$x - 2 = -5y + 2 - 2$$
$$x - 2 = -5y$$
$$\frac{x-2}{-5} = y$$

I solved for y by subtracting 2 from both sides and dividing both sides by -5.

$$f^{-1}(x) = \frac{x-2}{-5} \quad \text{or} \quad f^{-1}(x) = \frac{2-x}{5}$$

I wrote the equation in function notation.

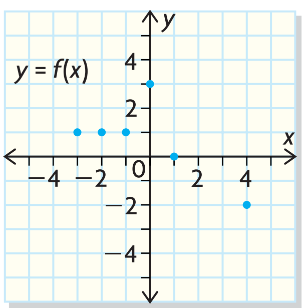
The inverse is a function.

The graph of $y = f^{-1}(x)$ is a straight line with slope $= -\frac{1}{5}$. The graph passes the vertical-line test.

EXAMPLE 2 Relating the graphs of functions and their inverses

Use the graph of each function to obtain the graph of the inverse. Is the inverse a function? Explain.

a)



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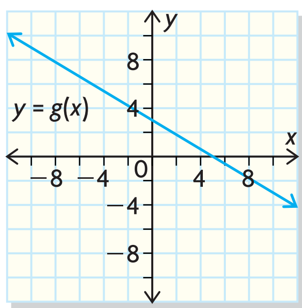


Carlos's Solution

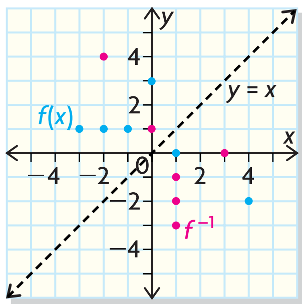
a) $f(x)$ is a function represented by the set of points $\{(-3, 1), (-2, 1), (-1, 1), (0, 3), (1, 0), (4, -2)\}$.

So $f^{-1}(x)$ is $\{(1, -3), (1, -2), (1, -1), (3, 0), (0, 1), (-2, 4)\}$.

I wrote the coordinates of the points in the graph and then switched the x- and y-coordinates of each point. That gave me the inverse.

Plot the points for the inverse and draw the line $y = x$ to check for symmetry.

I plotted the points in red.

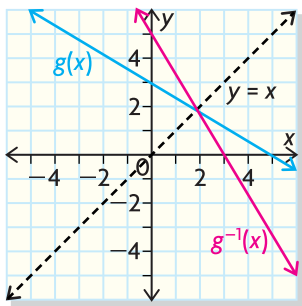


I checked that the points on one side of the line $y = x$ were mirror images of the points on the other side.

The inverse is not a function: The graph fails the vertical-line test at $x = 1$.

There are three red points for $x = 1$, so a vertical line drawn here would go through three points.

b)



I wrote the coordinates of the x- and y-intercepts of $g(x)$: (5, 0) and (0, 3).

Then I switched the coordinates to find the two points (0, 5) and (3, 0) of $g^{-1}(x)$. I noticed that they were the intercepts.

I plotted the two points of $g^{-1}(x)$ and joined them with a straight line.

I drew the line $y = x$ and checked that the graphs of $g(x)$ and $g^{-1}(x)$ crossed on that line.

The inverse is a function.

The inverse is a function because it passes the vertical-line test.

EXAMPLE 3 Using the inverse of a linear function to solve a problem

Recall from Lesson 1.2 that the temperature below Earth's surface is a function of depth and can be defined by $T(d) = 11 + 0.015d$.

a) State the domain and range of $T(d)$.
b) Determine the inverse of this function.
c) State the domain and range of $T^{-1}(d)$.
d) Explain what the inverse represents.

Erynn's Solution

a) Domain $= \{d \in \mathbf{R} \mid 0 \le d \le 5000\}$

I realized that d is 0 m on the surface. This is the beginning of the domain. The deeper mine has a depth of 4100 m, so I chose to end the domain at 5000.

Range $=$
$\{T(d) \in \mathbf{R} \mid 11 \le T(d) \le 86\}$

I calculated the beginning and end of the range by substituting $d = 0$ and $d = 5000$ into the equation for $T(d)$.

b)
$$T(d) = 11 + 0.015d$$

I wrote the temperature function with y and x instead of $T(d)$ and d.

$$y = 11 + 0.015x$$
$$x = 11 + 0.015y$$
$$x - 11 = 0.015y$$

I switched x and y and solved for y to get the inverse equation.

$$\frac{x - 11}{0.015} = y$$

$d(T) = \dfrac{T - 11}{0.015}$ is the inverse function.

Because I had switched the variables, I knew that y was now distance and x was temperature. I wrote the inverse in function notation.

c) Domain $\{T \in \mathbf{R} \mid 11 \le T \le 86\}$

Range $=$
$\{d(T) \in \mathbf{R} \mid 0 \le d(T) \le 5000\}$

The domain of the inverse is the same as the range of the original function, and the range of the inverse is the same as the domain of the original function.

d) The inverse shows the depth as a function of the temperature.

The inverse function is used to determine how far down a mine you would have to go to reach a temperature of, for example, 22 °C. I substituted 22 for T in the equation to get the answer.

$$d(22) = \frac{22 - 11}{0.015}$$
$$\doteq 733$$

When the temperature is 22 °C, the depth is about 733 m.

Someone planning a geothermal heating system would need this kind of information.

In Summary

Key Ideas

- The inverse of a linear function is the reverse of the original function. It undoes what the original has done and can be found using the inverse operations of the original function in reverse order. For example, to apply the function defined by $f(x) = 5x + 8$, multiply x by 5 and then add 8. To reverse this function, subtract 8 from x and then divide the result by 5: $f^{-1}(x) = \frac{x-8}{5}$.
- The inverse of a function is not necessarily a function itself.

Need to Know

- A way to determine the inverse function is to switch the two variables and solve for the previously independent variable. For example, if $y = 5x + 8$, rewrite this equation as $x = 5y + 8$ and solve for y to get $y = \frac{x-8}{5}$.
- If the original function is linear (with the exception of a horizontal line), the inverse is also a linear function.
- f^{-1} is the notation for the inverse function of f.
- If (a, b) is a point on the graph of $y = f(x)$, then (b, a) is a point on the graph of $y = f^{-1}(x)$. This implies that the domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.
- The graph of the inverse is the reflection of the graph of $y = f(x)$ in the line $y = x$.

CHECK Your Understanding

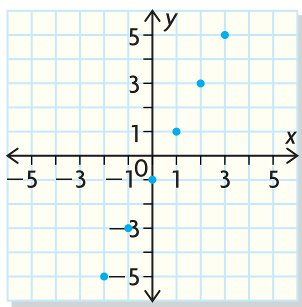
1. Determine the inverse relation for each set of ordered pairs. Graph each relation and its inverse. Which of the relations and inverse relations are functions?

a) $\{(-2, 3), (0, 4), (2, 5), (4, 6)\}$

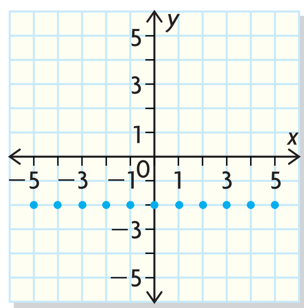
b) $\{(2, 5), (2, -1), (3, 1), (5, 1)\}$

2. Copy the graph of each function and graph its inverse. For each graph, identify the points that are common to the function and its inverse. Which inverse relations are functions?

a)



b)



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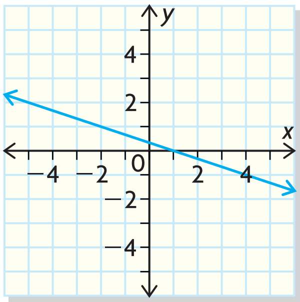

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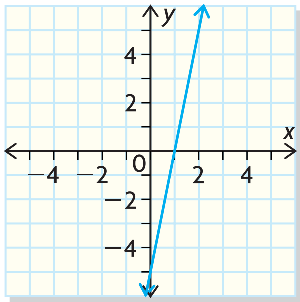
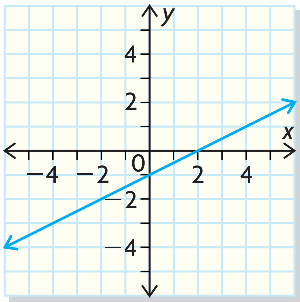
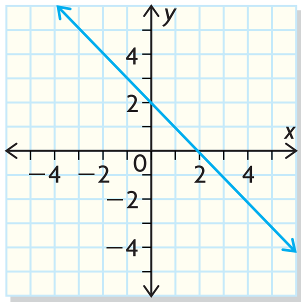


d)



f)



3. Determine whether each pair of functions described in words are inverses.
   a) f: Multiply by 3, then add 1; g: Divide by 3, then subtract 1.
   b) f: Multiply by 5, then subtract 2; g: Add 2, then divide by 5.

4. For each linear function, interchange x and y. Then solve for y to determine the inverse.
   a) $y = 4x - 3$
   b) $y = 2 - \frac{1}{2}x$
   c) $3x + 4y = 6$
   d) $2y - 10 = 5x$

PRACTISING

5. Determine the inverse of each linear function by reversing the operations.
   a) $f(x) = x - 4$
   b) $f(x) = 3x + 1$
   c) $f(x) = 5x$
   d) $f(x) = \frac{1}{2}x - 1$
   e) $f(x) = 6 - 5x$
   f) $f(x) = \frac{3}{4}x + 2$

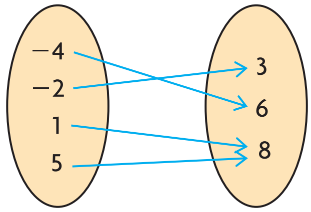
6. Determine the inverse of each linear function by interchanging the variables.
   a) $f(x) = x + 7$
   b) $f(x) = 2 - x$
   c) $f(x) = 5$
   d) $f(x) = -\frac{1}{5}x - 2$
   e) $f(x) = x$
   f) $f(x) = \frac{x - 3}{4}$

7. Sketch the graph of each function in questions 5 and 6, and sketch its inverse. Is each inverse linear? Is each inverse a function? Explain.

8. **K** For each function, determine the inverse, sketch the graphs of the function and its inverse, and state the domain and range of both the function and its inverse. In each case, how do the domain and range of the function compare with the domain and range of the inverse?

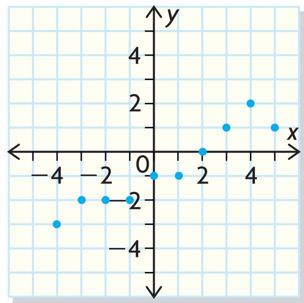
a) $\{(-1, 2), (1, 4), (2, 6), (3, 8)\}$

b)



c) $f(x) = 1 - 3x$

d)



9. **a)** Determine f^{-1} for the linear function $f(x) = 5x - 2$.
b) Graph f and f^{-1} on the same axes.
c) Explain how you can tell that f^{-1} is also a linear function.
d) State the coordinates of any points that are common to both f and f^{-1}.
e) Compare the slopes of the two lines.
f) Repeat parts (a) to (e) for $g(x) = -\frac{1}{2}x + 3$, $h(x) = 2x - 1$, $p(x) = 6 - x$, and $q(x) = 2$.

10. For $g(t) = 3t - 2$, determine each value.

a) $g(13)$
b) $g(7)$
c) $\dfrac{g(13) - g(7)}{13 - 7}$
d) $g^{-1}(13)$
e) $g^{-1}(7)$
f) $\dfrac{g^{-1}(13) - g^{-1}(7)}{13 - 7}$

11. Explain what parts (c) and (f) represent in question 10.

12. **A** The formula for converting a temperature in degrees Celsius into degrees Fahrenheit is $F = \frac{9}{5}C + 32$. Shirelle, an American visitor to Canada, uses a simpler rule to convert from Celsius to Fahrenheit: Double the Celsius temperature, then add 30.
a) Use function notation to write an equation for this rule. Call the function f and let x represent the temperature in degrees Celsius.
b) Write f^{-1} as a rule. Who might use this rule?
c) Determine $f^{-1}(x)$.
d) One day, the temperature was 14 °C. Use function notation to express this temperature in degrees Fahrenheit.
e) Another day, the temperature was 70 °F. Use function notation to express this temperature in degrees Celsius.

13. Ben, another American visitor to Canada, uses this rule to convert centimetres to inches: Multiply by 4 and then divide by 10. Let the function g be the method for converting centimetres to inches, according to Ben's rule.
a) Write g^{-1} as a rule.
b) Describe a situation in which the rule for g^{-1} might be useful.

c) Determine $g(x)$ and $g^{-1}(x)$.
d) One day, 15 cm of snow fell. Use function notation to represent this amount in inches.
e) Ben is 5 ft 10 in. tall. Use function notation to represent his height in centimetres.

14. Ali did his homework at school with a graphing calculator. He determined that the equation of the line of best fit for some data was $y = 2.63x - 1.29$. Once he got home, he realized he had mixed up the independent and dependent variables. Write the correct equation for the relation in the form $y = mx + b$.

15. Tiffany is paid \$8.05/h, plus 5% of her sales over \$1000, for a 40 h work week. For example, suppose Tiffany sold \$1800 worth of merchandise. Then she would earn \$8.05(40) + 0.05(\$800) = \$362.

a) Graph the relation between Tiffany's total pay for a 40 h work week and her sales for that week.
b) Write the relation in function notation.
c) Graph the inverse relation.
d) Write the inverse relation in function notation.
e) Write an expression in function notation that represents her sales if she earned \$420 one work week. Then evaluate.

16. The ordered pair (1, 5) belongs to a function f. Explain why the ordered pair (2, 1) cannot belong to f^{-1}.

17. **T** Given $f(x) = k(2 + x)$, find the value of k if $f^{-1}(-2) = -3$.

18. **C** Use a chart like the one shown to summarize what you have learned about the inverse of a linear function.

Definition:	Methods:
Inverse of a Linear Function	
Examples:	Properties:

Extending

19. *Self-inverse* functions are their own inverses. Find three linear functions that are self-inverse.

20. Determine the inverse of the inverse of $f(x) = 3x + 4$.

1.6 Exploring Transformations of Parent Functions

YOU WILL NEED

- graphing calculator or graphing software

Communication *Tip*

The function defined by $g(x) = af(x - d) + c$ describes a transformation of the graph of f.

When $f(x) = x^2$,
$g(x) = a(x - d)^2 + c$.

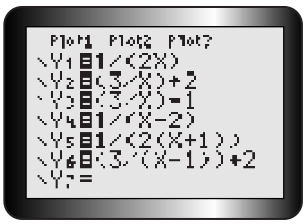
When $f(x) = \sqrt{x}$,
$g(x) = a\sqrt{x - d} + c$.

When $f(x) = \dfrac{1}{x}$,
$g(x) = \dfrac{a}{x - d} + c$.

When $f(x) = |x|$,
$g(x) = a|x - d| + c$.

Tech *Support*

Use brackets when entering transformed versions of $y = \dfrac{1}{x}$:



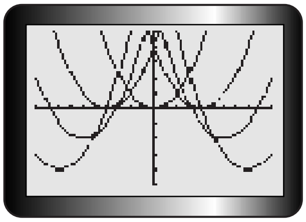
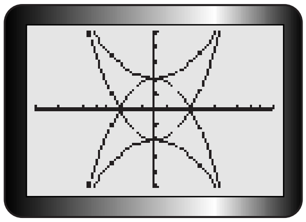
To enter $f(x) = |x|$, press MATH ▶ 1 X,T,Θ,n.

GOAL

Investigate transformations of parent functions.

EXPLORE the Math

Anastasia and Shelby made patterns with parabolas by applying **transformations** to the graph of the parent quadratic function $y = x^2$.

Anastasia thinks they could make more interesting patterns by applying transformations to other parent functions as well. Shelby wonders whether the transformations will have the same effect on the other functions as they do on quadratic functions.

? Do transformations of other parent functions behave in the same way as transformations of quadratic functions?

A. Graph the parent functions $f(x) = x^2$, $g(x) = \sqrt{x}$, $h(x) = \dfrac{1}{x}$, and $j(x) = |x|$. Sketch and label each graph.

B. Without using a calculator, use what you know about transformations of quadratic functions to sketch the graphs of $y = 3x^2$, $y = \frac{1}{2}x^2$, and $y = -2x^2$. Describe the transformations in words.

C. Predict what the graphs of $y = 3\sqrt{x}$, $y = \frac{1}{2}\sqrt{x}$, and $y = -2\sqrt{x}$ will look like. Use a graphing calculator to verify your predictions. Sketch and label each curve on the same axes, along with a sketch of the parent function. Compare the effect of these transformations with the effect of the same transformations on quadratic functions.

D. Repeat part C for $y = \dfrac{3}{x}$, $y = \dfrac{1}{2x}$, and $y = -\dfrac{2}{x}$, and for $y = 3|x|$, $y = \frac{1}{2}|x|$, and $y = -2|x|$.

E. Sketch $y = 3x^2 + 2$ and $y = 3x^2 - 1$ without a calculator. Describe the transformations in words. Predict what the graphs of $y = 3f(x) + 2$ and $y = 3f(x) - 1$ for each of the other parent functions will look like. Verify your predictions with a graphing calculator. Make labelled sketches and compare them with transformations on quadratic functions as before.

F. Repeat part E for $y = f(x - 2)$, $y = \frac{1}{2}f(x + 1)$, and $y = 3f(x - 1) + 2$.

G. Examine your sketches for each type of transformation. Did the transformations have the same effect on the new parent functions as they had on quadratic functions? Explain.

Reflecting

H. How did the effect of transformations on parent functions compare with that on quadratic functions?

I. When you graphed $y = af(x - d) + c$, what were the effects of c and d?

J. How did the graphs with $a \geq 0$ compare with the graphs with $a \leq 0$?

K. How did the graphs for which $a > 1$ compare with the graph for which $0 < a < 1$?

In Summary

Key Idea

- In functions of the form $g(x) = af(x - d) + c$, the constants a, c, and d each change the location or shape of the graph of $f(x)$. The shape of the graph of $g(x)$ depends on the graph of the parent function $g(x)$ and on the value of a.

FURTHER Your Understanding

1. The graph of the equation $y = (x - 1)^2 + 2$ is the graph of a parabola that opens up and has its vertex at (1, 2). What do you know about the graphs of the following equations?

a) $y = \sqrt{x - 1} + 2$ **b)** $y = |x - 1| + 2$ **c)** $y = \dfrac{1}{x - 1} + 2$

2. The graph of $y = x^2$ opens up and the graph of $y = -x^2$ opens down. How would you compare the graphs of the following pairs of equations?

a) $y = \sqrt{x}$ and $y = -\sqrt{x}$

b) $y = |x|$ and $y = -|x|$

c) $y = \dfrac{1}{x}$ and $y = -\dfrac{1}{x}$

3. The graph of $y = 2x^2$ is narrower than the graph of $y = x^2$. How do the following graphs compare?

a) $y = 2\sqrt{x}$ and $y = \sqrt{x}$

b) $y = 2|x|$ and $y = |x|$

c) $y = \dfrac{2}{x}$ and $y = \dfrac{1}{x}$

4. Experiment with each of the parent functions to create patterns on a graphing calculator screen.

1.7 Investigating Horizontal Stretches, Compressions, and Reflections

YOU WILL NEED

- graph paper (optional)
- graphing calculator

GOAL

Investigate and apply horizontal stretches, compressions, and reflections to parent functions

INVESTIGATE the Math

The function $p(L) = 2\pi\sqrt{\frac{1}{10}L}$ describes the time it takes a pendulum to complete one swing, from one side to the other and back, as a function of its length. In this formula,

- $p(L)$ represents the time in seconds
- L represents the pendulum's length in metres

Shannon wants to sketch the graph of this function. She knows that the parent function is $f(x) = \sqrt{x}$ and that the 2π causes a vertical stretch. She wonders what transformation is caused by multiplying x by $\frac{1}{10}$.

? What transformation must be applied to the graph of $y = f(x)$ to get the graph of $y = f(kx)$?

A. Copy and complete tables of values for $y = \sqrt{x}$ and $y = \sqrt{2x}$.

$y = \sqrt{x}$	
x	y
0	
1	
4	
9	
10	

$y = \sqrt{2x}$	
x	y
0	
0.5	
2	
4.5	
8	

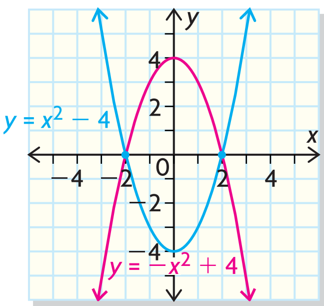
Graph both functions on the same set of axes. State the domain and range of each function.

B. Compare the position and shape of the two graphs. Are there any **invariant points** on the graphs? Explain.

C. How could you transform the graph of $y = \sqrt{x}$ to obtain the graph of $y = \sqrt{2x}$?

invariant point

a point on a graph (or figure) that is unchanged by a transformation—for example, $(-2, 0)$ and $(2, 0)$ for this graph and transformation



D. Compare the points in the tables of values. How could you use the first table to obtain the second? What happens to the point (x, y) under this transformation? This transformation is called a *horizontal compression of factor* $\frac{1}{2}$. Explain why this is a good description.

E. Repeat parts A through D for $y = \sqrt{x}$ and $y = \sqrt{\frac{1}{2}x}$. What happens to the point (x, y) under this transformation? Describe the transformation in words.

F. Repeat parts A through D for $y = \sqrt{x}$ and $y = \sqrt{-x}$.

G. Using a graphing calculator, investigate the effect of varying k in $y = f(kx)$ on the graphs of the given parent functions. In each case, try values of k that are **i)** between 0 and 1, **ii)** greater than 1, and **iii)** less than 0.

a) $f(x) = x^2$ **b)** $f(x) = \frac{1}{x}$ **c)** $f(x) = |x|$

H. Write a summary of the results of your investigations. Explain how you would use the graph of $y = f(x)$ to sketch the graph of $y = f(kx)$.

Communication *Tip*

In describing vertical stretches/compressions $af(x)$, the scale factor is a, but for horizontal stretches/compressions $f(kx)$, the scale factor is $\frac{1}{k}$. In both cases, for a scale factor greater than 1, a stretch occurs, and for a scale factor between 0 and 1, a compression occurs.

Reflecting

I. What transformation is caused by multiplying L by $\frac{1}{10}$ in the pendulum function $p(L) = 2\pi\sqrt{\frac{1}{10}L}$?

J. How is the graph of $y = f(2x)$ different from the graph of $y = 2f(x)$?

K. How is the graph of $y = f(-x)$ different from the graph of $y = -f(x)$?

L. What effect does k in $y = f(kx)$ have on the graph of $y = f(x)$ when

i) $|k| > 1$? **ii)** $0 < |k| < 1$? **iii)** $k < 0$?

APPLY the Math

EXAMPLE 1 Applying horizontal stretches, compressions, and reflections

For each pair of functions, identify the parent function, describe the transformations required to graph them from the parent function, and sketch all three graphs on the same set of axes.

a) $y = (4x)^2; y = \left(\frac{1}{5}x\right)^2$ **b)** $y = |0.25x|; y = |-x - 3|$

Ana's Solution

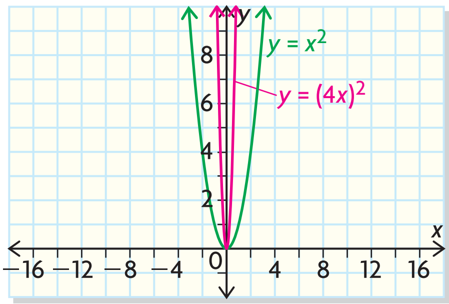
a) These functions are of the form $y = f(kx)^2$. The quadratic function $f(x) = x^2$ is the parent function.

I saw that these functions were $y = x^2$, with x multiplied by a number.

To graph $y = (4x)^2$, compress the graph of $y = (x)^2$ horizontally by the factor $\frac{1}{4}$.

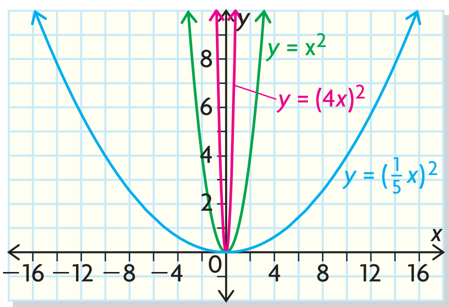
When x is multiplied by a number greater than 1, the graph is compressed horizontally. That makes sense, since the x-value required to make $y = 1$ is ± 1 for $y = x^2$, but is $\pm\frac{1}{4}$ for $y = (4x)^2$.

I multiplied the x-coordinates of the points (1, 1), (2, 4), and (3, 9) on $y = x^2$ by $\frac{1}{4}$ to find three points, $\left(\frac{1}{4}, 1\right)$, $\left(\frac{1}{2}, 4\right)$, and $\left(\frac{3}{4}, 9\right)$, on $y = (4x)^2$.



I plotted these points and joined them to the invariant point (0, 0) to graph one-half of the parabola. Then I used symmetry to complete the other half of the graph.

To graph $y = \left(\frac{1}{5}x^2\right)$, stretch the graph of $y = x^2$ horizontally by the factor 5.



This time, x is multiplied by a number between 0 and 1, so the graph is stretched horizontally. Instead of using an x-value of ± 1 to get a y-value of 1, I need an x-value of ± 5.

I used the same x-coordinates as before and multiplied by 5, which gave me points (5, 1), (10, 4), and (15, 9) to plot.

I used the invariant point (0, 0) and symmetry to complete the graph of $y = \left(\frac{1}{5}x\right)^2$.

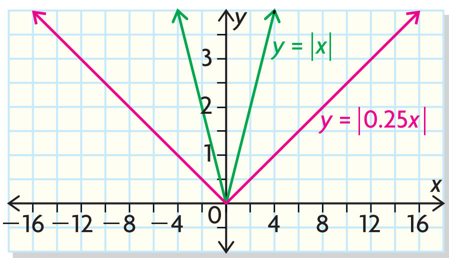
b) The parent function is $f(x) = |x|$.

To graph $y = |0.25x|$, stretch the graph of $y = |x|$ horizontally by the factor 4.

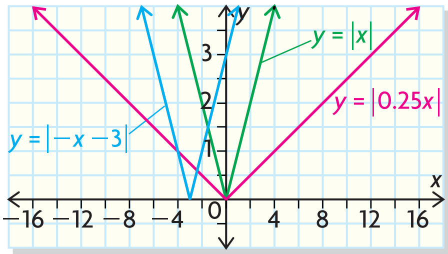
I knew from the absolute value signs that the parent function was the absolute value function.

I knew that the stretch factor was $\frac{1}{0.25} = 4$. The point that originally was (1, 1) corresponded to the new point (4, 1). So I multiplied the x-coordinates of the points (1, 1), (2, 2), and (3, 3) on $y = |x|$ by 4 to find the points (4, 1), (8, 2), and (12, 3) on the new graph.

I joined these points to the invariant point (0, 0) to graph one-half of the stretched absolute value function. I used symmetry to complete the graph.



To graph $y = |-x - 3|$, reflect the parent function graph in the y-axis, then translate it 3 units left.



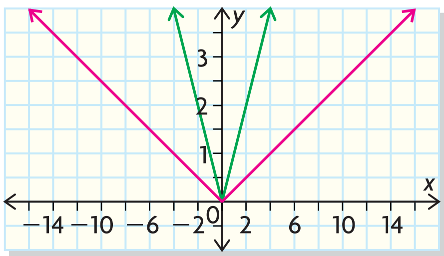
First I thought about the graph of $y = |-x|$. For this graph, I switched the values of x and $-x$, so I really reflected the graph of $y = |x|$ in the y-axis. The points that were originally $(-1, 1)$, $(0, 0)$, and $(1, 1)$ changed to $(1, 1)$, $(0, 0)$, and $(-1, 1)$; so the graph didn't change.

Next, I reasoned that since $|-x| = |x|$, $|-x - 3| = |x + 3|$. Therefore, I shifted the graph of $y = |x|$ 3 units left. The points that were originally $(-1, 1)$, $(0, 0)$, and $(1, 1)$ changed to $(-4, 1)$, $(-3, 0)$, and $(-2, 1)$.

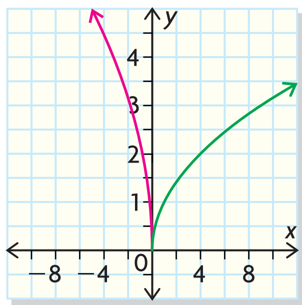
EXAMPLE 2 Using a graph to determine the equation of a transformed function

In the graphs shown, three parent functions have been graphed in green. The functions graphed in red have equations of the form $y = f(kx)$. Determine the equations. Explain your reasoning.

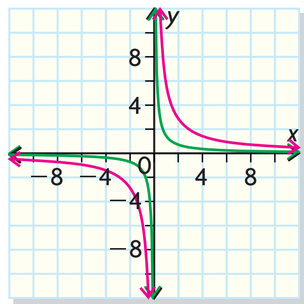
a)



b)



c)



Robert's Solution

a) The parent function is $f(x) = |x|$.

I recognized the V shape of the absolute value function.

Point $(1, 1)$ on $y = |x|$ corresponds to point $(4, 1)$ on the red graph.

I knew that $y = f(kx)$ represents a horizontal stretch or compression and/or a reflection in the y-axis.

Point (2, 2) corresponds to point (8, 2), and point (3, 3) corresponds to point (12, 3).

The red graph is a stretched-out version of the green graph, so k must be between 0 and 1.

The red graph is the green graph stretched horizontally by the factor 4.

The x-coordinates of points on the red graph are 4 times the ones on the green graph.

The equation is $y = |\frac{1}{4}x|$.

Since the stretch scale factor is 4, and $0 < k < 1$, it follows that $k = \frac{1}{4}$. So I could complete the equation.

b) The green graph is a graph of the square root function $f(x) = \sqrt{x}$.

The green graph is the square root function because it begins at (0, 0) and has the shape of a half parabola on its side.

The green graph has been compressed horizontally and reflected in the y-axis to produce the red graph.

The red graph is a compressed version of the green graph that had been flipped over the y-axis. Therefore, k is negative and less than -1.

(1, 1) corresponds to $(-0.25, 1)$.

(4, 2) corresponds to $(-1, 2)$.

(16, 4) corresponds to $(-4, 4)$.

Each x-coordinate has been divided by -4.

The equation is $y = \sqrt{-4x}$.

I divided the corresponding x-coordinates to find k:

$1 \div -0.25 = -4$

$4 \div -1 = -4$

$16 \div -4 = -4, \quad \text{so} \quad k = -4$

c) The parent function is $f(x) = \frac{1}{x}$.

I recognized the reciprocal function because the graph was in two parts and had asymptotes.

The graph has been stretched horizontally.

The red graph is further away from the asymptotes than the green graph, so it must have been stretched.

(1, 1) corresponds to (6, 1).

$(\frac{1}{2}, 2)$ corresponds to (3, 2).

$(-1, -1)$ corresponds to $(-6, -1)$.

$(-\frac{1}{2}, -2)$ corresponds to $(-3, -2)$.

Each x-coordinate has been multiplied by 6. The equation is $y = \frac{1}{(\frac{1}{6}x)}$.

Since the stretch scale factor is 6, and $0 < k < 1$, it follows that $k = \frac{1}{6}$.

EXAMPLE 3 Using transformations to sketch the graph for a real situation

Use transformations to sketch the graph of the pendulum function $p(L) = 2\pi\sqrt{\frac{1}{10}L}$, where $p(L)$ is the time, in seconds, that it takes for a pendulum to complete one swing and L is the length of the pendulum, in metres.

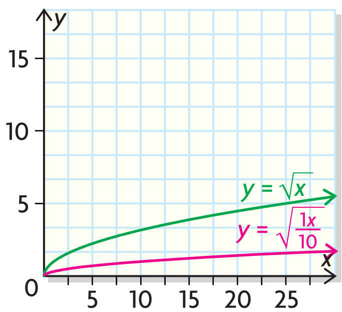
Shannon's Solution

The graph of $y = 2\pi\sqrt{\frac{1}{10}x}$ is the graph of the parent function $y = \sqrt{x}$ stretched horizontally by the factor 10 and vertically by the factor 2π.

The original equation was in the form $y = af(kx)$.

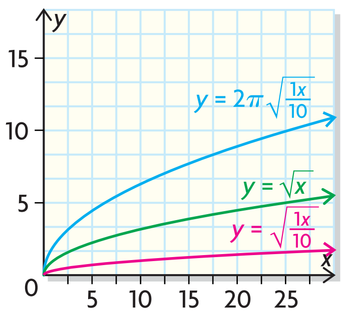
Since $0 < k < 1$, the graph is stretched horizontally by a scale factor of $\frac{1}{\frac{1}{10}} = 10$.

Because $a = 2\pi$ and $2\pi > 1$, the graph is stretched vertically by a scale factor of 2π.

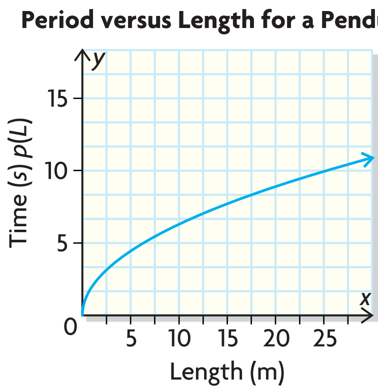


I applied the horizontal stretch.

I multiplied the x-coordinates by 10 to find points on the horizontally stretched graph:
(1, 1) moves to (10, 1).
(4, 2) moves to (40, 2).



Then I applied the vertical stretch to the red graph. I multiplied the y-coordinates by 2π, which is approximately 6.3 (to one decimal place):
(10, 1) moves to (10, 6.3).
(40, 2) moves to (40, 12.6) .



I drew a correctly labelled graph of the situation. I copied the sketch onto a graph with length L on the x-axis and time $p(L)$ on the y-axis.

In Summary

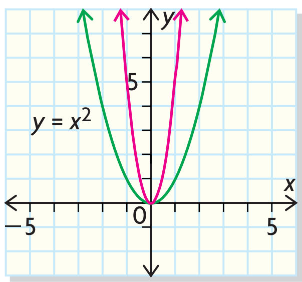
Key Idea

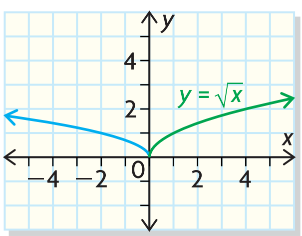
- Functions of the form $g(x) = f(kx)$ have graphs that are not congruent to the graph of $f(x)$. The differences in shape are a result of stretching or compressing in a horizontal direction.

Need to Know

- The image of the point (x, y) on the graph of $f(x)$ is the point $\left(\frac{x}{k}, y\right)$ on the graph of $f(kx)$.
- If $g(x) = f(kx)$, then the value of k has the following effect on the graph of $f(x)$:
  - When $|k| > 1$, the graph is compressed horizontally by the factor $\frac{1}{|k|}$.
  - When $0 < |k| < 1$, the graph is stretched horizontally by the factor $\frac{1}{|k|}$.
  - When $k < 0$, the graph is also reflected in the y-axis.

CHECK *Your Understanding*





1. **a)** The red graph has been compressed horizontally by the factor $\frac{1}{3}$ relative to the graph of $y = x^2$. Write the equation of the red graph.

b) The blue graph has been stretched horizontally by the factor 2 relative to the graph of $y = \sqrt{x}$ and then reflected in the y-axis. Write the equation of the blue graph.

2. For each function, identify the parent function and describe how the graph of the function can be obtained from the graph of the parent function. Then sketch both graphs on the same set of axes.

a) $y = |0.5x|$ **b)** $y = \left(\frac{1}{4}x\right)^2$ **c)** $y = \sqrt{-2x}$ **d)** $y = \frac{1}{(5x)}$

PRACTISING

3. The point (3, 4) is on the graph of $y = f(x)$. State the coordinates of the image of this point on each graph.

a) $y = f(2x)$ **b)** $y = f(0.5x)$ **c)** $y = f\left(\frac{1}{3}x\right)$ **d)** $y = f(-4x)$

4. Sketch graphs of each pair of transformed functions, along with the graph of the parent function, on the same set of axes. Describe the transformations in words and note any invariant points.

a) $y = (2x)^2, y = (5x)^2$

b) $y = \sqrt{3x}, y = \sqrt{4x}$

c) $y = \frac{1}{(2x)}, y = \frac{1}{(3x)}$

d) $y = |3x|, y = |5x|$

5. Repeat question 4 for each pair of transformed functions.

a) $y = (-2x)^2, y = (-5x)^2$

b) $y = \sqrt{-3x}, y = \sqrt{-4x}$

c) $y = \frac{1}{(-2x)}, y = \frac{1}{(-3x)}$

d) $y = |-3x|, y = |-5x|$

6. Repeat question 4 for each pair of transformed functions.

a) $y = \left(\frac{1}{2}x\right)^2, y = \left(\frac{1}{3}x\right)^2$

b) $y = \sqrt{\frac{1}{2}x}, y = \sqrt{\frac{1}{3}x}$

c) $y = \frac{1}{\left(\frac{1}{2}x\right)}, y = \frac{1}{\left(\frac{1}{4}x\right)}$

d) $y = \left|\frac{1}{3}x\right|, y = \left|\frac{1}{5}x\right|$

7. Repeat question 4 for each pair of transformed functions.

a) $y = \left(-\frac{1}{2}x\right)^2, y = \left(-\frac{1}{3}x\right)^2$

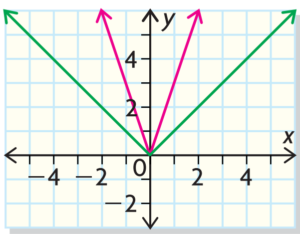
b) $y = \sqrt{-\frac{1}{2}x}, y = \sqrt{-\frac{1}{3}x}$

c) $y = \frac{1}{\left(-\frac{1}{2}x\right)}, y = \frac{1}{\left(-\frac{1}{4}x\right)}$

d) $y = \left|-\frac{1}{3}x\right|, y = \left|-\frac{1}{5}x\right|$

8. **K** In each graph, one of the parent functions $f(x) = x^2, f(x) = \sqrt{x}, f(x) = \frac{1}{x}$, and $f(x) = |x|$ has undergone a transformation of the form $f(kx)$. Determine the equations of the transformed functions graphed in red.

a)



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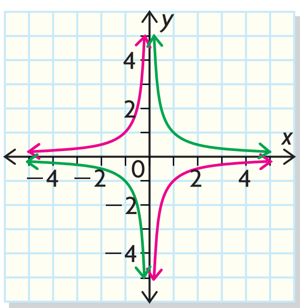
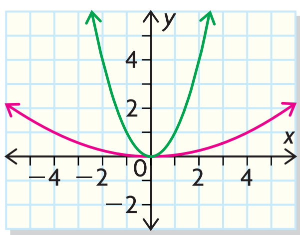
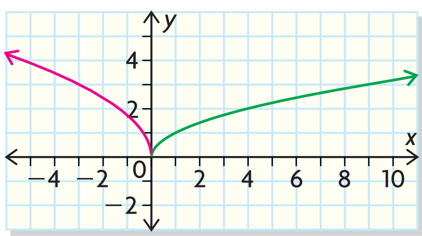


c)



d)



9. When an object is dropped from a height, the time it takes to reach the ground is a function of the height from which it was dropped. An equation for this function is $t(h) = \sqrt{\frac{h}{4.9}}$, where h is in metres and t is in seconds.

A

a) Describe the domain and range of the function.

b) Sketch the graph by applying a transformation to the graph of $t(h) = \sqrt{h}$.

10. For each set of functions, transform the graph of $f(x)$ to sketch $g(x)$ and $h(x)$, and state the domain and range of each function.

a) $f(x) = x^2, g(x) = \left(\frac{1}{4}x^2\right), h(x) = (-4x^2)$

b) $f(x) = \sqrt{x}, g(x) = \sqrt{\frac{1}{5}x}, h(x) = \sqrt{-5x}$

c) $f(x) = \frac{1}{x}, g(x) = \frac{1}{4x}, h(x) = \frac{1}{\left(-\frac{1}{3}x\right)}$

d) $f(x) = |x|, g(x) = |-2x|, h(x) = \left|\frac{1}{2}x\right|$

11. The function $y = f(x)$ has been transformed to $y = f(kx)$. Determine the value of k for each transformation.

a) a horizontal stretch by the factor 4

b) a horizontal compression by the factor $\frac{1}{2}$

c) a reflection in the y-axis

d) a horizontal compression by the factor $\frac{1}{5}$ and a reflection in the y-axis

12. A quadratic function has equation $f(x) = x^2 - x - 6$. Determine the x-intercepts for each function.

T

a) $y = f(2x)$

b) $y = f\left(\frac{1}{3}x\right)$

c) $y = f(-3x)$

13. a) Describe how the graph of $y = f(kx)$ can be obtained from the graph of $y = f(x)$. Include examples that show how the transformations vary with the value of k.

C

b) Compare the graph of $y = f(kx)$ with the graph of $y = kf(x)$ for different values of k and different functions $f(x)$. How are the transformations alike? How are they different?

Extending

14. a) Graph the function $y = \frac{1}{x}$.

b) Apply a horizontal stretch with factor 2.

c) Apply a vertical stretch with factor 2. What do you notice?

d) Write the equations of the functions that result from the transformations in parts (b) and (c). Explain why these equations are the same.

15. Suppose you are asked to graph $y = f(2x + 4)$. What two transformations are required? Does the order in which you apply these transformations make a difference? Choose one of the parent functions and investigate. If you get two different results, use a graphing calculator to verify which graph is correct.

1.8 Using Transformations to Graph Functions of the Form $y = af[k(x - d)] + c$

GOAL

Apply combinations of transformations, in a systematic order, to sketch graphs of functions.

YOU WILL NEED

- graph paper or graphing calculator

LEARN ABOUT the Math

Neil wants to sketch the graph of $f(x) = -3\sqrt{2(x + 4)} - 1$.

? How can Neil apply the transformations necessary to sketch the graph?

EXAMPLE 1 Applying a combination of transformations

Sketch the graph of $f(x) = -3\sqrt{2(x + 4)} - 1$. State the domain and range of the transformed function.

Neil's Solution

The parent function is $f(x) = \sqrt{x}$.

The function is a transformed square root function.

Vertical stretch by a factor of 3

Horizontal translation 4 units left

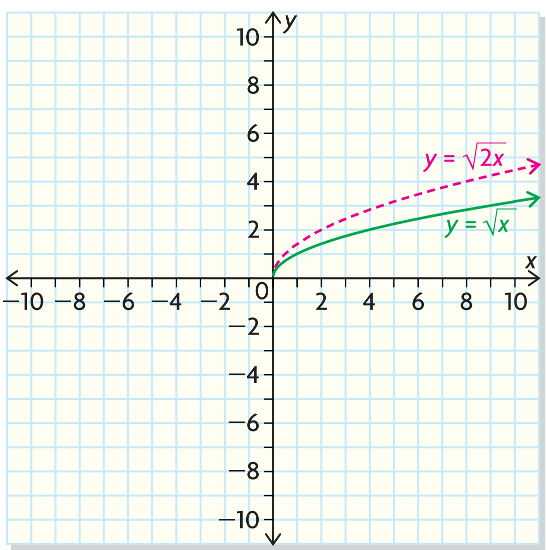
$$f(x) = -3\sqrt{2(x + 4)} - 1$$

Vertical translation 1 unit down

Reflection in the x-axis

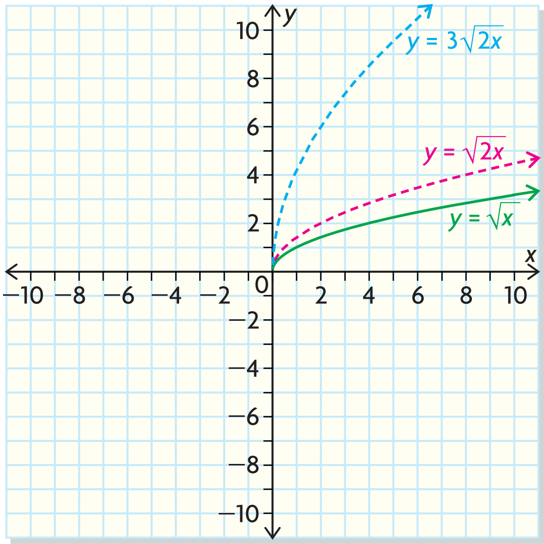
Horizontal compression by a factor of $\frac{1}{2}$

I looked at each part of the function and wrote down all the transformations I needed to apply.



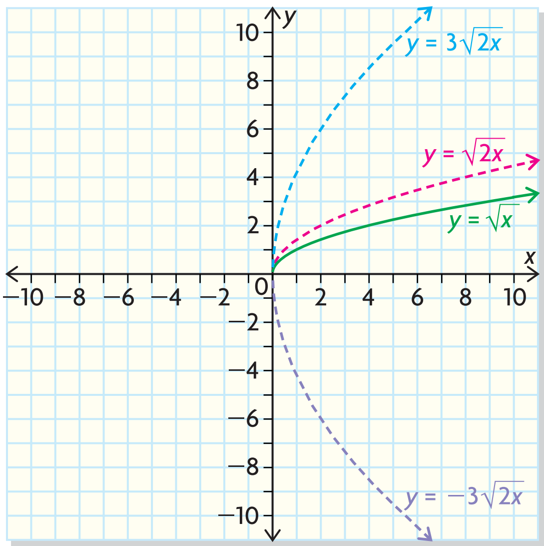
First I divided the x-coordinates of points on $y = \sqrt{x}$ by 2 to compress the graph horizontally by the factor $\frac{1}{2}$.

$f(x)$	$f(2x)$
(0, 0)	(0, 0)
(1, 1)	$\left(\frac{1}{2}, 1\right)$
(4, 2)	(2, 2)
(9, 3)	$\left(4\frac{1}{2}, 3\right)$



I multiplied the y-coordinates of $y = \sqrt{2x}$ by 3 to stretch the graph vertically by the factor 3.

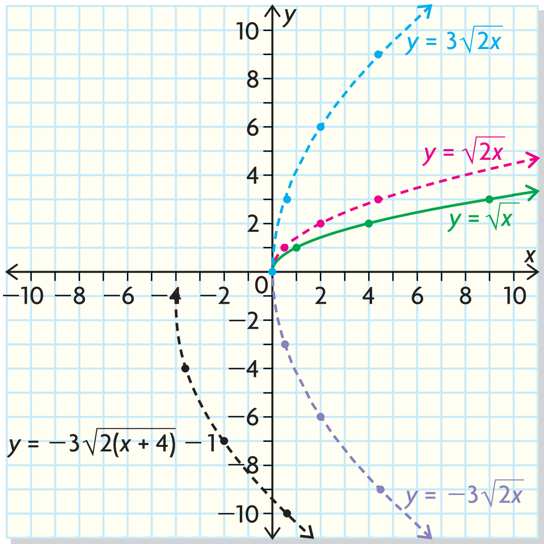
$f(x)$	$f(2x)$	$3f(2x)$
(0, 0)	(0, 0)	(0, 0)
(1, 1)	$\left(\frac{1}{2}, 1\right)$	$\left(\frac{1}{2}, 3\right)$
(4, 2)	(2, 2)	(2, 6)
(9, 3)	$\left(4\frac{1}{2}, 3\right)$	$\left(4\frac{1}{2}, 9\right)$



I flipped the graph of $y = 3\sqrt{2x}$ over the x-axis.

$f(x)$	$f(2x)$	$3f(2x)$	$-3f(2x)$
(0, 0)	(0, 0)	(0, 0)	(0, 0)
(1, 1)	$\left(\frac{1}{2}, 1\right)$	$\left(\frac{1}{2}, 3\right)$	$\left(\frac{1}{2}, -3\right)$
(4, 2)	(2, 2)	(2, 6)	(2, −6)
(9, 3)	$\left(4\frac{1}{2}, 3\right)$	$\left(4\frac{1}{2}, 9\right)$	$\left(4\frac{1}{2}, -9\right)$

Translate the graph 4 units left and 1 unit down.



I did both shifts together. I subtracted 4 from each of the x-coordinates and subtracted 1 from each of the y-coordinates of the graph of $y = -3\sqrt{2x}$.

$f(x)$	$f(2x)$	$3f(2x)$	$-3f(2x)$	$-3f(2(x + 4)) - 1$
$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(-4, -1)$
$(1, 1)$	$\left(\frac{1}{2}, 1\right)$	$\left(\frac{1}{2}, 3\right)$	$\left(\frac{1}{2}, -3\right)$	$\left(-3\frac{1}{2}, -4\right)$
$(4, 2)$	$(2, 2)$	$(2, 6)$	$(2, -6)$	$(-2, -7)$
$(9, 3)$	$\left(4\frac{1}{2}, 3\right)$	$\left(4\frac{1}{2}, 9\right)$	$\left(4\frac{1}{2}, -9\right)$	$\left(\frac{1}{2}, -10\right)$

Domain $= \{x \in \mathbf{R} \mid x \geq -4\}$

Range $= \{y \in \mathbf{R} \mid y \leq -1\}$

From the final graph, $x \geq -4$ and $y \leq -1$.

Reflecting

A. How do the numbers in the function $f(x) = -3\sqrt{2(x + 4)} - 1$ affect the x- and y-coordinates of each point on the parent function ?

B. How did Neil determine the domain and range of the final function?

C. How does the order in which Neil applied the transformations compare with the order of operations for numerical expressions?

D. Sarit says that she can graph the function in two steps. She would do both stretches or compressions and any reflections to the parent function first and then both translations. Do you think this will work? Explain.

APPLY the Math

EXAMPLE 2 Applying transformations to the equation and the graph

Some transformations are applied, in order, to the reciprocal function $f(x) = \frac{1}{x}$:

- horizontal stretch by the factor 3
- vertical stretch by the factor 2
- reflection in the y-axis
- translation 5 units right and 4 units up

a) Write the equation for the final transformed function $g(x)$.
b) Sketch the graphs of $f(x)$ and $g(x)$.
c) State the domain and range of both functions.

Lynn's Solution

a)
$$\begin{aligned} g(x) &= af[k(x-d)] + c \\ &= 2f\left[-\frac{1}{3}(x-5)\right] + 4 \\ &= \frac{2}{\left(-\frac{1}{3}(x-5)\right)} + 4 \end{aligned}$$

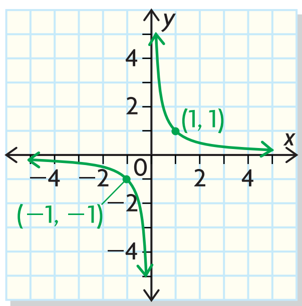
I built up the equation from the transformations.

A horizontal stretch by the factor 3 and a reflection in the y-axis means that $k = -\frac{1}{3}$.

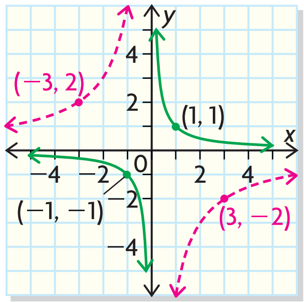
$a = 2$, because there is a vertical stretch by the factor 2.

$d = 5$ and $c = 4$, because the translation is 5 units right and 4 units up.

b) Graph of $f(x)$:



I sketched the graph of $f(x)$ and labelled the points (1, 1) and $(-1, -1)$. The vertical asymptote is $x = 0$ and the horizontal asymptote is $y = 0$.

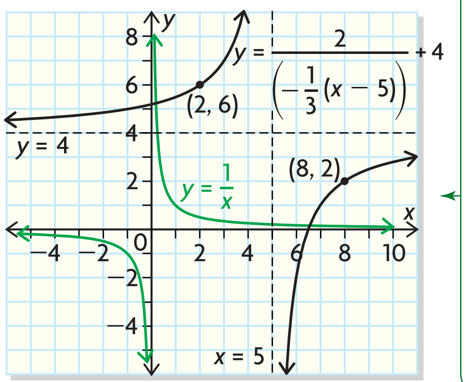


I applied the stretches and reflection to the labelled points by multiplying the x-coordinates by -3 and the y-coordinates by 2.

$(1, 1)$ became $(-3, 2)$ and $(-1, -1)$ became $(3, -2)$. The asymptotes did not change, since x and y still couldn't be 0.

I made a sketch of the stretched and reflected graph before applying the translation.

Graph of $g(x)$:



To apply the translations, 5 right and 4 up, I drew in the translated asymptotes first.

Since all the points moved 5 right, the new vertical asymptote is $x = 5$.

Since all the points moved up 4, the new horizontal asymptote is $y = 4$.

Then I drew the stretched and reflected graph in the new position after the translation.

I labelled the graphs and wrote the equations for the asymptotes.

c) For $f(x)$,

Domain $= \{x \in \mathbf{R} \mid x \neq 0\}$

Range $= \{y \in \mathbf{R} \mid y \neq 0\}$

For $g(x)$,

Domain $= \{x \in \mathbf{R} \mid x \neq 5\}$

Range $= \{y \in \mathbf{R} \mid y \neq 4\}$

I used the equations of the asymptotes to help determine the domain and range.

The graphs do not meet their asymptotes, so for $f(x)$, x cannot be 0 and y cannot be 0. Also, for $g(x)$, x cannot be 5 and y cannot be 4.

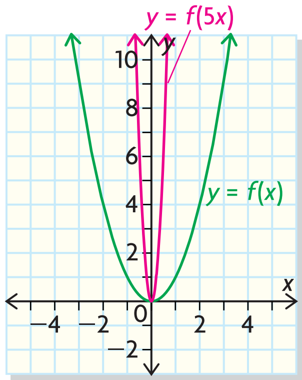
EXAMPLE 3 Factoring out k before applying transformations

For $f(x) = x^2$, sketch the graph of $g(x) = f(-5x + 10)$.

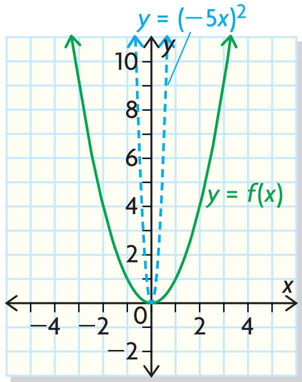
Stefan's Solution

$$g(x) = f(-5x + 10)$$
$$= f[-5(x - 2)]$$

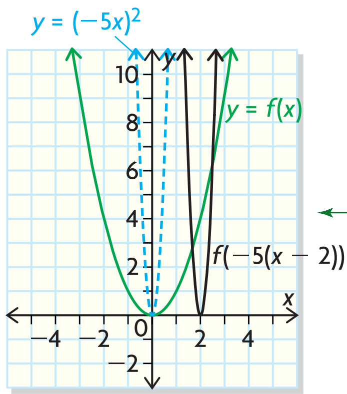
I wrote $g(x)$ in $af[k(x - d)] + c$ form by factoring out $k = -5$.



I graphed $y = f(x)$ and compressed the graph horizontally by the factor $\frac{1}{5}$. This gave me the graph of $y = f(5x)$.



I reflected $y = f(5x)$ in the y-axis. The graph of $y = f(-5x)$ looked the same because the y-axis is the axis of symmetry for $y = f(5x)$.



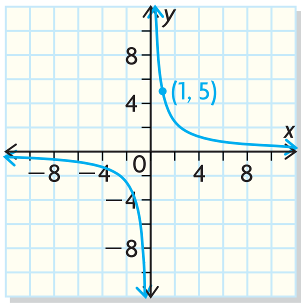
I translated the compressed and reflected graph 2 units right. This gave me the graph of $y = f[-5(x - 2)]$.

EXAMPLE 4 Identifying the equation of a transformed function from its graph

Match each equation to its graph. Explain your reasoning.

1. $y = \dfrac{1}{0.3(x + 1)} - 2$
2. $y = -4|x + 2| + 1$
3. $y = -\sqrt{3(x + 2)} - 1$
4. $y = \sqrt{-0.4(x - 4)} + 3$
5. $y = (0.5(x - 4))^2 + 2$
6. $y = \dfrac{5}{x}$
7. $y = -3(x + 1)^2 + 4$

A



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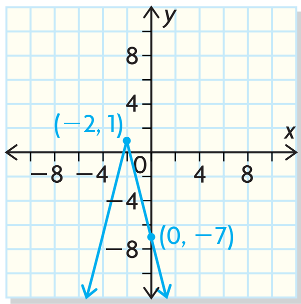

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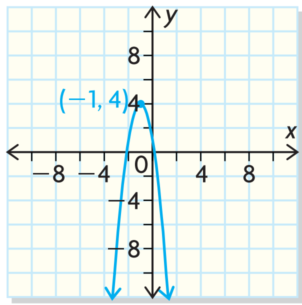

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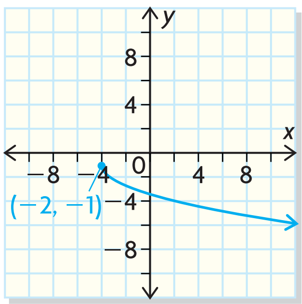

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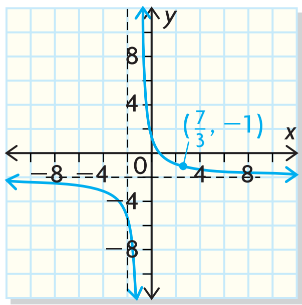

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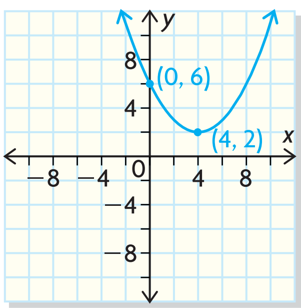

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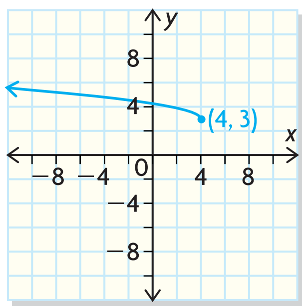


Donna's Solution

Graph A matches equation 6.

Graph A is like the graph of $y = \frac{1}{x}$, but it has been stretched vertically. The point (1, 1) has been stretched to (1, 5), so the scale factor is 5. The equation really is $y = \frac{5}{x}$.

Graph B matches equation 3.

This is the graph of a square root function that has been flipped over the x-axis, so, in the equation, a will be negative.

The parent square root graph has been compressed horizontally or stretched vertically. It starts at $(-2, -1)$ instead of (0, 0), so it has been translated 2 units left and 1 unit down. So, $c = -2$ and $d = -1$.

Graph C matches equation 5.

Graph C is a parabola, so it has to match equation 5 or equation 7. Since $a > 0$, the parabola opens upward, so the answer can't be equation 7 and has to be equation 5.

I checked: The vertex is (4, 2), so $d = 4$ and $c = 2$.

Graph C is wider than the parent function, so it has been stretched horizontally or compressed vertically. Equation 5 is the equation of a parabola with vertex (4, 2), that opens up, and that has been stretched horizontally by the factor $\frac{1}{0.5} = 2$.

Graph D matches equation 2.

This is the graph of an absolute value function. The parent graph has been reflected in the x-axis, stretched vertically, and shifted 2 units left and 1 unit up. The equation must have $a < -1$, $d = -2$, and $c = 1$.

Graph E matches equation 1.

This is a transformation of the graph of $y = \frac{1}{x}$, so the answer has to be equation 1 or equation 6. The equations for the asymptotes are $x = -1$ and $y = -2$, so $d = -1$ and $c = -2$. This matches equation 1.

Also, in the equation, $k = 0.3$ means that the parent graph has been stretched horizontally by the factor $\frac{1}{0.3}$. The point (1, 1) on the parent graph becomes $\left(\frac{10}{3}, 1\right)$, when you multiply the x-coordinate by $\frac{1}{0.3}$. Then, this point becomes $\left(\frac{7}{3}, -1\right)$, when you apply the translations by subtracting 1 from the x-coordinate and 2 from the y-coordinate.

Graph F matches equation 4.

This is another square root function. The parent function has been flipped over the y-axis, so $k < 0$. It has been stretched horizontally, so $-1 < k < 0$, and translated 4 units right and 3 units up, so $d = 4$ and $c = 3$.

Graph G matches equation 7.

Graph G is a parabola that opens down, so it has to match equation 7 because it is vertically stretched (narrow) and has vertex at $(-1, 4)$. Equation 7 has $a = -3$, which means that the parabola opens down and is vertically stretched by the factor 3. Also, $c = -1$ and $d = 4$, which means that the vertex is $(-1, 4)$, as in graph G.

In Summary

Key Ideas

- You can graph functions of the form $g(x) = af[k(x - d)] + c$ by applying the appropriate transformations to the key points of the parent function, one at a time, making sure to apply a and k before c and d. This order is like the order of operations for numerical expressions, since multiplications (stretches, compressions, and reflections) are done before additions and subtractions (translations).
- When using transformations to graph, you can apply a and k together, then c and d together, to get the desired graph in fewer steps.

Need to Know

- The value of a determines the vertical stretch or compression and whether there is a reflection in the x-axis:
  - When $|a| > 1$, the graph of $y = f(x)$ is stretched vertically by the factor $|a|$.
  - For $0 < |a| < 1$, the graph is compressed vertically by the factor $|a|$.
  - When $a < 0$, the graph is also reflected in the x-axis.
- The value of k determines the horizontal stretch or compression and whether there is a reflection in the y-axis:
  - When $|k| > 1$, the graph is compressed horizontally by the factor $\frac{1}{|k|}$.
  - When $0 < |k| < 1$, the graph is stretched horizontally by the factor $\frac{1}{|k|}$.
  - When $k < 0$, the graph is also reflected in the y-axis.
- The value of d determines the horizontal translation:
  - For $d > 0$, the graph is translated d units right.
  - For $d < 0$, the graph is translated d units left.
- The value of c determines the vertical translation:
  - For $c > 0$, the graph is translated c units up.
  - For $c < 0$, the graph is translated c units down.

CHECK Your Understanding

1. Use words from the list to describe the transformations indicated by the arrows.

horizontal	x-axis
vertical	y-axis
stretch	factor
compression	up
reflection	down
translation	right
	left

$$f(x) = \overset{A}{5}\sqrt{\underset{B}{-}\underset{C}{3}(x - \overset{D}{2})} + 4 \longleftarrow E$$

2. Match each operation to one of the transformations from question 1.

Divide the x-coordinates by 3.	A
Multiply the y-coordinates by 5.	B
Multiply the x-coordinates by -1.	C
Add 4 to the y-coordinate.	D
Add 2 to the x-coordinate.	E

3. Complete the table for the point (1, 1).

$f(x)$	$f(3x)$	$f(-3x)$	$5f(-3x)$	$5f(-3(x-2)) + 4$
(1, 1)				

PRACTISING

4. **K** Explain what transformations you would need to apply to the graph of $y = f(x)$ to graph each function.

a) $y = 3f(x) - 1$

b) $y = f(x - 2) + 3$

c) $y = f(2x) - 5$

d) $y = -f\left(\frac{1}{2}x\right) - 2$

e) $y = \frac{2}{3}f(x + 3) + 1$

f) $y = 4f(-x) - 4$

5. Sketch each set of functions on the same set of axes.

a) $y = x^2, y = 3x^2, y = 3(x - 2)^2 + 1$

b) $y = \sqrt{x}, y = \sqrt{3x}, y = \sqrt{-3x}, y = \sqrt{-3(x + 1)} - 4$

c) $y = \frac{1}{x}, y = \frac{2}{x}, y = -\frac{2}{x}, y = -\frac{2}{x - 1} + 3$

d) $y = |x|, y = \left|\frac{1}{2}x\right|, y = -\left|\frac{1}{2}x\right|, y = -\left|\frac{1}{2}(x + 3)\right| - 2$

6. Explain what transformations you would need to apply to the graph of $y = f(x)$ to graph each function.

a) $y = f\left(\frac{1}{3}(x + 4)\right)$

b) $y = 2f(-(x - 3)) + 1$

c) $y = -3f(2(x - 1)) - 3$

7. If $f(x) = x^2$, sketch the graph of each function and state the domain and range.

a) $y = f(x - 2) + 3$

b) $y = -f\left(\frac{1}{4}(x + 1)\right) + 2$

c) $y = 0.5f(3(x - 4)) - 1$

8. If $f(x) = \sqrt{x}$, sketch the graph of each function and state the domain and range.

a) $y = f(x - 1) + 4$

b) $y = f\left(-\frac{1}{2}(x + 4)\right) - 3$

c) $y = -2f(-(x - 2)) + 1$

9. If $f(x) = |x|$, sketch the graph of each function and state the domain and range.

a) $y = 2f(x - 3)$

b) $y = 4f(2(x - 1)) - 2$

c) $y = -\frac{1}{2}f(3\,(x + 2)) + 4$

10. Describe the transformations that you would apply to the graph of $f(x) = \frac{1}{x}$ to transform it into each of these graphs.

a) $y = \dfrac{1}{x - 2}$

b) $y = \frac{1}{x} + 2$

c) $y = 0.5\left(\frac{1}{x}\right)$

d) $y = \frac{2}{x}$

e) $y = \frac{1}{2x}$

f) $y = -\frac{1}{x}$

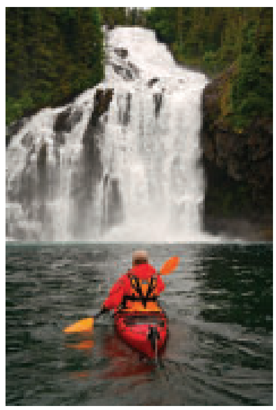
11. For $f(x) = x^2$, sketch the graph of $g(x) = f(2x + 6)$.

12. For $f(x) = \sqrt{x}$, sketch the graph of $h(x) = f(-3x - 12)$.

13. For $f(x) = |x|$, sketch the graph of $p(x) = f(4x + 8)$.

14. **A** Low and high blood pressure can both be dangerous. Doctors use a special index, P_d, to measure how far from normal someone's blood pressure is. In the equation $P_d = |P - \overline{P}|$, P is a person's systolic blood pressure and $\overline{P}$ is the normal systolic blood pressure. Sketch the graph of this index. Assume that normal systolic blood pressure is 120 mm(Hg).

15. Bhavesh uses the relationship Time $= \frac{\text{Distance}}{\text{Speed}}$ to plan his kayaking trips. Tomorrow Bhavesh plans to kayak 20 km across a calm lake. He wants to graph the relation $T(s) = \frac{20}{s}$ to see how the time, T, it will take varies with his kayaking speed, s. The next day, he will kayak 15 km up a river that flows at 3 km/h. He will need the graph of $T(s) = \frac{15}{s - 3}$ to plan this trip. Use transformations to sketch both graphs.



16. The graph of $g(x) = \sqrt{x}$ is reflected across the y-axis, stretched vertically by the factor 3, and then translated 5 units right and 2 units down. Draw the graph of the new function and write its equation.

17. The graph of $y = f(x)$ is reflected in the y-axis, stretched vertically by the factor 3, and then translated up 2 units and 1 unit left. Write the equation of the new function in terms of f.

18. Match each equation to its graph. Explain your reasoning.

a) $y = \dfrac{3}{-(x-2)} + 1$

b) $y = 2|x-3| - 2$

c) $y = -2\sqrt{x+3} - 2$

d) $y = (0.25(x-2))^2 - 3$

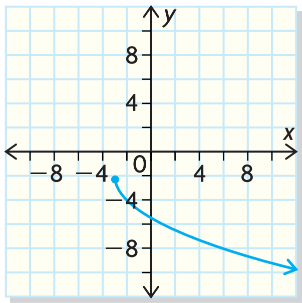
e) $y = -\dfrac{4}{x} - 3$

f) $y = -0.5|x+4| + 2$

g) $y = -0.5\sqrt{1-x} + 1$

h) $y = -\dfrac{1}{2}(x+4)^2 + 1$

A



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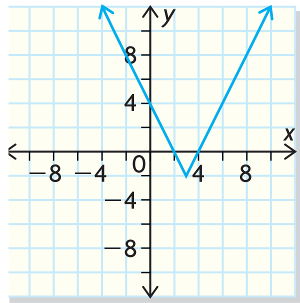

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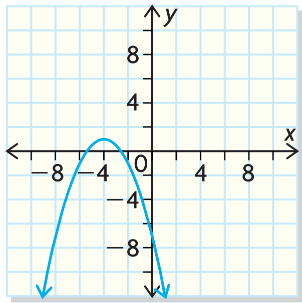

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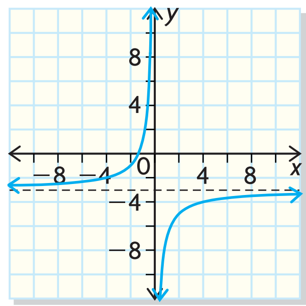

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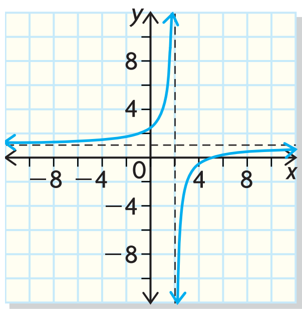

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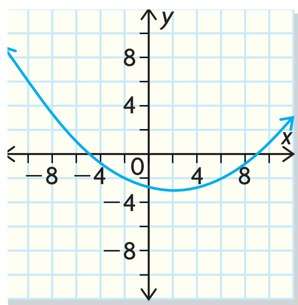

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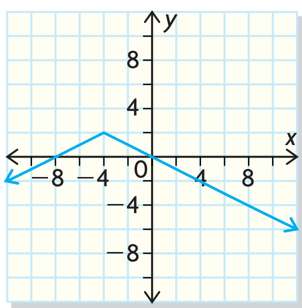

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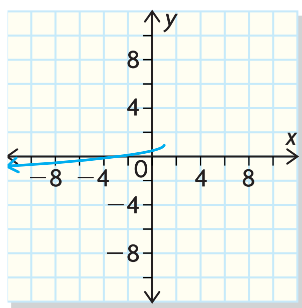


19. The function $y = f(x)$ has been transformed to $y = af[k(x - d)] + c$. Determine a, k, c, and d; sketch the graph; and state the domain and range for each transformation.

a) A vertical stretch by the factor 2, a reflection in the x-axis, and a translation 4 units right are applied to $y = \sqrt{x}$.

b) A vertical compression by the factor $\frac{1}{2}$, a reflection in the y-axis, a translation 3 units left, and a translation 4 units down are applied to $f(x) = \dfrac{1}{x}$.

c) A horizontal compression by the factor $\frac{1}{3}$, a vertical stretch by the factor 3, a translation 1 unit right, and a translation 6 units down are applied to $y = |x|$.

20. If $f(x) = (x - 2)(x + 5)$, determine the x-intercepts for each function.

a) $y = f(x)$

b) $y = -4f(x)$

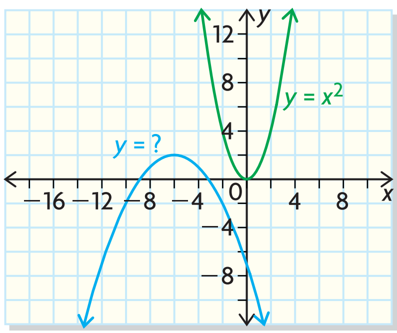
c) $y = f\left(-\dfrac{1}{3}x\right)$

d) $y = f(-(x + 2))$

21. **C** List the steps you would take to sketch the graph of a function of the form $y = af(k(x - d)) + c$ when $f(x)$ is one of the parent functions you have studied in this chapter. Discuss the roles of a, k, d, and c, and the order in which they would be applied.

Extending

22. The graphs of $y = x^2$ and another parabola are shown.



a) Determine a combination of transformations that would produce the second parabola from the first.

b) Determine a possible equation for the second parabola.

23. Compare the graphs and the domains and ranges of $f(x) = x^2$ and $g(x) = \sqrt{x}$. How are they alike? How are they different? Develop a procedure to obtain the graph of $g(x)$ from the graph of $f(x)$.

1 Chapter Review

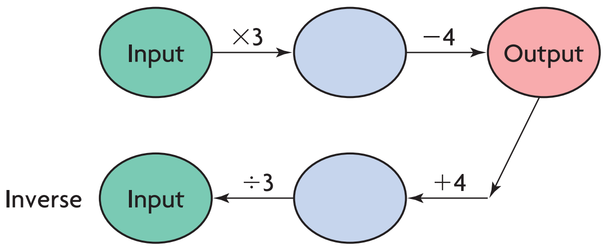
FREQUENTLY ASKED Questions

Study *Aid*

- See Lesson 1.5, Examples 1 and 2.
- Try Chapter Review Questions 10 and 11.

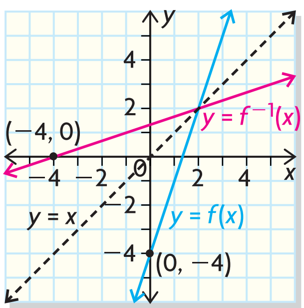
Q: How can you determine the inverse of a linear function?

A1: The inverse of a linear function is the reverse of the original function. It undoes what the original has done. This means that you can find the equation of the inverse by reversing the operations on x. For example, if $f(x) = 3x - 4$, the operations on x are as follows: Multiply by 3 and then subtract 4. To reverse these operations, you add 4 and divide by 3, so the inverse function is $f^{-1}(x) = \frac{x+4}{3}$.



A2: If (x, y) is on the graph of $f(x)$, then (y, x) is on the inverse graph, so you can switch x and y in the equation to find the inverse equation. For example, if $f(x) = 3x - 4$, you can write this as $y = 3x - 4$. Then switch x and y and solve for y.

$$x = 3y - 4$$
$$x + 4 = 3y - 4 + 4$$
$$x + 4 = 3y$$
$$\frac{x+4}{3} = y, \text{ so the inverse function is } f^{-1}(x) = \frac{x+4}{3}.$$



A3: If you have the graph of a linear function, you can graph the inverse function by reflecting in the line $y = x$.

The inverse of a linear function is another linear function, unless the original function represents a horizontal line.

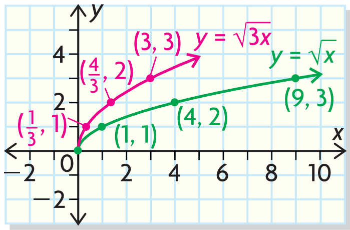
Study *Aid*

- See Lesson 1.7, Example 1.
- Try Chapter Review Questions 12 and 13.

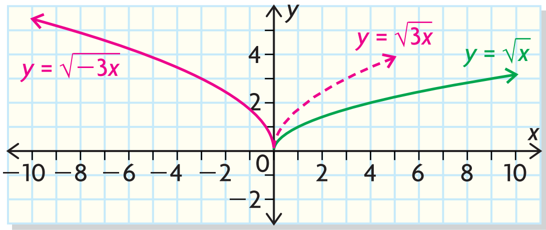
Q: How do you apply a horizontal stretch, compression, or reflection to the graph of a function?

A: The graph of $y = f(kx)$ is the graph of $y = f(x)$ after a horizontal stretch, compression, or reflection. When k is a number greater than 1 or less than -1, the graph is compressed horizontally by the factor $\frac{1}{k}$. When k is a number between -1 and 1, the graph is stretched horizontally by the factor $\frac{1}{k}$. Whenever k is negative, the graph is also reflected in the y-axis.

You apply a horizontal compression by dividing the x-coordinates of points on the original graph by k. For example, to graph $y = \sqrt{3x}$, graph $y = \sqrt{x}$ and then divide the x-coordinates of the points (0, 0), (1, 1), (4, 2), (9, 3) by 3 (or multiply by $\frac{1}{3}$) to get the points (0, 0), $(\frac{1}{3}, 1)$, $(\frac{4}{3}, 2)$, (3, 3) on the transformed graph.



When k is negative, you also reflect the graph in the y-axis. For example, $y = \sqrt{-3x}$.



Q: How do you sketch the graph of $y = af[k(x - d)] + c$ when you have the graph of $y = f(x)$?

A1: You can graph the parent function and then apply the transformations one by one, starting with the compressions, stretches, and reflections and leaving the translations until last. For example, to sketch the graph of $y = 3f(6 - 2x) - 5$ when $f(x) = \frac{1}{x}$, begin by putting the equation into the form $y = af[k(x - d)] + c$ by factoring. This gives $y = 3f[-2(x - 3)] - 5$. Then identify all the transformations you need to apply:

- $a = 3$ means a vertical stretch by the factor 3.
- $k = -2$ means a horizontal compression by the factor $\frac{1}{2}$ and a reflection in the y-axis.
- $d = 3$ means a horizontal translation 3 units right.
- $c = -5$ means a vertical translation 5 units down.

A2: You can graph the function in two steps: Apply both stretches or compressions and any reflections to the parent function first, and then both translations.

Study *Aid*

- See Lesson 1.8, Examples 1, 2, and 3.
- Try Chapter Review Questions 14 to 18.

PRACTICE Questions

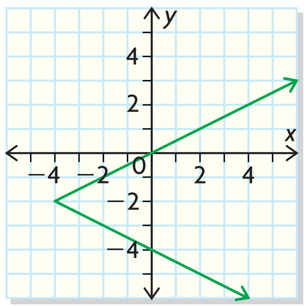
Lesson 1.1

1. For each relation, determine the domain and range and whether the relation is a function. Explain your reasoning.

a) $\{(-3, 0), (-1, 1), (0, 1), (4, 5), (0, 6)\}$

b) $y = 4 - x$

c)



d) $x^2 + y^2 = 16$

2. What rule can you use to determine, from the graph of a relation, whether the relation is a function? Graph each relation and determine which are functions.

a) $\{(-2. 1), (1, 1), (0, 0), (1, -1), (1, -2), (2, -2)\}$

b) $y = 4 - 3x$

c) $y = (x - 2)^2 + 4$

d) $x^2 + y^2 = 1$

e) $y = \dfrac{1}{x}$

f) $y = \sqrt{x}$

3. Sketch the graph of a function whose domain is the set of real numbers and whose range is the set of real numbers less than or equal to 3.

Lesson 1.2

4. If $f(x) = x^2 + 3x - 5$ and $g(x) = 2x - 3$, determine each.

a) $f(-1)$

b) $f(0)$

c) $g\left(\dfrac{1}{2}\right)$

d) $f(2b)$

e) $g(1 - 4a)$

f) x when $f(x) = g(x)$

5. a) Graph the function $f(x) = -2(x - 3)^2 + 4$, and state its domain and range.

b) What does $f(1)$ represent on the graph? Indicate, on the graph, how you would find $f(1)$.

c) Use the equation to determine each of the following.

i) $f(3) - f(2)$

ii) $2f(5) + 7$

iii) $f(1 - x)$

6. If $f(x) = x^2 - 4x + 3$, determine the input(s) for x whose output is $f(x) = 8$.

Lesson 1.4

7. A ball is thrown upward from the roof of a building 60 m tall. The ball reaches a height of 80 m above the ground after 2 s and hits the ground 6 s after being thrown.

a) Sketch a graph that shows the height of the ball as a function of time.

b) State the domain and range of the function.

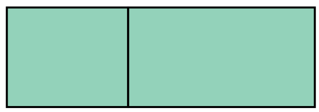
c) Determine an equation for the function.

8. State the domain and range of each function.

a) $f(x) = 2(x - 1)^2 + 3$

b) $f(x) = \sqrt{2x + 4}$

9. A farmer has 540 m of fencing to enclose a rectangular area and divide it into two sections as shown.



a) Write an equation to express the total area enclosed as a function of the width.

b) Determine the domain and range of this area function.

c) Determine the dimensions that give the maximum area.

Lesson 1.5

10. Using the functions listed as examples, describe three methods for determining the inverse of a linear function. Use a different method for each function.

a) $f(x) = 2x - 5$

b) $f(x) = \dfrac{x + 3}{7}$

c) $f(x) = 4 - \dfrac{1}{2}x$

11. For a fundraising event, a local charity organization expects to receive \$15 000 from corporate sponsorship, plus \$30 from each person who attends the event.

a) Use function notation to express the total income from the event as a function of the number of people who attend.

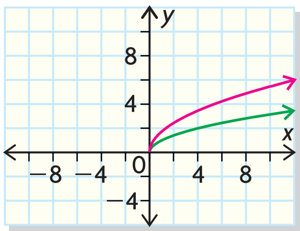
b) Suggest a reasonable domain and range for the function in part (a). Explain your reasoning.

c) The organizers want to know how many tickets they need to sell to reach their fundraising goal. Create a function to express the number of people as a function of expected income. State the domain of this new function.

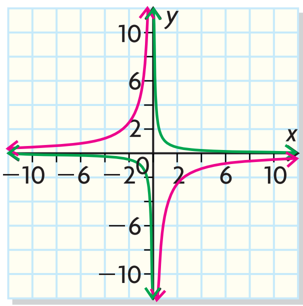
Lesson 1.7

12. In each graph, a parent function has undergone a transformation of the form $f(kx)$. Determine the equations of the transformed functions graphed in red. Explain your reasoning.

a)



b)



13. For each set of functions, transform the graph of $f(x)$ to sketch $g(x)$ and $h(x)$, and state the domain and range of each function.

a) $f(x) = x^2$, $g(x) = \left(\frac{1}{2}x\right)^2$, $h(x) = -(2x)^2$

b) $f(x) = |x|$, $g(x) = |-4x|$, $h(x) = \left|\frac{1}{4}x\right|$

Lesson 1.8

14. Three transformations are applied to $y = x^2$: a vertical stretch by the factor 2, a translation 3 units right, and a translation 4 units down.

a) Is the order of the transformations important?

b) Is there any other sequence of these transformations that could produce the same result?

15. The point $(1, 4)$ is on the graph of $y = f(x)$. Determine the coordinates of the image of this point on the graph of $y = 3f[-4(x + 1)] - 2$.

16. a) Explain what you would need to do to the graph of $y = f(x)$ to graph the function $y = -2f[(\frac{1}{3}x + 4)] - 1$.

b) Graph the function in part (a) for $f(x) = x^2$.

17. In each case, write the equation for the transformed function, sketch its graph, and state its domain and range.

a) The graph of $f(x) = \sqrt{x}$ is compressed horizontally by the factor $\frac{1}{2}$, reflected in the y-axis, and translated 3 units right and 2 units down.

b) The graph of $y = \frac{1}{x}$ is stretched vertically by the factor 3, reflected in the x-axis, and translated 4 units left and 1 unit up.

18. If $f(x) = (x - 4)(x + 3)$, determine the x-intercepts of each function.

a) $y = f(x)$

b) $y = -2f(x)$

c) $y = f\left(-\frac{1}{2}x\right)$

d) $y = f(-(x + 1))$

19. A function $f(x)$ has domain $\{x \in \mathbf{R} \mid x \geq -4\}$ and range $\{y \in \mathbf{R} \mid y < -1\}$. Determine the domain and range of each function.

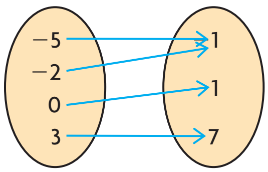
a) $y = 2f(x)$

b) $y = f(-x)$

c) $y = 3f(x + 1) + 4$

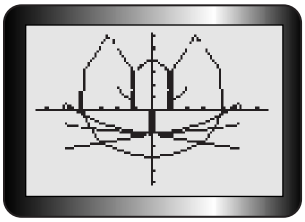
d) $y = -2f(-x + 5) + 1$

1 Chapter Self-Test



1. For each relation, determine the domain and range and whether the relation is a function. Explain your reasoning.
   a) The function shown at the left.
   b) $y = \sqrt{x + 2}$
2. An incandescent light bulb costs \$0.65 to buy and \$0.004/h for electricity to run. A fluorescent bulb costs \$3.50 to buy and \$0.001/h to run.
   a) Use function notation to write a cost equation for each type of bulb.
   b) State the domain and range of each function.
   c) After how long is the fluorescent bulb cheaper than the regular bulb?
   d) Determine the difference in costs after one year. Assume the light is on for an average of 6 h/day.
3. Determine the domain and range of each function. Show your steps.
   a) $f(x) = \dfrac{1}{x - 2}$
   b) $f(x) = \sqrt{3 - x} - 4$
   c) $f(x) = -|x + 1| + 3$
4. Explain what the term *inverse* means in relation to a linear function. How are the domain and range of a linear function related to the domain and range of its inverse?
5. For each function, determine the inverse, sketch the graphs of the function and its inverse, and state the domain and range of both the function and its inverse.
   a) $\{(-2, 3), (0, 5), (2, 6), (4, 8)\}$
   b) $f(x) = 3 - 4x$
6. At Phoenix Fashions, Rebecca is paid a monthly salary of \$1500, plus 4% commission on her sales over \$2500.
   a) Graph the relation between monthly earnings and sales.
   b) Use function notation to write an equation of the relation.
   c) Graph the inverse relation.
   d) Use function notation to write an equation of the inverse.
   e) Use the equation in part (d) to express Rebecca's sales if she earned \$1740 one month. Then evaluate.
7. The function $y = f(x)$ has been transformed to $y = f(kx)$. Determine the value of k for each transformation.
   a) a horizontal stretch by the factor 5
   b) a horizontal compression by the factor $\frac{1}{3}$ and a reflection in the y-axis
8. The function $y = f(x)$ has been transformed to $y = af[k(x - d)] + c$. Determine a, k, d, and c; write the equation; sketch the graph; and state the domain and range of each transformed function.
   a) vertical compression by the factor $\frac{1}{2}$, reflection in the y-axis, and translation 2 units right, applied to $y = \sqrt{x}$
   b) vertical stretch by the factor 4, reflection in the x-axis, translation 2 units left, and translation 3 units down, applied to $y = \dfrac{1}{x}$
   c) horizontal compression by the factor $\frac{1}{4}$, vertical stretch by the factor $\frac{3}{2}$, reflection in the x-axis, translation 3 units right, and translation 2 units down, applied to $y = |x|$

Functional Art



Parts of transformed parent functions were used to make this cat's face on a graphing calculator. The functions used are listed in the table that follows.

$$-6.58 \leq X \leq 6.58, \text{ Xscl } 1$$

$$-6.2 \leq Y \leq 6.2, \text{ Yscl } 1$$

Feature	Function	Domain
ears	$y = -\lvert 2x - 5 \rvert + 6$ $y = -\lvert -2x - 5 \rvert + 6$	$1 \leq x \leq 4$ $-4 \leq x \leq -1$
top of head	$y = -x^2 + 4$	$-1 \leq x \leq 1$
chin	$y = \dfrac{5}{x + 5} - 5$ $y = \dfrac{5}{-x + 5} - 5$	$-4 \leq x \leq 0$ $0 \leq x \leq 4$
eyes	$y = -\sqrt{4 - 2x} + 2$ $y = -\sqrt{4 - 2x} + 2$	$1 \leq x \leq 3$ $-3 \leq x \leq -1$
whiskers	$y = 0.1x^2 - 2$ $y = 0.03x^2 - 2$ $y = -0.25\lvert x \rvert - 2$	$-5 \leq x \leq 5$ $-5 \leq x \leq 5$ $-5 \leq x \leq 5$

? How can you use transformations of parent functions to create other pictures?

A. Re-create the cat's face on a graphing calculator. Begin by putting the calculator in DOT mode. Then enter each function listed in the table, along with its domain. The first three entries are shown.

B. Describe how transformations were used to create the cat's features. For each feature, describe
- which properties of the parent function were useful for that feature
- which transformations were used and why
- how symmetry was used and which transformations ensured symmetry

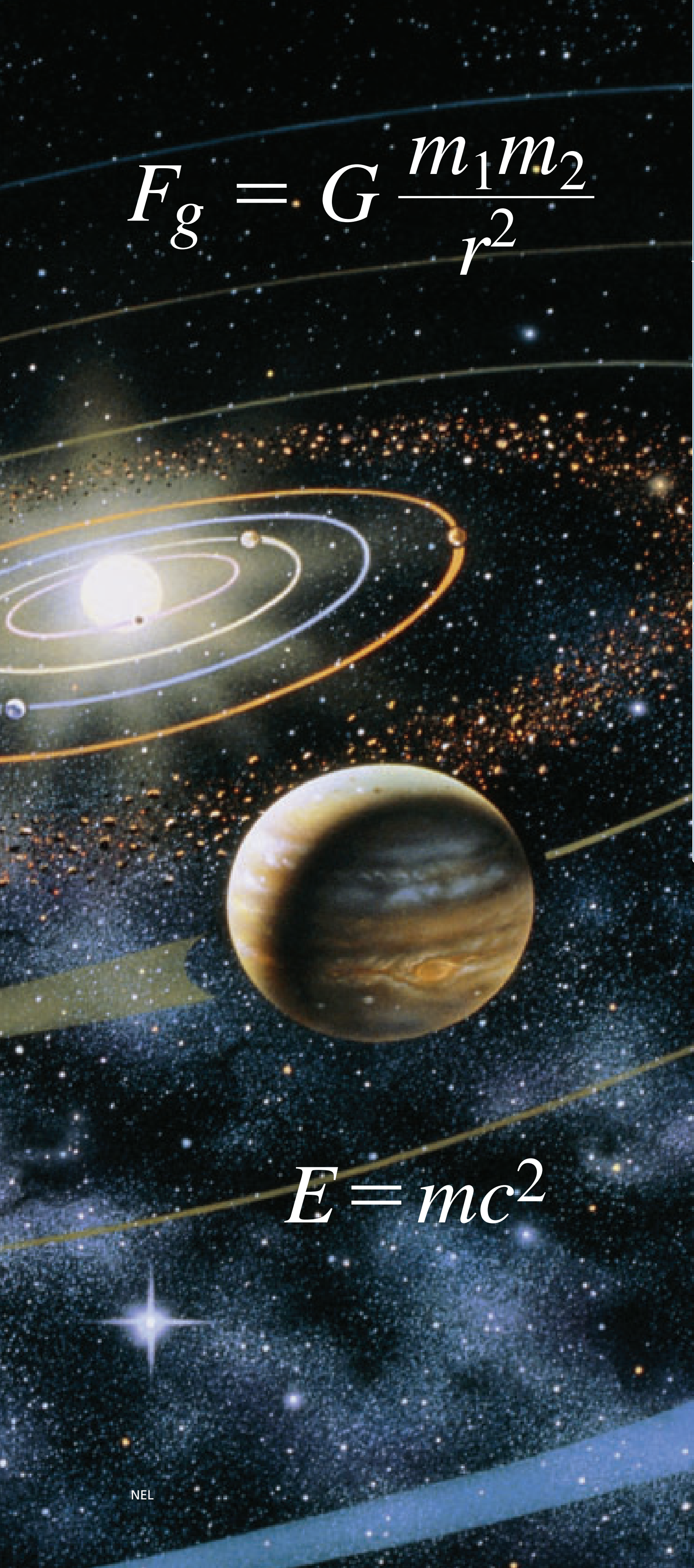
C. Create your own picture, using transformations of the parent functions $f(x) = x^2$, $f(x) = \sqrt{x}$, $y = \dfrac{1}{x}$, and $f(x) = \lvert x \rvert$. You must use each parent function at least once.

D. List the parts of your picture, the functions used, and the corresponding domains in a table. Explain why you chose each function and each transformation.

Task *Checklist*

- ✔ Did you use each parent function at least once?
- ✔ Did you list the transformed functions and the corresponding domains?
- ✔ Did you explain why you chose each function and each transformation?
- ✔ Did you describe the transformations in appropriate math language?


$E = M + \sum_{x=1}^{\infty} a_n e^n$
$s = \frac{1}{2} a t^2$
$F = ma$



$$F_g = G\frac{m_1 m_2}{r^2}$$

$$E = mc^2$$

Chapter 2

Equivalent Algebraic Expressions

GOALS

You will be able to

- Determine whether algebraic expressions are equivalent
- Add, subtract, multiply, and factor polynomials
- Simplify rational expressions
- Add, subtract, multiply, and divide rational expressions

? Many great scientists and mathematicians, such as Hypatia, Nicolaus Copernicus, Johannes Kepler, Isaac Newton, Caroline Herschel, Albert Einstein, and Steven Hawking, have spent their lives trying to explain the nature of the universe. What role does mathematics play in this quest for knowledge?

2 Getting Started

SKILLS AND CONCEPTS You Need

	Polynomial	Type	Degree
	$3x^2 + 2x - 1$	trinomial	2
a)	$4x - 7$		
b)	3		
c)	$7 + 2x^2$		
d)	$2xy$		
e)	$x^2 - 3xy + y^2$		

Study *Aid*

For help, see Essential Skills Appendix.

Question	Appendix
2	A-8
4	A-9
5	A-2
6	A-3

1. For each polynomial in the table at the left, state the type and the degree. The first one has been done for you.

2. Expand, where necessary, and then simplify.

a) $(2x + 3) + (7x - 5)$

b) $(4x^2 - 7x + 1) - (2x^2 - 3x + 10)$

c) $(2x - 3)(4x + 5)$

d) $(2x - 1)^2$

3. How are expanding and factoring related to each other? Use an example in your explanation.

4. Factor, where possible.

a) $6xy^3 - 8x^2y^3$

b) $a^2 - 7a + 10$

c) $12n^2 + 7n - 10$

d) $9 - 25x^2$

e) $x^2 + 5x + 8$

f) $y^2 - 5y - 36$

5. Simplify.

a) $\frac{3}{4} + \frac{1}{6}$

b) $\frac{-2}{5} - \frac{1}{10}$

c) $\left(\frac{-12}{25}\right)\left(\frac{-10}{9}\right)$

d) $\left(\frac{4}{3}\right) \div \left(\frac{-2}{15}\right)$

6. Simplify.

a) $\left(\frac{2x^2}{3}\right)\left(\frac{5x^3}{4}\right)$

b) $\left(\frac{3x}{2}\right) \div \left(\frac{x^3}{5}\right)$

c) $(2x^2y^3)(4xy^2)$

d) $(25x^5y^3) \div (5x^2y)$

7. Determine where each function is undefined, then state its domain.

a) $f(x) = x$

b) $g(x) = 2x^2$

c) $m(x) = \sqrt{x}$

d) $h(x) = \frac{1}{x}$

e) $j(x) = \frac{3}{x - 4}$

f) $n(x) = \sqrt{x + 10}$

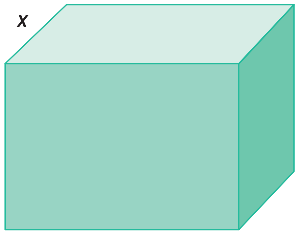
8. Complete the chart to show what you know about polynomials.

Definition:	Characteristics:
Polynomials	
Examples:	Non-examples:

APPLYING What You Know

Doubling Dimensions

A rectangular box has dimensions x, $x + 1$, and $x + 2$.



? Suppose each dimension of the box is doubled. By what factors will the surface area and volume of the box increase?

A. What are the formulas for the volume and surface area of a rectangular box of width w, length l, and height h?

B. Use the given dimensions to write expressions for the volume and surface area of the box.

C. Write expressions for the volume and surface area of the new box after the dimensions are doubled.

D. By what factor has the volume increased? By what factor has the surface area increased? Explain how you know.

2.1 Adding and Subtracting Polynomials

YOU WILL NEED

- graphing calculator

Communication | *Tip*

The numbers 1 and 2 in $a_1(t)$ and $a_2(t)$ are called subscripts. In this case, they are used to distinguish one function from the other. This distinction is necessary because both functions are named with the letter *a*.

GOAL

Determine whether polynomial expressions are equivalent.

LEARN ABOUT the Math

Fred enjoys working with model rockets. He wants to determine the difference in altitude of two different rockets when their fuel burns out and they begin to coast.

The altitudes, in metres, are given by these equations:

$$a_1(t) = -5t^2 + 100t + 1000$$

and

$$a_2(t) = -5t^2 + 75t + 1200$$

where t is the elapsed time, in seconds.

The difference in altitude, $f(t)$, is given by

$$f(t) = (-5t^2 + 100t + 1000) - (-5t^2 + 75t + 1200)$$

Fred simplified $f(t)$ to $g(t) = 175t + 2200$.

? Are the functions $f(t)$ and $g(t)$ equivalent?

EXAMPLE 1 | Selecting a strategy to determine equivalence

Determine if $f(t)$ and $g(t)$ are equivalent functions.

Anita's Solution: Simplifying the Polynomial in $f(t)$

$$\begin{aligned} f(t) &= (-5t^2 + 100t + 1000) \\ &\quad - (-5t^2 + 75t + 1200) \\ &= -5t^2 + 100t + 1000 + 5t^2 - 75t - 1200 \end{aligned}$$

Polynomials behave like numbers because, for any value of the variable, the result is a number. I know that with numbers, subtraction is equivalent to adding the opposite, so I subtracted the polynomials by adding the opposite of the second expression.

$$= 25t - 200$$

Then I collected like terms.

Since $g(t) \neq 25t - 200$, the functions are not equivalent.

If two expressions are not equivalent, then, for most values of t, their function values are different. The exception is when the functions intersect.

Maria's Solution: Evaluating the Functions for the Same Value of the Variable

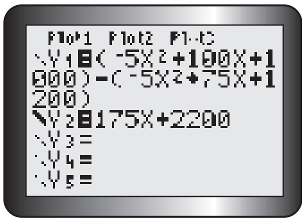
$$
\begin{aligned}
f(t) &= (-5t^2 + 100t + 1000) \\
&\quad - (-5t^2 + 75t + 1200) \\
f(0) &= (-5(0)^2 + 100(0) + 1000) \\
&\quad - (-5(0)^2 + 75(0) + 1200) \\
&= 1000 - 1200 \\
&= -200 \\
g(0) &= 175(0) + 2200 \\
&= 2200
\end{aligned}
$$

I used $t = 0$ because it makes the calculations to find $f(0)$ and $g(0)$ easier.

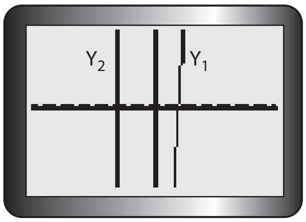
Since $f(0) \neq g(0)$, the functions are not equivalent.

If two functions are equivalent, then their graphs should be identical.

Sam's Solution: Graphing Both Functions



I graphed Fred's original function, $f(t)$, in Y_1 and his final function, $g(t)$, in Y_2. I zoomed out until I could see both functions.



Since the graphs are different, the functions must be different.

The functions are not equivalent.

Reflecting

A. How is subtracting two polynomials like subtracting integers? How is it different?

B. If Fred had not made an error when he simplified, whose method would have shown that his original and final expressions are equivalent?

C. What are the advantages and disadvantages of the three methods used to determine whether two polynomials are equivalent?

APPLY the Math

EXAMPLE 2 Reasoning whether two polynomials are equivalent

Nigel and Petra are hosting a dinner for 300 guests. Cheers banquet hall has quoted these charges:

- \$500, plus \$10 per person, for food,
- \$200, plus \$20 per person, for drinks, and
- a discount of \$5 per person if the number of guests exceeds 200.

Nigel and Petra have created two different functions for the total cost, where n represents the number of guests and $n > 200$.

Nigel's cost function: $C_1(n) = (10n + 500) + (20n + 200) - 5n$
Petra's cost function: $C_2(n) = (10n + 20n - 5n) + (500 + 200)$

Are the functions equivalent?

Lee's Solution

$$C_1(n) = (10n + 500) + (20n + 200) - 5n$$
$$= 10n + 20n - 5n + 500 + 200$$

Nigel's cost function was developed using the cost of each item.

The cost of the food is $(10n + 500)$. The cost of the drinks is $(20n + 200)$. The discount is $5n$.

$$= 25n + 700$$

I simplified by collecting like terms.

$$C_2(n) = (10n + 20n - 5n) + (500 + 200)$$
$$= 25n + 700$$

Petra's cost function was developed by adding the variable costs $(10n + 20n - 5n)$ and the fixed costs $(500 + 200)$. The result is two groups of like terms, which I then simplified.

$$C_1(n) = 25n + 700$$
$$C_2(n) = 25n + 700$$

Both cost functions simplify to the same function.

The two cost functions are equivalent.

EXAMPLE 3 Reasoning about the equivalence of expressions

Are the expressions $xy + xz + yz$ and $x^2 + y^2 + z^2$ equivalent?

Dwayne's Solution

To check for non-equivalence, I substituted some values for x, y, and z.

The values 0, 1, and -1 are often good choices, since they usually make expressions easy to evaluate.

$$xy + xz + yz = 0(0) + 0(1) + 0(1) = 0$$

I tried $x = 0$, $y = 0$, and $z = 1$ and evaluated the first expression.

$$x^2 + y^2 + z^2 = 0^2 + 0^2 + 1^2 = 1$$

I evaluated the second expression, using the same values for x, y, and z.

The expressions are not equivalent.

The expressions result in different values.

In Summary

Key Ideas

- Two polynomial functions or expressions are equivalent if
  - they simplify algebraically to give the same function or expression
  - they produce the same graph
- Two polynomial functions or expressions are not equivalent if
  - they result in different values when they are evaluated with the same numbers substituted for the variable(s)

Need to Know

- If you notice that two functions are equivalent at one value of a variable, it does not necessarily mean they are equivalent at *all* values of the variable. Evaluating both functions at a single value is sufficient to demonstrate non-equivalence, but it isn't enough to demonstrate equivalence.
- The sum of two or more polynomial functions or expressions can be determined by writing an expression for the sum of the polynomials and collecting like terms.
- The difference of two polynomial functions or expressions can be determined by adding the opposite of one polynomial and collecting like terms.

CHECK Your Understanding

1. Simplify.

a) $(3x^2 - 7x + 5) + (x^2 - x + 3)$

b) $(x^2 - 6x + 1) - (-x^2 - 6x + 5)$

c) $(2x^2 - 4x + 3) - (x^2 - 3x + 2) + (x^2 - 1)$

2. Show that $f(x)$ and $g(x)$ are equivalent by simplifying each.

$$f(x) = (2x - 1) - (3 - 4x) + (x + 2)$$
$$g(x) = (-x + 6) + (6x - 9) - (-2x - 1)$$

3. Show that $f(x)$ and $g(x)$ are not equivalent by evaluating each function at a suitable value of x.

$$f(x) = 2(x - 3) + 3(x - 3)$$
$$g(x) = 5(2x - 6)$$

PRACTISING

4. Simplify.

a) $(2a + 4c + 8) + (7a - 9c - 3)$

b) $(3x + 4y - 5z) + (2x^2 + 6z)$

c) $(6x + 2y + 9) + (-3x - 5y - 8)$

d) $(2x^2 - 7x + 6) + (x^2 - 2x - 9)$

e) $(-4x^2 - 2xy) + (6x^2 - 3xy + 2y^2)$

f) $(x^2 + y^2 + 8) + (4x^2 - 2y^2 - 9)$

5. Simplify.

a) $(m - n + 2p) - (3n + p - 7)$

b) $(-6m - 2q + 8) - (2m + 2q + 7)$

c) $(4a^2 - 9) - (a^3 + 2a - 9)$

d) $(2m^2 - 6mn + 8n^2) - (4m^2 - mn - 7n^2)$

e) $(3x^2 + 2y^2 + 7) - (4x^2 - 2y^2 - 8)$

f) $5x^2 - (2x^2 - 30) - (-20)$

6. Simplify.

a) $(2x - y) - (-3x + 4y) + (6x - 2y)$

b) $(3x^2 - 2x) + (x^2 - 7x) - (7x + 3)$

c) $(2x^2 + xy - y^2) - (x^2 - 4xy - y^2) + (3x^2 - 5xy)$

d) $(xy - xz + 4yz) + (2x - 3yz) - (4y - xz)$

e) $\left(\frac{1}{2}x + \frac{1}{3}y\right) - \left(\frac{1}{5}x - y\right)$

f) $\left(\frac{3}{4}x + \frac{1}{2}y\right) - \left(\frac{2}{3}x + \frac{1}{4}y - 1\right)$

7. Use two different methods to show that the expressions
K $(3x^2 - x) - (5x^2 - x)$ and $-2x^2 - 2x$ are not equivalent.

8. Determine whether each pair of functions is equivalent.
 a) $f(x) = (2x^2 + 7x - 2) - (3x + 7)$ and $g(x) = (x^2 + 12) + (x^2 + 4x - 17)$
 b) $s_1(t) = (t + 2)^3$ and $s_2(t) = t^3 + 8$
 c) $y_1 = (x - 1)(x)(x + 2)$ and $y_2 = 3x(x^2 - 1)$
 d) $f(n) = 0.5n^2 + 2n - 3 + (1.5n^2 - 6)$ and $g(n) = n^2 - n + 1 - (-n^2 - 3n + 10)$
 e) $y_1 = 3p(q - 2) + 2p(q + 5)$ and $y_2 = p(q + 4)$
 f) $f(m) = m(5 - m) - 2(2m - m^2)$ and $g(m) = 4m^2(m - 1) - 3m^2 + 5m$

9. Determine two non-equivalent polynomials, $f(x)$ and $g(x)$, such that $f(0) = g(0)$ and $f(2) = g(2)$.

10. Kosuke wrote a mathematics contest consisting of 25 multiple-choice questions. The scoring system gave 6 points for a correct answer, 2 points for not answering a question, and 1 point for an incorrect answer. Kosuke got x correct answers and left y questions unanswered.
 a) Write an expression for the number of questions he answered incorrectly.
 b) Write an expression, in simplified form, for Kosuke's score.
 c) Use the expressions you wrote in parts (a) and (b) to determine Kosuke's score if he answered 13 questions correctly and 7 incorrectly.

11. **A** The two equal sides of an isosceles triangle each have a length of $2x + 3y - 1$. The perimeter of the triangle is $7x + 9y$. Determine the length of the third side.

12. Tino owns a small company that produces and sells cellphone cases. The revenue and cost functions for Tino's company are shown below, where x represents the selling price in dollars.

$$\text{Revenue: } R(x) = -50x^2 + 2500x$$

$$\text{Cost: } C(x) = 150x + 9500$$

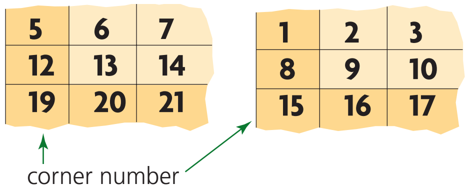
 a) Write the simplified form of the profit function, $P(x) = R(x) - C(x)$.
 b) What profit will the company make if it sells the cases for \$12 each?

13. **T** For each pair of functions, label the pairs as equivalent, non-equivalent, or cannot be determined.
 a) $f(2) = g(2)$
 b) $h(3) = g(4)$
 c) $j(8) \neq k(8)$
 d) $l(5) \neq m(7)$
 e) $n(x) = p(x)$ for all values of x in their domain

14. **C** Ramy used his graphing calculator to graph three different polynomial functions on the same axes. The equations of the functions all appeared to be different, but his calculator showed only two different graphs. He concluded that two of his functions were equivalent.
 a) Is his conclusion correct? Explain.
 b) How could he determine which, if any, of the functions were equivalent without using his graphing calculator?

Extending

15. Sanya noticed an interesting property of numbers that form a five-square capital-L pattern on a calendar page:



In each case that she checked, the sum of the five numbers was 18 less than five times the value of the number in the corner of the L. For example, in the calendars shown,

$$5 + 12 + 19 + 20 + 21 = 5(19) - 18$$

$$1 + 8 + 15 + 16 + 17 = 5(15) - 18$$

a) Show that this pattern holds for any numbers on the calendar page.
b) The sum of certain numbers in this pattern is 112. Determine the value of the corner number.
c) Write expressions for the sum of the five numbers, for the other three orientations of the L, when x is the corner number.

16. Since $70 = 5 \times 14$, 70 has 5 as a divisor. The number 70 can also be expressed as the sum of five consecutive natural numbers:

$$70 = 12 + 13 + 14 + 15 + 16$$

a) $105 = 5 \times 21$. Express 105 as the sum of five consecutive natural numbers.
b) Suppose m is a natural number that is greater than 2 and $n = 5m$. Express n as the sum of five consecutive natural numbers.
c) Express 91 as a sum of seven consecutive natural numbers.

17. a) Consider the linear functions $f(x) = ax + b$ and $g(x) = cx + d$. Suppose that $f(2) = g(2)$ and $f(5) = g(5)$. Show that the functions must be equivalent.
b) Consider the two quadratic functions $f(x) = ax^2 + bx + c$ and $g(x) = px^2 + qx + r$. Suppose that $f(2) = g(2)$, $f(3) = g(3)$, and $f(4) = g(4)$. Show that the functions must be equivalent.

2.2 Multiplying Polynomials

GOAL

Simplify polynomials by multiplying.

LEARN ABOUT the Math

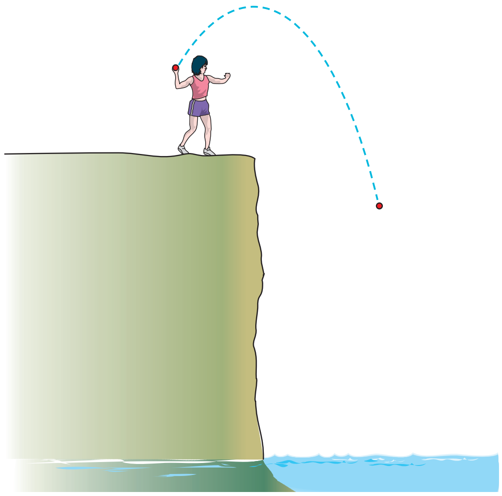
In a physics textbook, Kristina reads about an experiment in which a ball is thrown upward from the top of a cliff, ultimately landing in the water below the cliff. The height of the ball above the cliff, $h(t)$, and its velocity, $v(t)$, at time t are respectively given by

$$h(t) = -5t^2 + 5t + 2.5$$

and

$$v(t) = -10t + 5.$$

Kristina learns that the product of the two functions allows her to determine when the ball moves away from, and when the ball moves toward, the top of the cliff.



? How can she simplify the expression for $v(t) \times h(t)$?

EXAMPLE 1 Selecting a strategy to simplify a product: The distributive property

Simplify the expression $v(t) \times h(t) = (-10t + 5)(-5t^2 + 5t + 2.5)$.

Sam's Solution

$$\begin{aligned} v(t)h(t) &= (-10t + 5)(-5t^2 + 5t + 2.5) \\ &= (-10t)(-5t^2 + 5t + 2.5) + (5)(-5t^2 + 5t + 2.5) \\ &= (50t^3 - 50t^2 - 25t) + (-25t^2 + 25t + 12.5) \\ &= 50t^3 - 50t^2 - 25t^2 - 25t + 25t + 12.5 \\ &= 50t^3 - 75t^2 + 12.5 \end{aligned}$$

I used the **commutative property** for multiplication to create an equivalent expression.

I used the **distributive property** to expand the product of the polynomials. I multiplied each of the three terms in the trinomial by each of the terms in the binomial.

Then I grouped and collected like terms.

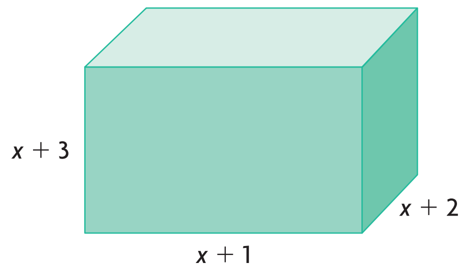
Reflecting

1. How does the simplified expression differ from the original?
2. Sam grouped together like terms in order to simplify. Is this always necessary? Explain.
3. Would Sam have gotten a different answer if he multiplied $(-5t^2 + 5t + 2.5)(-10t + 5)$ by multiplying each term in the second factor by each term in the first factor? Explain.

WORK WITH the Math

EXAMPLE 2 Selecting a strategy to multiply three binomials

Determine a simplified function that represents the volume of the given box.



Fred's Solution: Starting with the First Two Binomials

$V = lwh$

$V = (x + 1)(x + 2)(x + 3)$

$= (x^2 + 3x + 2)(x + 3)$

I know that multiplication is **associative,** so I can multiply in any order. I multiplied the first two binomials together and got a trinomial.

$= (x^2 + 3x + 2)(x) + (x^2 + 3x + 2)(3)$

Then I took the x-term from $(x + 3)$ and multiplied it by the trinomial. Next, I took the 3 term from $(x + 3)$ and multiplied it by the trinomial.

$= x^3 + 3x^2 + 2x + 3x^2 + 9x + 6$

$= x^3 + 6x^2 + 11x + 6$

Finally, I simplified by collecting like terms and arranged the terms in descending order.

Atish's Solution: Starting with the Last Two Binomials

$V = (x + 1)(x + 2)(x + 3)$

I multiplied the last two binomials together and got a trinomial.

$= (x + 1)(x^2 + 5x + 6)$

Then I took the x-term from $(x + 1)$ and multiplied it by the trinomial. Next, I took the 1 term from $(x + 1)$ and multiplied it by the trinomial.

$= x^3 + 5x^2 + 6x + x^2 + 5x + 6$

$= x^3 + 6x^2 + 11x + 6$

Finally, I simplified by collecting like terms and arranged the terms in descending order.

EXAMPLE 3 Selecting a strategy to determine non-equivalence

Is $(2x + 3y + 4z)^2 = 4x^2 + 9y^2 + 16z^2$?

Mathias's Solution: Using Substitution and then Evaluating

Let $x = y = z = 1$

$$\begin{aligned} \text{L.S.} &= (2x + 3y + 4z)^2 \\ &= (2 + 3 + 4)^2 \\ &= 9^2 \\ &= 81 \end{aligned} \qquad \begin{aligned} \text{R.S.} &= 4x^2 + 9y^2 + 16z^2 \\ &= 4 + 9 + 16 \\ &= 29 \end{aligned}$$

I substituted 1 for each of the variables in each expression to see if the results would be different.

Since the left side did not equal the right side, the expressions are not equivalent.

Lee's Solution: Expanding and Simplifying

$(2x + 3y + 4z)^2 = (2x + 3y + 4z)(2x + 3y + 4z)$

I wrote the left side of the equation as the product of two identical factors. I multiplied directly, multiplying each term in one polynomial by each term in the other.

$= 4x^2 + 6xy + 8xz + 6xy + 9y^2 + 12yz + 8xz + 12yz + 16z^2$

$= 4x^2 + 12xy + 16xz + 9y^2 + 24yz + 16z^2$

I simplified by collecting like terms.

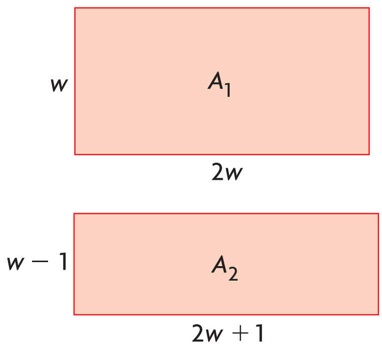
The expressions are not equivalent.

The simplified expression does not result in $4x^2 + 9y^2 + 16z^2$.

EXAMPLE 4 Representing changes in area as a polynomial

A rectangle is twice as long as it is wide. Predict how the area will change if the length of the rectangle is increased by 1 and the width is decreased by 1. Write an expression for the change in area and interpret the result.

Kim's Solution



Since we are increasing the length and decreasing the width by the same amount, I predict that there will be no change in the area.

$$A_1 = (2w)w$$
$$= 2w^2$$

To check my prediction, I let w represent the width and $2w$ the length. Their product gives the original area, A_1.

$$A_2 = (2w + 1)(w - 1)$$
$$= 2w^2 - w - 1$$

I increased the length by 1 and decreased the width by 1 by adding and subtracting 1 to my previous expressions. The product gives the area of the new rectangle, A_2.

$$\text{change in area} = A_2 - A_1$$
$$= (2w^2 - w - 1) - (2w^2)$$
$$= -w - 1$$

I took the difference of the new area, A_2, and the original area, A_1, to represent the change in area.

My prediction was wrong.

w represents width, which is always positive. Substituting any positive value for w into $-w - 1$ results in a negative number. This means that the new rectangle must have a smaller area than the original one.

In Summary

Key Idea

- The product of two or more expressions, one of which contains at least two terms, can be found by using the distributive property (often called expanding) and then collecting like terms.

Need to Know

- Since polynomials behave like numbers, the multiplication of polynomials has the same properties:
  For any polynomials a, b, and c:

$$ab = ba \text{ (commutative property)}$$
$$(ab)c = a(bc) \text{ (associative property)}$$
$$a(b + c) = ab + ac \text{ (distributive property)}$$

  With the use of the distributive property, the product of two polynomials can be found by multiplying each term in one polynomial by each term in the other.

CHECK Your Understanding

1. Expand and simplify.

a) $2x(3x - 5x^2 + 4y)$

b) $(3x - 4)(2x + 5)$

c) $(x + 4)^2$

d) $(x + 1)(x^2 + 2x - 3)$

2. a) Is $(3x + 2)^2 = 9x^2 + 4$? Justify your decision.

b) Write the simplified expression that is equivalent to $(3x + 2)^2$.

3. Expand and simplify $(2x + 4)(3x^2 + 6x - 5)$ by

a) multiplying from left to right

b) multiplying from right to left

PRACTISING

4. Expand and simplify.

a) $5x(5x^2 + 3x - 4)$

b) $(x - 6)(2x + 5)$

c) $(x + 3)(x - 3) + (5x - 6)(3x - 7)$

d) $4(n - 4)(3 + n) - 3(n - 5)(n + 8)$

e) $3(2x - 1)^2 - 5(4x + 1)^2$

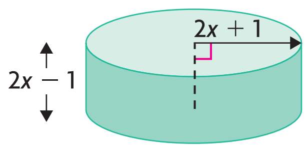
f) $2(3a + 4)(a - 6) - (3 - a)^2 + 4(5 - a)$

5. Expand and simplify.
   a) $4x(x + 5)(x - 5)$
   b) $-2a(a + 4)^2$
   c) $(x + 2)(x - 5)(x - 2)$
   d) $(2x + 1)(3x - 5)(4 - x)$
   e) $(9a - 5)^3$
   f) $(a - b + c - d)(a + b - c - d)$

6. Determine whether each pair of expressions is equivalent.
   a) $(3x - 2)(2x - 1)$ and $3x(2x - 1) - 2(2x - 1)$
   b) $(x - 4)(2x^2 + 5x - 6)$ and $2x^2(x - 4) + 5x(x - 4) - 6(x - 4)$
   c) $(x + 2)(3x - 1) - (1 - 2x)^2$ and $x^2 + 9x - 3$
   d) $2(x - 3)(2x^2 - 4x + 5)$ and $4x^3 - 20x^2 + 34x - 30$
   e) $(4x + y - 3)^2$ and $16x^2 - 8xy + 24x + y^2 - 6y + 9$
   f) $3(y - 2x)^3$ and $-24x^3 + 36x^2y - 18xy^2 + 3y^3$

7. **T** Is the equation $(x - 1)(x^4 + x^3 + x^2 + x + 1) = x^5 - 1$ true for all, some, or no real numbers? Explain.

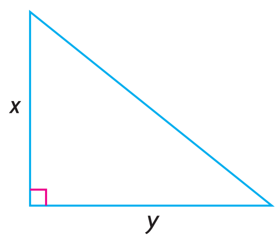
8. **K** Recall that the associative property of multiplication states that $(ab)c = a(bc)$.
   a) Verify this property for the product $19(5x + 7)(3x - 2)$ by expanding and simplifying in two different ways.
   b) Which method did you find easier?



9. **A** A cylinder with a top and bottom has radius $2x + 1$ and height $2x - 1$. Write a simplified expressions for its
   a) surface area, where $SA = 2\pi r^2 + 2\pi rh$
   b) volume, where $V = \pi r^2 h$

10. a) Is $(x - 3)^2 = (3 - x)^2$? Explain.
    b) Is $(x - 3)^3 = (3 - x)^3$? Explain.

11. Expand and simplify.
    a) $(x^2 + 2x - 1)^2$
    b) $(2 - a)^3$
    c) $(x^3 + x^2 + x + 1)(x^3 - x^2 - x - 1)$
    d) $2(x + 1)^2 - 3(2x - 1)(3x - 5)$



12. The two sides of the right triangle shown at the left have lengths x and y. Represent the change in the triangle's area if the length of one side is doubled and the length of the other side is halved.

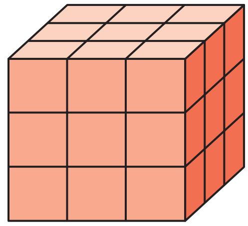
13. The kinetic energy of an object is given by $E = \frac{1}{2}mv^2$, where m represents the mass of the object and v represents its speed. Write a simplified expression for the kinetic energy of the object if
    a) its mass is increased by x
    b) its speed is increased by y

14. **a)** If $f(x)$ has two terms and $g(x)$ has three terms, how many terms will the product of $f(x)$ and $g(x)$ have before like terms are collected? Explain and illustrate with an example.

b) In general, if two or more polynomials are to be multiplied, how can you determine how many terms the product will have before like terms are collected? Explain and illustrate with an example.

Extending

15. Suppose a $3 \times 3 \times 3$ cube is painted red and then divided into twenty-seven $1 \times 1 \times 1$ cubes.



a) How many of the 27 smaller cubes are coloured red on

i) three faces? **iii)** one face?

ii) two faces? **iv)** no faces?

b) Answer (i) to (iv) from part (a) for a $10 \times 10 \times 10$ cube divided into one thousand $1 \times 1 \times 1$ cubes.

c) Answer (i) to (iv) from part (a) for an $n \times n \times n$ cube.

d) Check your results for parts (a) and (b) and by substituting 3 and then 10 into the expressions obtained in your answers to part (c).

16. Many tricks in mental arithmetic involve algebra. For instance, Cynthie claims to have an easy method for squaring any two-digit number whose last digit is 5; for example, 75. Here are her steps:

i) Remove the last digit from the number you wish to square.

ii) Add the resulting number from part (i) to its square.

iii) Write the digits 25 at the end of the number you obtained in part (ii). The number that results will be the answer you want.

a) Choose a two-digit number whose last digit is 5, and determine whether Cynthie's method for squaring works for that number.

b) Show algebraically that Cynthie's method always works.

Cynthie's steps for 75^2

75

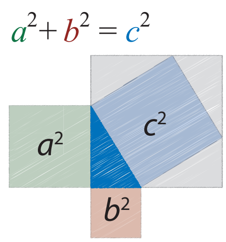
7

$7 + 7^2 = 56$

$5625 = 75^2$

Curious *Math*

Pythagorean Triples



A Pythagorean triple is three natural numbers that satisfy the equation of the Pythagorean theorem, that is, $a^2 + b^2 = c^2$.

An example of a Pythagorean triple is 3, 4, 5, since $3^2 + 4^2 = 5^2$. The numbers 6, 8, 10 also work, since each number is just twice the corresponding number in the example 3, 4, 5.

1. Show that 5, 12, 13 and 8, 15, 17 are Pythagorean triples.
2. Show that, for any value of p, $p^2 + \left(\frac{1}{2}p^2 - \frac{1}{2}\right)^2 = \left(\frac{1}{2}p^2 + \frac{1}{2}\right)^2$.
3. Use the relationship in question 2 to produce three new Pythagorean triples.
4. Show that, for any values of p and q, $(2pq)^2 + (p^2 - q^2)^2 = (p^2 + q^2)^2$. This relationship was known to the Babylonians about 4000 years ago!
5. Use the relationship in question 4 to identify two more new Pythagorean triples.

2.3 Factoring Polynomials

GOAL

Review and extend factoring skills.

LEARN ABOUT the Math

Mai claims that, for any natural number n, the function $f(n) = n^3 + 3n^2 + 2n + 6$ always generates values that are not **prime**.

? Is Mai's claim true?

EXAMPLE 1 Selecting a factoring strategy: Testing values of n to determine a pattern

Show that $f(n) = n^3 + 3n^2 + 2n + 6$ can be factored for any natural number, n.

Sally's Solution

$f(1) = 12 = 4 \times 3$

$f(2) = 30 = 5 \times 6$

$f(3) = 66 = 6 \times 11$

$f(4) = 126 = 7 \times 18$

$f(5) = 216 = 8 \times 27$

After some calculations and guess and check, I found a pattern. The first factor was of the form $n + 3$ and the second factor was of the form $n^2 + 2$.

$$f(n) = (n + 3)(n^2 + 2)$$
$$= n^3 + 3n^2 + 2n + 6$$

To confirm the pattern, I multiplied $(n + 3)$ by $(n^2 + 2)$.

Since both factors produce numbers greater than 1, $f(n)$ can never be expressed as the product of 1 and itself. So Mai's claim is true.

Sometimes an expression that doesn't appear to be factorable directly can be factored by grouping terms of the expression and dividing out common factors.

EXAMPLE 2 Selecting a factoring strategy: Grouping

Factor $f(n) = n^3 + 3n^2 + 2n + 6$ by grouping.

Noah's Solution

$$f(n) = n^3 + 3n^2 + 2n + 6$$
$$= (n^3 + 3n^2) + (2n + 6)$$

I separated $f(n)$ into two groups: the first two terms and the last two terms.

$$= n^2(n + 3) + 2(n + 3)$$

I factored each group by dividing by its common factor.

$$= (n + 3)(n^2 + 2)$$

Then I factored by dividing each term by the common factor $n + 3$.

Both factors produce numbers greater than 1, so $f(n)$ can never be expressed as the product of 1 and itself. So Mai's claim is true.

Reflecting

A. Why is Noah's method called factoring by grouping?

B. What are the advantages and disadvantages of Sally's and Noah's methods of factoring?

APPLY the Math

EXAMPLE 3 Selecting factoring strategies: Quadratic expressions

Factor.

a) $x^2 - x - 30$

b) $18x^2 - 50$

c) $10x^2 - x - 3$

d) $9x^2 + 30x + 25$

e) $2x^2 + x + 3$

Winnie's Solution

a) $x^2 - x - 30$

$$= (x + 5)(x - 6)$$

This is a trinomial of the form $ax^2 + bx + c$, where $a = 1$. I can factor it by finding two numbers whose sum is -1 and whose product is -30. These numbers are 5 and -6.

Check: $(x + 5)(x - 6)$

$= (x + 5)x - (x + 5)6$

$= x^2 + 5x - 6x - 30$

$= x^2 - x - 30$

I checked the answer by multiplying the two factors.

b) $18x^2 - 50$

$= 2(9x^2 - 25)$

First I divided each term by the common factor, 2. This left a difference of squares.

$= 2(3x + 5)(3x - 5)$

I took the square root of $9x^2$ and 25 to get $3x$ and 5, respectively.

c) $10x^2 - x - 3$

This is a trinomial of the form $ax^2 + bx + c$, where $a \neq 1$, and it has no common factor.

$= 10x^2 + 5x - 6x - 3$

I used decomposition by finding two numbers whose sum is -1 and whose product is $(10)(-3) = -30$. These numbers are 5 and -6. I used them to "decompose" the middle term.

$= 5x(2x + 1) - 3(2x + 1)$

I factored the group consisting of the first two terms and the group consisting of the last two terms by dividing each group by its common factor.

$= (2x + 1)(5x - 3)$

I divided out the common factor of $2x + 1$ from each term.

d) $9x^2 + 30x + 25$

$= (3x + 5)^2$

I noticed that the first and last terms are perfect squares. The square roots are $3x$ and 5, respectively. The middle term is double the product of the two square roots, $2(3x)(5) = 30x$. So this trinomial is a perfect square, namely, the square of a binomial.

e) $2x^2 + x + 3$

Trinomials of this form may be factored by decomposition. I tried to come up with two integers whose sum is 1 and whose product is 6. There were no such integers, so the trinomial cannot be factored.

EXAMPLE 4 Selecting a factoring strategy: Grouping

Factor $f(x) = x^3 + x^2 + x + 1$.

Fred's Solution

$$\begin{aligned} f(x) &= x^3 + x^2 + x + 1 \\ &= (x^3 + x^2) + (x + 1) \\ &= x^2(x + 1) + (x + 1) \end{aligned}$$

I grouped pairs of terms. Then I factored the **greatest common factor** (GCF) from each pair.

$$= (x + 1)(x^2 + 1)$$

Then I factored out the greatest common factor, $(x + 1)$, to complete the factoring.

EXAMPLE 5 Selecting a factoring strategy: Grouping as a difference of squares

Factor $g(x) = x^2 - 6x + 9 - 4y^2$.

Fran's Solution

$$\begin{aligned} g(x) &= x^2 - 6x + 9 - 4y^2 \\ &= (x - 3)^2 - (2y)^2 \end{aligned}$$

I recognized that the group consisting of the first three terms was the square of the binomial $x - 3$ and the last term was the square of $2y$.

$$= (x - 3 - 2y)(x - 3 + 2y)$$

I factored the resulting expression by using a difference of squares.

In Summary

Key Ideas

- Factoring a polynomial means writing it as a product. So factoring is the opposite of expanding.

factoring

$$x^2 + 3x - 4 = (x + 4)(x - 1)$$

expanding

(*continued*)

- If a polynomial has more than three terms, you may be able to factor it by grouping. This is only possible if the grouping of terms allows you to divide out the same common factor from each group.

Need to Know

- To factor a polynomial fully means that only 1 and -1 remain as common factors in the factored expression.
- To factor polynomials fully, you can use factoring strategies that include
  - dividing by the greatest common factor (GCF)
  - recognizing a factorable trinomial of the form $ax^2 + bx + c$, where $a = 1$
  - recognizing a factorable trinomial of the form $ax^2 + bx + c$, where $a \neq 1$
  - recognizing a polynomial that can be factored as a difference of squares: $a^2 - b^2 = (a + b)(a - b)$
  - recognizing a polynomial that can be factored as a perfect square: $a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$
  - factoring by grouping

CHECK Your Understanding

1. Factor.

a) $x^2 - 6x - 27$

b) $25x^2 - 49$

c) $4x^2 + 20x + 25$

d) $6x^2 - x - 2$

2. Each expression given can be factored by grouping. Describe how you would group the terms to factor each.

a) $ac + bc - ad - bd$

b) $x^2 + 2x + 1 - y^2$

c) $x^2 - y^2 - 10y - 25$

3. Factor.

a) $x^2 - 3x - 28$

b) $36x^2 - 25$

c) $9x^2 - 42x + 49$

d) $2x^2 - 7x - 15$

PRACTISING

4. Factor.

a) $4x^3 - 6x^2 + 2x$

b) $3x^3y^2 - 9x^2y^4 + 3xy^3$

c) $4a(a + 1) - 3(a + 1)$

d) $7x^2(x + 1) - x(x + 1) + 6(x + 1)$

e) $5x(2 - x) + 4x(2x - 5) - (3x - 4)$

f) $4t(t^2 + 4t + 2) - 2t(3t^2 - 6t + 17)$

5. Factor.

a) $x^2 - 5x - 14$

b) $x^2 + 4xy - 5y^2$

c) $6m^2 - 90m + 324$

d) $2y^2 + 5y - 7$

e) $8a^2 - 2ab - 21b^2$

f) $16x^2 + 76x + 90$

6. Factor.

a) $x^2 - 9$
b) $4n^2 - 49$
c) $x^8 - 1$
d) $9(y - 1)^2 - 25$
e) $3x^2 - 27(2 - x)^2$
f) $-p^2q^2 + 81$

7. Factor.

a) $ax + ay + bx + by$
b) $2ab + 2a - 3b - 3$
c) $x^3 + x^2 - x - 1$
d) $1 - x^2 + 6x - 9$
e) $a^2 - b^2 + 25 + 10a$
f) $2m^2 + 10m + 10n - 2n^2$

8. **K** Andrij claims that the following statement is true:
$x^3 - y^3 = (x - y)(x^2 + y^2)$
Is Andrij correct? Justify your decision.

9. Factor.

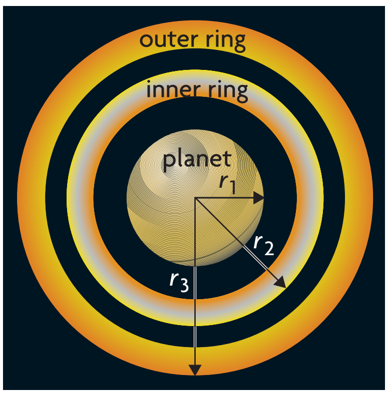
a) $2x(x - 3) + 7(3 - x)$
b) $xy + 6x + 5y + 30$
c) $x^3 - x^2 - 4x + 4$
d) $y^2 - 49 + 14x - x^2$
e) $6x^2 - 21x - 12x + 42$
f) $12m^3 - 14m^2 - 30m + 35$

10. **T** Show that the function $f(n) = 2n^3 + n^2 + 6n + 3$ always produces a number that is divisible by an odd number greater than 1, for any natural number, n.

11. Sedna has designed a fishpond in the shape of a right triangle with two sides of length a and b and hypotenuse of length c.

a) Write an expression in factored form for a^2.
b) The hypotenuse is 3 m longer than b, and the sum of the lengths of the hypotenuse and b is 11 m. What are the lengths of the sides of the pond?

12. **A** Saturn is the ringed planet most people think of, but Uranus and Neptune also have rings. In addition, there are ringed planets outside our solar system. Consider the cross-section of the ringed planet shown.



a) Write factored expressions for
  i) the area of the region between the planet and the inner ring
  ii) the area of the region between the planet and the outer ring
  iii) the difference of the areas from parts (i) and (ii)
b) What does the quantity in part (iii) represent?

13. **C** Create a flow chart that will describe which strategies you would use to try to factor a polynomial. For each path through the flow chart, give an example of a polynomial that would follow that path and show its factored form. Explain how your flow chart could describe how to factor or show the non-factorability of any polynomial in this chapter.

Extending

14. The polynomial $x^4 - 5x^2 + 4$ is not factorable, but it can be factored by a form of completing the square:

$$\begin{aligned}
&x^4 - 5x^2 + 4\\
&\quad = x^4 + 4x^2 + 4 - 4x^2 - 5x^2 \\
&\quad = (x^2 + 2)^2 - 9x^2 \\
&\quad = (x^2 + 2 - 3x)(x^2 + 2 + 3x)\\
&\quad = (x^2 - 3x + 2)(x^2 + 3x + 2)\\
&\quad = (x - 2)(x - 1)(x + 2)(x + 1)
\end{aligned}$$

The first three terms form a perfect square.

This is now a difference of squares.

Use this strategy to factor each polynomial by creating a perfect square.

a) $x^4 + 3x^2 + 36$ **b)** $x^4 - 23x^2 + 49$

15. Expanding confirms that $x^2 - 1 = (x - 1)(x + 1)$ and also that $x^3 - 1 = (x - 1)(x^2 + x + 1)$.

Make conjectures and determine similar factorings for each expression.

a) $x^4 - 1$ **c)** $x^n - 1$

b) $x^5 - 1$ **d)** $x^n - y^n$

16. Mersenne numbers are numbers of the form $n = 2^m - 1$, where m is a natural number. For example, if $m = 6$, then $2^6 - 1 = 64 - 1 = 63$, and if $m = 5$, then $2^5 - 1 = 32 - 1 = 31$. $63 = 3 \times 21$ is a Mersenne number that is composite, and $31 = 1 \times 31$ is a Mersenne number that is prime. The French mathematician Mersenne was interested in finding the values of m that produced prime numbers, n.

a) $63 = 3(21)$ can also be expressed as $(2^2 - 1)(2^4 + 2^2 + 2^0)$, and $63 = 7(9)$ can also be expressed as $(2^3 - 1)(2^3 + 2^0)$. Expand the expressions that contain powers, treating them like polynomials, to show that you get $2^6 - 1$.

b) If $m = 9$, then $n = 2^9 - 1 = 511 = 7(73) = (2^3 - 1)(2^6 + 2^3 + 2^0)$. Using these types of patterns, show that $n = 2^{35} - 1$ is composite.

c) If m is composite, will the Mersenne number $n = 2^m - 1$ always be composite? Explain.

2 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How can you determine whether two polynomials are equivalent? For example, suppose that $f(x) = (x - 2)^2$, $g(x) = (2 - x)^2$, and $h(x) = (x + 2)(x - 2)$.

A1: You can simplify both polynomials. If their simplified versions are the same, the polynomials are equivalent; otherwise, they are not.

Simplifying yields
$f(x) = x^2 - 4x + 4, \quad g(x) = 4 - 4x + x^2, \quad \text{and} \quad h(x) = x^2 - 4.$

So $f(x)$ and $g(x)$ are equivalent, but $h(x)$ is not equivalent to either of them.

A2: If the domains of two functions differ in value for any number in both domains, then the functions are not equivalent.

For example, for the functions above, $f(0) = 4$ while $h(0) = -4$. So $f(x)$ and $h(x)$ are not equivalent.

A3: You can graph both functions. If the graphs are exactly the same, then the functions are equivalent; otherwise, they are not.

Q: How do you add, subtract, and multiply polynomials?

A: When the variables of a polynomial are replaced with numbers, the result is a number. The properties for adding, subtracting, and multiplying polynomials are the same as the properties for the numbers.

For any polynomials a, b, c:

Commutative Property

$a + b = b + a$; $ab = ba$

(*Note:* $a - b \neq b - a$, except in special cases.)

Associative Property

$(a + b) + c = a + (b + c)$; $(ab)c = a(bc)$

(*Note:* $(a - b) - c \neq a - (b - c)$, except in special cases.)

Distributive Property

$a(b + c) = ab + ac$; $a(b - c) = ab - ac$

Study Aid

- See Lesson 2.1, Examples 1, 2, and 3 and Lesson 2.2, Example 3.
- Try Mid-Chapter Review Question 2.

Study Aid

- See Lesson 2.1, Example 1 Anita's Solution and Example 2.
- See Lesson 2.2, Examples 1, 2, and 3 Lee's Solution.
- Try Mid-Chapter Review Questions 1 and 3 to 6.

Because of the distributive property, the product of two polynomials can be found by multiplying each term in one polynomial by each term in the other and can be simplified by collecting like terms. For example,

$$(2x + 3y - 5z)(2x + 3y + 4z)$$
$$= 4x^2 + 6xy + 8xz + 6xy + 9y^2 + 12yz - 10xz - 15yz - 20z^2$$
$$= 4x^2 + 12xy - 2xz + 9y^2 - 3yz - 20z^2$$

Study Aid

- See Lesson 2.3, Examples 1 to 5.
- Try Mid-Chapter Review Questions 7 to 10.

Q: What strategies can you use to factor polynomials?

A: The strategies include:

Common factoring

EXAMPLE

$$9x^2 - 18x$$
$$= 9x(x - 2)$$

Decomposition

EXAMPLE

$$6x^2 + 5x - 4$$
$$= 6x^2 - 3x + 8x - 4$$
$$= 3x(2x - 1) + 4(2x - 1)$$
$$= (2x - 1)(3x + 4)$$

Factoring a difference of squares

EXAMPLE

$$9x^2 - 16$$
$$= (3x + 4)(3x - 4)$$

Factoring by grouping

EXAMPLE

$$5b + 2ab + 4a + 10$$
$$= (5b + 2ab) + (4a + 10)$$
$$= b(5 + 2a) + 2(2a + 5)$$
$$= (2a + 5)(b + 2)$$

PRACTICE Questions

Lesson 2.1

1. Simplify.

a) $(4a^2 - 3a + 2) - (-2a^2 - 3a + 9)$
b) $(2x^2 - 4xy + y^2) - (4x^2 + 7xy - 2y^2) + (3x^2 + 6y^2)$
c) $-(3d^2 - 2cd + d) + d(2c - 5d) - 3c(2c + d)$
d) $3x(2x + y) - 4x[5 - (3x + 2)]$
e) $2a(3a - 5b + 4) - 6(3 - 2a - b)$
f) $7x(2x^2 + 3y - 3) - 3x(9 - 2x + 4y)$

2. Determine whether each pair of functions is equivalent.

a) $g(t) = (t - 2)^5$ and $h(t) = (2 - t)^5$
b) $f(x) = (x^2 - 6x) - (x^2 + x - 4) + (2x^2 + 1)$ and $g(x) = (4x^2 - 7x - 3) - (2x^2 - 8)$
c) $h(x) = (x - 4)(x + 7)(x + 4)$ and $d(x) = (x + 7)(x^2 - 16)$
d) $b(t) = (3t + 1)^3$ and $c(t) = 27t^3 + 27t^2 + 9t - 1$

Lesson 2.2

3. If you multiply a linear polynomial by a quadratic one, what is the degree of the product polynomial? Justify your answer.

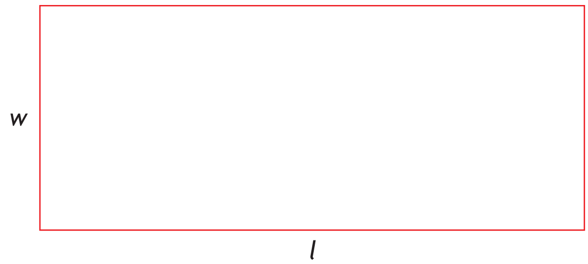
4. The sum of the ages of Pam, Dion, and their three children, in years, is $5x - 99$, where x is Dion's age. Pam is five years younger than Dion. What is the sum of the ages of their children?

5. Expand and simplify.

a) $2(x - 5)(3x - 4)$
b) $(3x - 1)^3$
c) $2(x^2 - 3x + 4)(-x^2 + 3x - 4)$
d) $(5x - 4)(3x - 5) - (2x - 3)^2$
e) $3(2x - 5) - 9(4x - 5)$
f) $-(x - y)^3$

6. If the length of the rectangle shown is increased by 2 and the width is decreased by 1, determine the change in

a) the perimeter **b)** the area



Lesson 2.3

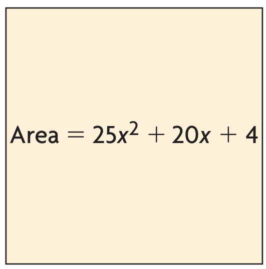
7. Factor.

a) $x(x - 2) - 3(x - 2)$
b) $x^2 - 11x + 28$
c) $3a^2 - 10a - 8$
d) $30x^2 - 9x - 3$
e) $16 - 25x^2$
f) $4(2 - a)^2 - 81$

8. Factor.

a) $2n - 6m + 5n^2 - 15mn$
b) $y^2 + 9 - 6y - x^2$
c) $y - b - (y - b)^2$
d) $2x^2 - 8y^2 + 8x + 8$
e) $w^2 + wb - aw - ab$
f) $ab + b^2 + 6a + 6b$

9. What is the perimeter of the square shown?



10. The expression $3n^2 - 11n + k$ can be factored into two linear polynomials with integer coefficients. Determine the possible values of k.

2.4 Simplifying Rational Functions

GOAL

Define rational functions, and explore methods of simplifying the related rational expression.

LEARN ABOUT the Math

Adonis has designed a game called "2 and 1" to raise money at a charity casino. To start the game, Adonis announces he will draw n numbers from a set that includes all the natural numbers from 1 to $2n$.

The players then pick three numbers.

Adonis draws n numbers and announces them. The players check for matches. Any player who has at least two matches wins.

rational function

any function that is the ratio of two polynomials. A rational function can be expressed as $f(x) = \frac{R(x)}{S(x)}$, where R and S are polynomials and $S \neq 0$; for example,

$f(x) = \frac{x^2 - 2x + 3}{4x - 1}, x \neq \frac{1}{4}$

A rational expression is a quotient of polynomials; for example,

$\frac{2x - 1}{3x}, \ x \neq 0$

The probability of a player winning is given by the rational function

$$P(n) = \frac{3n^3 - 3n^2}{8n^3 - 12n^2 + 4n}$$

For example, if Adonis draws 5 numbers from the set 1 to 10, the probability of winning is

$$P(5) = \frac{3(5)^3 - 3(5)^2}{8(5)^3 - 12(5)^2 + 4(5)}$$

$$= \frac{5}{12}$$

The game is played at a rapid pace, and Adonis needs a fast way to determine the range he should use, based on the number of players and their chances of winning.

? What is the simplified expression for the probability of a player winning at "2 and 1"?

EXAMPLE 1 Simplifying rational functions

Write the simplified expression for the function defined by

$$P(n) = \frac{3n^3 - 3n^2}{8n^3 - 12n^2 + 4n}.$$

Faez's Solution

$$P(n) = \frac{3n^2(n-1)}{4n(2n^2 - 3n + 1)}$$

I knew that I could simplify rational numbers by first factoring numerators and denominators and dividing each by the common factor $\left(\text{e.g., } \frac{24}{27} = \frac{3(8)}{3(9)} = \frac{8}{9}\right)$. So I tried the same idea here. I factored the numerator and denominator of $P(n)$.

$$= \frac{3n^2\cancel{(n-1)}^{1}}{4n(2n-1)\cancel{(n-1)}_{1}}$$

Then I divided by the common factor, $(n-1)$.

$$= \frac{3n^2}{4n(2n-1)}$$

Restrictions:

When $4n(2n-1)(n-1) = 0$,

$$4n = 0 \quad (2n-1) = 0 \quad (n-1) = 0$$
$$n = 0 \quad n = \frac{1}{2} \quad n = 1$$

Since I cannot divide by zero, I determined the **restrictions** by calculating the values of n that make the factored denominator zero. I solved $4n(2n-1)(n-1) = 0$ by setting each factor equal to 0.

$$P(n) = \frac{3n^2}{4n(2n-1)}; n \neq 0, \frac{1}{2}, 1.$$

restrictions

the values of the variable(s) in a rational function or rational expression that cause the function to be undefined. These are the zeros of the denominator or, equivalently, the numbers that are not in the domain of the function.

Reflecting

A. How is working with rational expressions like working with rational numbers? How is it different?

B. How do the restrictions on the rational expression in $P(n)$ relate to the domain of this rational function?

C. How does factoring help to simplify and determine the restrictions on the variable?

APPLY the Math

EXAMPLE 2 Selecting a strategy for simplifying the quotient of a monomial and a monomial

Simplify and state any restrictions on the variables.

$$\frac{30x^4y^3}{-6x^7y}$$

Tanya's Solution

$$\frac{30x^4y^3}{-6x^7y} = \frac{\cancel{-6x^4y}^{\,1}(-5y^2)}{\cancel{-6x^4y}_{\,1}(x^3)}$$

I factored the numerator and denominator by dividing out the GCF $-6x^4y$. Then I divided the numerator and denominator by the GCF.

$$= \frac{-5y^2}{x^3};\ x, y \neq 0$$

I determined the restrictions by finding the zeros of the *original* denominator by solving $-6x^7y = 0$. This gives the restrictions x, $y \neq 0$.

EXAMPLE 3 Selecting a strategy for simplifying the quotient of a polynomial and a monomial

Simplify and state any restrictions on the variables.

$$\frac{10x^4 - 8x^2 + 4x}{2x^2}$$

Lee's Solution

$$\frac{10x^4 - 8x^2 + 4x}{2x^2}$$

$$= \frac{\cancel{2x}^{\,1}(5x^3 - 4x + 2)}{\cancel{2x}_{\,1}(x)}$$

I factored the numerator and denominator by dividing out the GCF $2x$. Then I divided both the numerator and denominator by $2x$.

$$= \frac{5x^3 - 4x + 2}{x};\ x \neq 0$$

I determined the restrictions by solving $2x^2 = 0$ to get the zeros of the original denominator. The only restriction is $x \neq 0$.

EXAMPLE 4 Selecting a strategy for simplifying a function involving the quotient of a trinomial and a binomial

Simplify $f(x)$ and state the domain, where $f(x) = \dfrac{x^2 + 7x - 8}{2 - 2x}$.

Michel's Solution

$$f(x) = \frac{x^2 + 7x - 8}{2 - 2x}$$

$$= \frac{(x - 1)(x + 8)}{2(1 - x)}$$

I factored the numerator and denominator and noticed that there were two factors that were similar, but with opposite signs.

$$= \frac{-\overset{1}{\cancel{(1 - x)}}(x + 8)}{2\underset{1}{\cancel{(1 - x)}}}$$

I divided out the common factor, -1, from $(x - 1)$ in the numerator, so that it became identical to $(1 - x)$ in the denominator. I divided the numerator and denominator by the GCF $1 - x$.

$$= \frac{-(x + 8)}{2}; x \neq 1$$

The domain is $\{x \in \mathbf{R} \mid x \neq 1\}$.

I determined the restrictions by solving $2(1 - x) = 0$. The only restriction is $x \neq 1$. This means that $f(x)$ is undefined when $x = 1$, so $x = 1$ must be excluded from the domain.

EXAMPLE 5 Selecting a strategy for simplifying the quotient of quadratics in two variables

Simplify and state any restrictions on the variables: $\dfrac{4x^2 - 16y^2}{x^2 + xy - 6y^2}$.

Hermione's Solution

$$\frac{4x^2 - 16y^2}{x^2 + xy - 6y^2}$$

$$= \frac{4\overset{1}{\cancel{(x - 2y)}}(x + 2y)}{(x + 3y)\underset{1}{\cancel{(x - 2y)}}}$$

I factored the numerator and denominator and then divided by the GCF $x - 2y$.

$$= \frac{4(x + 2y)}{x + 3y}; x \neq -3y, 2y$$

I determined the restrictions by finding the zeros of the factored denominator by solving $(x + 3y)(x - 2y) = 0$.

So, $(x + 3y) = 0$ and $(x - 2y) = 0$.

The restrictions are $x \neq -3y, 2y$.

In Summary

Key Ideas

- A rational function can be expressed as the ratio of two polynomial functions. For example,

$$f(x) = \frac{6x + 2}{x - 1}; x \neq 1$$

A rational expression is the ratio of two polynomials.
For example,

$$\frac{6x + 2}{x - 1}; x \neq 1$$

- Both rational functions and rational expressions are undefined for numbers that make the denominator zero. These numbers must be excluded or restricted from being possible values for the variables. As a result, for all rational functions, the domain is the set of all real numbers, except those numbers that make the denominator equal zero.

Need to Know

- Rational functions and rational expressions can be simplified by factoring the numerator and denominator and then dividing both by their greatest common factor.
- The restrictions are found by determining all the zeros of the denominator. If the denominator contains two or more terms, the zeros can be determined from its factored form before the function or expression is simplified.

CHECK Your Understanding

1. Simplify. State any restrictions on the variables.

a) $\frac{6 - 4t}{2}$ **b)** $\frac{9x^2}{6x^3}$ **c)** $\frac{7a^2b^3}{21a^4b}$

2. Simplify. State any restrictions on the variables.

a) $\frac{5(x + 3)}{(x + 3)(x - 3)}$ **b)** $\frac{6x - 9}{2x - 3}$ **c)** $\frac{4a^2b - 2ab^2}{(2a - b)^2}$

3. Simplify. State any restrictions on the variables.

a) $\frac{(x - 1)(x - 3)}{(x + 2)(x - 1)}$ **b)** $\frac{5x^2 + x - 4}{25x^2 - 40x + 16}$ **c)** $\frac{x^2 - 7xy + 10y^2}{x^2 + xy - 6y^2}$

PRACTISING

4. Simplify. State any restrictions on the variables.

a) $\dfrac{14x^3 - 7x^2 + 21x}{7x}$

b) $\dfrac{-5x^3y^2}{10xy^3}$

c) $\dfrac{2t(5 - t)}{5t^2(t - 5)}$

d) $\dfrac{5ab}{15a^4b - 10a^2b^2}$

e) $\dfrac{2x^2 + 10x}{-3x - 15}$

f) $\dfrac{2ab - 6a}{9a - 3ab}$

5. Simplify. State any restrictions on the variables.

a) $\dfrac{a + 4}{a^2 + 3a - 4}$

b) $\dfrac{x^2 - 9}{15 - 5x}$

c) $\dfrac{x^2 - 5x + 6}{x^2 + 3x - 10}$

d) $\dfrac{10 + 3p - p^2}{25 - p^2}$

e) $\dfrac{t^2 - 7t + 12}{t^3 - 6t^2 + 9t}$

f) $\dfrac{6t^2 - t - 2}{2t^2 - t - 1}$

6. State the domain of each function. Explain how you found each answer.

a) $f(x) = \dfrac{2 + x}{x}$

b) $g(x) = \dfrac{3}{x(x - 2)}$

c) $h(x) = \dfrac{-3}{(x + 5)(x - 5)}$

d) $f(x) = \dfrac{1}{x^2 - 1}$

e) $g(x) = \dfrac{1}{x^2 + 1}$

f) $h(x) = \dfrac{x - 1}{x^2 - 1}$

7. Determine which pairs of functions are equivalent. Explain your reasoning.

a) $f(x) = 2x^2 + x - 7$ and $g(x) = \dfrac{6x^2 + 3x - 21}{3}$

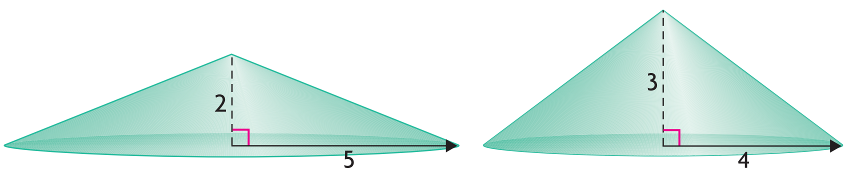
b) $h(x) = 3x^2 + 5x + 1$ and $j(x) = \dfrac{3x^3 + 5x^2 + x}{x}, x \neq 0$

8. A An isosceles triangle has two sides of length $9x + 3$. The perimeter of the triangle is $30x + 10$.

a) Determine the ratio of the base to the perimeter, in simplified form. State the restriction on x.

b) Explain why the restriction on x in part (a) is necessary in this situation.

9. Two cones have radii in the ratio 5:4 and heights in the ratio 2:3. Determine the ratio of their volumes, where $V = \frac{1}{3}\pi r^2 h$.



10. K Simplify. State any restrictions on the variables.

a) $\dfrac{20t^3 + 15t^2 - 5t}{5t}$

b) $\dfrac{5(4x - 2)}{8(2x - 1)^2}$

c) $\dfrac{x^2 - 9x + 20}{16 - x^2}$

d) $\dfrac{2x^2 - xy - y^2}{x^2 - 2xy + y^2}$

11. A rectangle is six times as long as it is wide. Determine the ratio of its area to its perimeter, in simplest form, if its width is w.

12. The quotient of two polynomials is $3x - 2$. Give two examples of a rational expression equivalent to this polynomial that has the restriction $x \neq 4$.

13. Give an example of a rational function that could have three restrictions that are consecutive numbers.

14. **T** Consider the rational expression $\frac{2x + 1}{x - 4}$.

a) Identify, if possible, a rational expression with integer coefficients that simplifies to $\frac{2x + 1}{x - 4}$, for each set of restrictions.

i) $x \neq -1, 4$ **ii)** $x \neq 0, 4$ **iii)** $x \neq \frac{2}{3}, 4$ **iv)** $x \neq -\frac{1}{2}, 4$

b) Is there a rational expression with denominator of the form $ax^2 + bx + c, a \neq 0$, that simplifies to $\frac{2x + 1}{x - 4}$, and has only the restriction $x \neq 4$? Explain.

15. **C** Can two different rational expressions simplify to the same polynomial? Explain using examples.

Extending

Communication *Tip*

"As x approaches positive infinity" is commonly written as $x \to \infty$, and "as x approaches negative infinity" is commonly written as $x \to -\infty$.

The limiting end behaviour of the function $f(x) = \frac{1}{x}$, which approaches zero as x approaches infinity, is written as

$$\lim_{x \to \infty} f(x) = 0$$

16. Mathematicians are often interested in the "end behaviour" of functions, that is, the value of the output as the input, x, gets greater and greater and approaches infinity and as the input, x, gets lesser and lesser and approaches negative infinity. For example, the output of $f(x) = \frac{1}{x}$ as x approaches infinity gets closer and closer to 0. By calculating values of the function, make a conjecture about both end behaviours of each rational function.

a) $f(x) = \dfrac{50x + 73}{x^2 - 10x - 400}$

b) $g(x) = \dfrac{4x^3 - 100}{5x^3 + 87x + 28}$

c) $h(x) = \dfrac{-7x^2 + 3x}{200x + 9999}$

17. Simplify. State any restrictions on the variables.

a) $a(t) = \dfrac{-2(1 + t^2)^2 + 2t(2)(1 + t^2)(2t)}{(1 + t^2)^4}$

b) $f(x) = \dfrac{2(2x + 1)(2)(3x - 2)^3 - (2x + 1)^2(3)(3x - 2)^2}{(3x - 2)^6}$

2.5 Exploring Graphs of Rational Functions

GOAL

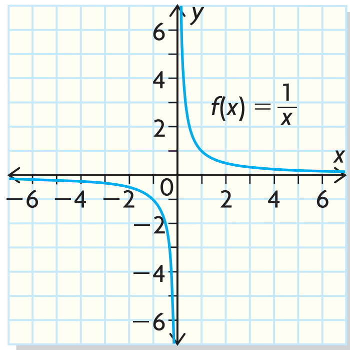
Explore some features of rational functions.

YOU WILL NEED

- graphing calculator

EXPLORE the Math

The graph of the rational function $f(x) = \frac{1}{x}$ is shown at the right. Its domain is $\{x \in \mathbf{R} \mid x \neq 0\}$, and it has a vertical **asymptote** at $x = 0$ and a horizontal asymptote at $y = 0$.



? What are some features of the graphs of rational functions, at or near numbers that are not in their domain?

A. Some rational functions simplify to polynomials. For example, $f(x) = \frac{x^2 - 4}{x - 2}$ can be simplified by factoring from $f(x) = \frac{(x + 2)(x - 2)}{x - 2}$ to $f(x) = x + 2$, where $x \neq 2$. Graph $f(x)$ prior to simplifying it, and zoom in and trace near $x = 2$. Describe what happens to the graph at $x = 2$.

B. Determine another rational function that simplifies to a polynomial with domain $\{x \in \mathbf{R} \mid x \neq 1\}$. Describe what happens to the graph at $x = 1$.

C. Some rational functions cannot be simplified; for example, $g(x) = \frac{1}{x - 3}$. Graph $g(x)$ and zoom in near $x = 3$. Describe what happens to the graph near $x = 3$.

D. Determine another rational function with domain $\{x \in \mathbf{R} \mid x \neq 2\}$ that can't be simplified. Graph your function and describe what happens to the graph at $x = 2$.

E. Determine the equation of a simplified rational function that has two vertical asymptotes: $x = -1$ and $x = 2$. Graph your function.

F. Determine the equation of a rational function that has both a vertical asymptote and a "hole." Graph your function.

G. The rational function $h(x) = \frac{1}{x}$ has a horizontal asymptote $y = 0$. Apply a transformation to $h(x)$ that will result in a rational function that has the horizontal asymptote $y = 2$. Determine the equation of this function and graph it.

H. Determine the equation of a rational function without any "holes," vertical asymptotes, or horizontal asymptotes. Graph your function.

I. Review what you have discovered and summarize your findings.

Reflecting

J. What determines where a rational function has a hole? A vertical asymptote?

K. When does a rational function have the horizontal asymptote $y = 0$? When does a rational function have another horizontal line as a horizontal asymptote?

L. Some rational functions have asymptotes, others have holes, and some have both. Explain how you can identify, without graphing, which graphical features a rational function will have.

In Summary

Key Idea

- The restricted values of rational functions correspond to two different kinds of graphical features: holes and vertical asymptotes.

Need to Know

- Holes occur at restricted values that result from a factor of the denominator that is also a factor of the numerator. For example,

$$g(x) = \frac{x^2 + 7x + 12}{x + 3}$$

has a hole at $x = -3$, since $g(x)$ can be simplified to the polynomial

$$g(x) = \frac{(x + 3)(x + 4)}{(x + 3)} = x + 4$$

- Vertical asymptotes occur at restricted values that are still zeros of the denominator after simplification. For example,

$$h(x) = \frac{5}{x - 8}$$

has a vertical asymptote at $x = 8$.

FURTHER Your Understanding

1. Identify a rational function whose graph is a horizontal line except for two holes. Graph the function.

2. Identify a rational function whose graph lies entirely above the x-axis and has a single vertical asymptote. Graph the function.

3. Identify a rational function whose graph has the horizontal asymptote $y = 2$ and two vertical asymptotes. Graph the function.

2.6 Multiplying and Dividing Rational Expressions

GOAL

Develop strategies for multiplying and dividing rational expressions.

LEARN ABOUT the Math

Tulia is telling Daisy about something that her chemistry teacher was demonstrating. It is about the variables X, Y, and Z.

Tulia: I didn't catch how X, Y, and Z are related.
Daisy: Tell me what the units of the three quantities are.
Tulia: They were $\frac{\text{mol}}{\text{L}\cdot\text{s}}$, $\frac{\text{L}}{\text{mol}}$, and s^{-1}, respectively.
Daisy: I have no idea what any of those mean.
Tulia: I guess I'll have to look for help elsewhere.
Daisy: Hold on. All of the units are like rational expressions, so maybe there is some operation that relates them.
Tulia: Like what?

? How are the three quantities related?

EXAMPLE 1 Selecting a strategy for multiplying rational expressions

Use multiplication to show how the expressions $\frac{\text{mol}}{\text{L}\cdot\text{s}}$, $\frac{\text{L}}{\text{mol}}$, and s^{-1} are related.

Luke's Solution

$$\frac{\overset{1}{\cancel{\text{mol}}}}{\underset{1}{\cancel{\text{L}}}\cdot\text{s}} \times \frac{\overset{1}{\cancel{\text{L}}}}{\underset{1}{\cancel{\text{mol}}}} = \frac{1}{\text{s}}$$

Daisy suggested that the quantities are related by multiplication. I wrote the quantities as a product and then simplified the expression.

$$\text{s}^{-1} = \frac{1}{\text{s}}$$

So, $\frac{\text{mol}}{\text{L}\cdot\text{s}} \times \frac{\text{L}}{\text{mol}} = \text{s}^{-1}$

I wrote the variable with a negative exponent as a rational expression with a positive exponent. This showed that the quantities are related by multiplication.

Reflecting

A. Was Daisy correct in saying that the units of *X*, *Y*, and *Z* were rational expressions?

B. Explain why Daisy's method for multiplying the rational expressions was correct.

APPLY the Math

EXAMPLE 2 Selecting a strategy for multiplying simple rational expressions

Simplify and state the restrictions.

$$\frac{6x^2}{5xy} \times \frac{15xy^3}{8xy^4}$$

Buzz's Solution

$$\frac{6x^2}{5xy} \times \frac{15xy^3}{8xy^4}$$

When I substituted values for the variables, the result was a fraction, so I multiplied the rational expressions the same way as when I multiply fractions. For $x = 1$, $y = 1$ the expression becomes

$$\frac{6(1)^2}{5(1)(1)} \times \frac{15(1)(1)^3}{8(1)(1)^4} = \frac{6}{5} \times \frac{15}{8} = \frac{9}{4}$$

$$= \frac{90x^3y^3}{40x^2y^5}$$

I multiplied the numerators and then the denominators. I did this by multiplying the coefficients and adding the exponents when the base was the same.

$$= \frac{\overset{1}{\cancel{10x^2y^3}}(9x)}{\underset{1}{\cancel{10x^2y^3}}(4y^2)}$$

I factored the numerator and denominator by dividing out the GCF $10x^2y^3$. Then I divided both the numerator and denominator by the GCF.

$$= \frac{9x}{4y^2};\ x \neq 0,\ y \neq 0$$

I determined the restrictions by setting the original denominator to zero: $40x^2y^5 = 0$. So, neither x nor y can be zero.

EXAMPLE 3 Selecting a strategy for multiplying more complex rational expressions

Simplify and state the restrictions.

$$\frac{x^2 - 4}{(x + 6)^2} \times \frac{x^2 + 9x + 18}{2(2 - x)}$$

Willy's Solution

$$\frac{x^2 - 4}{(x + 6)^2} \times \frac{x^2 + 9x + 18}{2(2 - x)}$$

$$= \frac{(x - 2)(x + 2)}{(x + 6)^2} \times \frac{(x + 3)(x + 6)}{2(2 - x)}$$

I factored the numerators and denominators.

$$= \frac{(x - 2)(x + 2)}{(x + 6)^2} \times \frac{(x + 3)(x + 6)}{-2(-2 + x)}$$

I noticed that $(x - 2)$ in the numerator was the opposite of $(2 - x)$ in the denominator. I divided out the common factor -1 from $(-2 + x)$ to get the signs the same in these factors.

$$= \frac{\overset{1}{\cancel{(x - 2)}}(x + 2)(x + 3)\cancel{(x + 6)}}{-2(x + 6)^{\cancel{2}\,1}\underset{1}{\cancel{(-2 + x)}}}$$

I multiplied the numerators and denominators and then divided out the common factors.

$$= \frac{-(x + 2)(x + 3)}{2(x + 6)};\ x \neq -6, 2$$

I determined the restrictions on the denominators by solving the equations $(x + 6)^2 = 0$ and $(-2 + x) = 0$.

EXAMPLE 4 Selecting a strategy for dividing rational expressions

Simplify and state the restrictions.

$$\frac{21p - 3p^2}{16p + 4p^2} \div \frac{14 - 9p + p^2}{12 + 7p + p^2}$$

Aurora's Solution

$$\frac{21p - 3p^2}{16p + 4p^2} \div \frac{14 - 9p + p^2}{12 + 7p + p^2}$$

$$= \frac{21p - 3p^2}{16p + 4p^2} \times \frac{12 + 7p + p^2}{14 - 9p + p^2}$$

When I substituted values for the variables, the result was a fraction, so I divided by multiplying the first rational expression by the reciprocal of the second, just as I would for fractions.

$$= \frac{3p(7 - p)}{4p(4 + p)} \times \frac{(3 + p)(4 + p)}{(7 - p)(2 - p)}$$

I factored the numerators and the denominators.

$$= \frac{3\cancel{p}\overset{1}{\cancel{(7 - p)}}}{4\cancel{p}\underset{1}{\cancel{(4 + p)}}} \times \frac{(3 + p)\overset{1}{\cancel{(4 + p)}}}{\underset{1}{\cancel{(7 - p)}}(2 - p)}$$

I simplified by dividing the numerators and denominators by all of their common factors.

$$= \frac{3(3 + p)}{4(2 - p)}; p \neq 0, -4, 7, 2, -3$$

I used the factored form of each denominator to determine the zeros by solving for p in $4p = 0$, $(4 + p) = 0$, $(7 - p) = 0$, $(2 - p) = 0$, and $(3 + p) = 0$.

In Summary

Key Idea

- The procedures you use to multiply or divide rational numbers can be used to multiply and divide rational expressions. That is, if A, B, C, and D are polynomials, then

$$\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}, \text{ provided that } B, D \neq 0$$

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C} = \frac{AD}{BC}, \text{ provided that } B, D, \text{ and } C \neq 0$$

(*continued*)

Need to Know

- To multiply rational expressions,
  - factor the numerators and denominators, if possible
  - divide out any factors that are common to the numerator and denominator
  - multiply the numerators, multiply the denominators, and then write the result as a single rational expression
- To divide two rational expressions,
  - factor the numerators and denominators, if possible
  - multiply by the reciprocal of the divisor
  - divide out any factors common to the numerator and denominator
  - multiply the numerators and then multiply the denominators
  - write the result as a single rational expression
- To determine the restrictions, solve for the zeros of all of the denominators in the factored expression. In the case of division, both the numerator and denominator of the divisor must be used. Both are needed because the reciprocal of this expression is used in the calculation.

CHECK Your Understanding

1. Simplify. State any restrictions on the variables.

a) $\dfrac{2}{3} \times \dfrac{5}{8}$

b) $\dfrac{6x^2y}{5y^3} \times \dfrac{xy}{8}$

c) $\dfrac{(x+1)(x-5)}{(x+4)} \times \dfrac{(x+4)}{2(x-5)}$

d) $\dfrac{x^2}{2x+1} \times \dfrac{6x+3}{5x}$

2. Simplify. State any restrictions on the variables.

a) $\dfrac{2x}{3} \div \dfrac{x^2}{5}$

b) $\dfrac{x-7}{10} \div \dfrac{2x-14}{25}$

c) $\dfrac{3x(x-6)}{(x+2)(x-7)} \div \dfrac{(x-6)}{(x+2)}$

d) $\dfrac{x^2-1}{x-2} \div \dfrac{x+1}{12-6x}$

3. Simplify. State any restrictions on the variables.

a) $\dfrac{(x+1)^2}{x^2+2x-3} \times \dfrac{(x-1)^2}{x^2+4x+3}$

b) $\dfrac{2x+10}{x^2-4x+4} \div \dfrac{x^2-25}{x-2}$

PRACTISING

4. Simplify. State any restrictions on the variables.

a) $\frac{2x^2}{7} \times \frac{21}{x}$

b) $\frac{7a}{3} \div \frac{14a^2}{5}$

c) $\frac{2x^3y}{3xy^2} \times \frac{9x}{4x^2y}$

d) $\frac{3a^2b^3}{2ab^2} \div \frac{9a^2b}{14a^2}$

5. Simplify. State any restrictions on the variables.

a) $\frac{2(x + 1)}{3} \times \frac{x - 1}{6(x + 1)}$

b) $\frac{3a - 6}{a + 2} \div \frac{a - 2}{a + 2}$

c) $\frac{2(x - 2)}{9x^3} \times \frac{12x^4}{2 - x}$

d) $\frac{3(m + 4)^2}{2m + 1} \div \frac{5(m + 4)}{7m + 14}$

6. Simplify. State any restrictions on the variables.

a) $\frac{(x + 1)(x - 3)}{(x + 2)^2} \times \frac{2(x + 2)}{(x - 3)(x + 3)}$

b) $\frac{2(n^2 - 7n + 12)}{n^2 - n - 6} \div \frac{5(n - 4)}{n^2 - 4}$

c) $\frac{2x^2 - x - 1}{x^2 - x - 6} \times \frac{6x^2 - 5x + 1}{8x^2 + 14x + 5}$

d) $\frac{9y^2 - 4}{4y - 12} \div \frac{9y^2 + 12y + 4}{18 - 6y}$

7. Simplify. State any restrictions on the variables.

a) $\frac{x^2 - 5xy + 4y^2}{x^2 + 3xy - 28y^2} \times \frac{x^2 + 2xy + y^2}{x^2 - y^2}$

b) $\frac{2a^2 - 12ab + 18b^2}{a^2 - 7ab + 10b^2} \div \frac{4a^2 - 12ab}{a^2 - 7ab + 10b^2}$

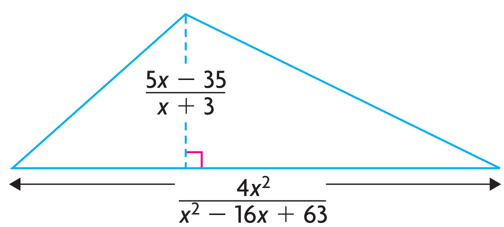
c) $\frac{10x^2 + 3xy - y^2}{9x^2 - y^2} \div \frac{6x^2 + 3xy}{12x + 4y}$

d) $\frac{15m^2 + mn - 2n^2}{2n - 14m} \times \frac{7m^2 - 8mn + n^2}{5m^2 + 7mn + 2n^2}$

8. **K** Simplify. State any restrictions on the variables.

$$\frac{x^2 + x - 6}{(2x - 1)^2} \times \frac{x(2x - 1)^2}{x^2 + 2x - 3} \div \frac{x^2 - 4}{3x}$$

9. **A** Determine the area of the triangle in simplified form. State the restrictions.



10. An object has mass $m = \dfrac{p + 1}{3p + 1}$ and density $\rho = \dfrac{p^2 - 1}{9p^2 + 6p + 1}$.

Determine its volume v, where $\rho = \dfrac{m}{v}$. State the restrictions on any variables.

11. Liz claims that if $x = y$, she can show that $x + y = 0$ by following these steps:

T

Since $x = y$,

$x^2 = y^2$ — I squared both sides of the equation.

So $x^2 - y^2 = 0$. — I rearranged terms in the equation.

$\dfrac{x^2 - y^2}{x - y} = \dfrac{0}{x - y}$ — I divided both sides by $x - y$.

$\dfrac{\overset{1}{\cancel{(x - y)}}(x + y)}{\underset{1}{\cancel{x - y}}} = \dfrac{0}{x - y}$ — I factored and simplified.

$x + y = 0$

Sarit says that's impossible because if $x = 1$, then $y = 1$, since $x = y$. Substituting into Liz's final equation, $x + y = 0$, gives $1 + 1 = 2$, not 0.

Explain the error in Liz's reasoning.

12. **a)** Why do you usually factor all numerators and denominators *before* multiplying rational functions?

C

b) Are there any exceptions to the rule in part (a)? Explain.

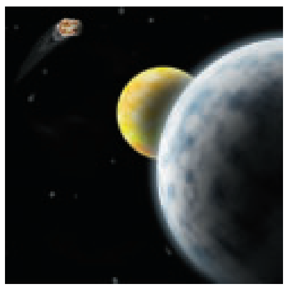
c) Sam says that dividing two rational functions and multiplying the first function by the reciprocal of the second will produce the same function. Is this true? Explain.

Extending

13. Simplify. State any restrictions on the variables.

$$\frac{\dfrac{m^2 - mn}{6m^2 + 11mn + 3n^2} \div \dfrac{m^2 - n^2}{2m^2 - mn - 6n^2}}{\dfrac{4m^2 - 7mn - 2n^2}{3m^2 + 7mn + 2n^2}}$$

14. Newton's law of gravitation states that any two objects exert a gravitational force on each other due to their masses, $F_g = G\dfrac{m_1 m_2}{r^2}$, where F is the gravitational force, G is a constant (the universal gravitational constant), m_1 and m_2 are the masses of the objects, and r is the separation distance between the centres of objects. The mass of Mercury is 2.2 times greater than the mass of Pluto. Pluto is 102.1 times as far from the Sun as Mercury. How many times greater is the gravitational force between the Sun and Mercury than the gravitational force between the Sun and Pluto?

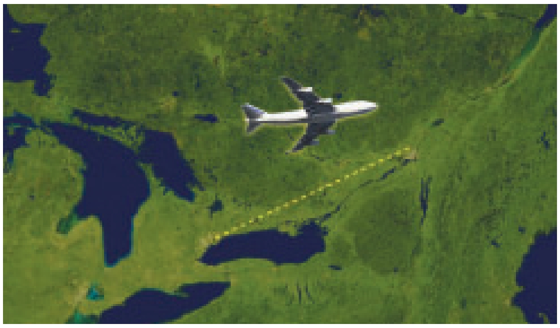


2.7 Adding and Subtracting Rational Expressions

GOAL

Develop strategies for adding and subtracting rational expressions.

LEARN ABOUT the Math



A jet flies along a straight path from Toronto to Montreal and back again. The straight-line distance between these cities is 540 km. On Monday, the jet made the round trip when there was no wind. On Friday, it made the round trip when there was a constant wind blowing from Toronto to Montreal at 80 km/h. While travelling in still air, the jet travels at constant speed.

? Which round trip takes less time?

EXAMPLE 1 Selecting a strategy for adding and subtracting rational expressions

Write expressions for the length of time required to fly from Toronto to Montreal in each situation. Determine which trip takes less time.

Basil's Solution

v is the jet's airspeed in still air.

$v + 80$ is the jet's airspeed from Toronto to Montreal.

$v - 80$ is the jet's airspeed from Montreal to Toronto.

I assigned a variable, v, to the jet's airspeed in still air, since its value is not given. So the speed with the wind from Toronto and the speed against the wind from Montreal are $v + 80$ and $v - 80$, respectively.

$\frac{540}{v}$ is the time elapsed when there is no wind.

$\frac{540}{v + 80}$ is the time elapsed from Toronto to Montreal.

$\frac{540}{v - 80}$ is the time elapsed from Montreal to Toronto.

Using the relation time $= \frac{\text{distance}}{\text{speed}}$, I determined expressions for the elapsed time for each way of the trip at each airspeed.

No wind

$$T_1 = \frac{540}{v} + \frac{540}{v}$$
$$= \frac{1080}{v}$$

I let T_1 represent the time on Monday, with no wind.

I let T_2 represent the time on Friday, with wind.

I found the round-trip times by adding the times for each way.

Wind

$$T_2 = \frac{540}{v+80} + \frac{540}{v-80}$$
$$= \frac{540(v-80) + 540(v+80)}{(v+80)(v-80)}$$
$$= \frac{1080v}{v^2 - 6400}$$

$$T_1 = \frac{1080}{v} \times \frac{v}{v}$$

I noticed that T_1 has the denominator v while T_2's denominator contains v^2. To compare T_1 with T_2, I need to have the same denominator, so I rewrote T_1 by multiplying its numerator and denominator by v.

Now the numerators are both the same.

$$= \frac{1080v}{v^2}$$

T_2 has a smaller denominator because 6400 is subtracted from v^2. Since I am dealing with division, the lesser of the two expressions is the one with the greater denominator, in this case T_1.

The trip without wind took less time.

Reflecting

A. Why were the expressions for time rational expressions?

B. How can you determine a common denominator of two rational functions?

C. How do the methods for adding and subtracting rational expressions compare with those for adding and subtracting rational numbers?

APPLY the Math

EXAMPLE 2 Using the lowest common denominator strategy to add rational expressions

Simplify and state any restrictions on the variables: $\frac{3}{8x^2} + \frac{1}{4x} - \frac{5}{6x^3}$.

Sheila's Solution

$$\text{LCD} = 24x^3$$

I found the **lowest common denominator** (LCD) by finding the least common multiple of $8x^2$, $4x$, and $6x^3$.

$$\frac{3}{8x^2} + \frac{1}{4x} - \frac{5}{6x^3}$$
$$= \frac{(3x)3}{(3x)8x^3} + \frac{(6x^2)1}{(6x^2)4x} - \frac{(4)5}{(4)6x^3}$$

I used the LCD to rewrite each term. For each term, I multiplied the denominator by the factor necessary to get the LCD. Then, I multiplied the numerator by the same factor.

$$= \frac{9x + 6x^2 - 20}{24x^3};\ x \neq 0$$

I added and subtracted the numerators.

I determined the restrictions on the denominator by solving $24x^3 = 0$.

EXAMPLE 3 Using a factoring strategy to add expressions with binomial denominators

Simplify and state any restrictions on the variables: $\frac{3n}{2n + 1} + \frac{4}{n - 3}$.

Tom's Solution

$$\text{LCD} = (2n + 1)(n - 3)$$

I found the lowest common denominator by multiplying both denominators.

$$\frac{3n}{2n + 1} + \frac{4}{n - 3}$$
$$= \frac{(n - 3)3n}{(2n + 1)(n - 3)} + \frac{(2n + 1)4}{(2n + 1)(n - 3)}$$

I used the lowest common denominator to rewrite each term.

$$= \frac{(n-3)3n + (2n+1)4}{(2n+1)(n-3)}$$

$$= \frac{3n^2 - 9n + 8n + 4}{(2n+1)(n-3)}$$

I simplified by expanding the numerators.

$$= \frac{3n^2 - n + 4}{(2n+1)(n-3)}; \ x \neq -\frac{1}{2}, 3$$

I collected like terms and determined the restrictions by solving $(2n+1)(n-3) = 0$.

EXAMPLE 4 Using a factoring strategy to add expressions with quadratic denominators

Simplify and state any restrictions on the variables: $\frac{2t}{t^2 - 1} - \frac{t+2}{t^2 + 3t - 4}$.

Frank's Solution

$$\frac{2t}{t^2 - 1} - \frac{t+2}{t^2 + 3t - 4}$$

$$= \frac{2t}{(t-1)(t+1)} - \frac{t+2}{(t+4)(t-1)}$$

I factored the denominators.

To find the LCD, I created a product by using the three unique factors:

$(t-1)(t+1)(t+4)$

$$= \frac{(t+4)2t}{(t-1)(t+1)(t+4)} - \frac{(t+1)(t+2)}{(t+1)(t-1)(t+4)}$$

I used the lowest common denominator to rewrite each term.

$$= \frac{2t^2 + 8t - t^2 - 2t - t - 2}{(t-1)(t+1)(t+4)}$$

I simplified by expanding the numerators.

$$= \frac{t^2 + 5t - 2}{(t-1)(t+1)(t+4)}; \ t \neq 1, -1, -4$$

I collected like terms and determined the restrictions by solving $(t-1)(t+1)(t+4) = 0$.

In Summary

Key Idea

- The procedures for adding or subtracting rational functions are the same as those for adding and subtracting rational numbers. When rational expressions are added or subtracted, they must have a common denominator.

Need to Know

- To add or subtract rational functions or expressions, determine the lowest common denominator (LCD). To do this, factor all the denominators. The LCD consists of the product of any common factors and all the unique factors.
- The LCD is not always the product of all the denominators.
- After finding the LCD, rewrite each term using the LCD as the denominator and then add or subtract numerators.
- Restrictions are found by finding the zeros of all denominators, that is, the zeros of the LCD.

CHECK Your Understanding

1. Simplify. State any restrictions on the variables.

a) $\frac{1}{3} + \frac{5}{4}$

b) $\frac{2x}{5} + \frac{6x}{2}$

c) $\frac{5}{4x^2} + \frac{1}{7x^3}$

d) $\frac{2}{x} + \frac{6}{x^2}$

2. Simplify. State any restrictions on the variables.

a) $\frac{5}{9} - \frac{2}{3}$

b) $\frac{5y}{3} - \frac{y}{2}$

c) $\frac{5}{3x^2} - \frac{7}{5}$

d) $\frac{6}{3xy} - \frac{5}{y^2}$

3. Simplify. State any restrictions on the variables.

a) $\frac{3}{x - 3} - \frac{7}{5x - 1}$

b) $\frac{2}{x + 3} + \frac{7}{x^2 - 9}$

c) $\frac{5}{x^2 - 4x + 3} - \frac{9}{x^2 - 2x + 1}$

4. a) Evaluate $\frac{2}{(x^2 - 9)} + \frac{3}{(x - 3)}$ when $x = 5$.

b) Simplify the original expression by adding.

c) Evaluate the simplified expression when $x = 5$. What do you notice?

PRACTISING

5. Simplify. State any restrictions on the variables.

a) $\frac{2x}{3} + \frac{3x}{4} - \frac{x}{6}$

b) $\frac{3}{t^4} + \frac{1}{2t^2} - \frac{3}{5t}$

c) $\frac{2x}{3y} - \frac{x^2}{4y^3} + \frac{3}{5y^4}$

d) $\frac{n}{m} + \frac{m}{n} - m$

6. Simplify. State any restrictions on the variables.

a) $\frac{7}{a - 4} + \frac{2}{a}$

b) $\frac{4}{3x - 2} + 6$

c) $\frac{5}{x + 4} + \frac{7}{x + 3}$

d) $\frac{6}{2n - 3} - \frac{4}{n - 5}$

e) $\frac{7x}{x + 4} + \frac{3x}{x - 6}$

f) $\frac{7}{2x - 6} + \frac{4}{10x - 15}$

7. Simplify. State any restrictions on the variables.

a) $\dfrac{3}{x + 1} + \dfrac{4}{x^2 - 3x - 4}$

b) $\dfrac{2t}{t - 4} - \dfrac{5t}{t^2 - 16}$

c) $\dfrac{3}{t^2 + t - 6} + \dfrac{5}{(t + 3)^2}$

d) $\dfrac{4x}{x^2 + 6x + 8} - \dfrac{3x}{x^2 - 3x - 10}$

e) $\dfrac{x - 1}{x^2 - 9} + \dfrac{x + 7}{x^2 - 5x + 6}$

f) $\dfrac{2t + 1}{2t^2 - 14t + 24} + \dfrac{5t}{4t^2 - 8t - 12}$

8. Simplify. State any restrictions on the variables.

a) $\dfrac{3}{4x^2 + 7x + 3} - \dfrac{5}{16x^2 + 24x + 9}$

b) $\dfrac{a - 1}{a^2 - 8a + 15} - \dfrac{a - 2}{2a^2 - 9a - 5}$

c) $\dfrac{3x + 2}{4x^2 - 1} + \dfrac{2x - 5}{4x^2 + 4x + 1}$

9. Simplify. State any restrictions on the variables. Remember the order of operations.

a) $\dfrac{2x^3}{3y^2} \times \dfrac{9y}{10x} - \dfrac{2y}{3x}$

b) $\dfrac{x + 1}{2x - 6} \div \dfrac{2(x + 1)^2}{2 - x} + \dfrac{11}{x - 2}$

c) $\dfrac{p + 1}{p^2 + 2p - 35} + \dfrac{p^2 + p - 12}{p^2 - 2p - 24} \times \dfrac{p^2 - 4p - 12}{p^2 + 2p - 15}$

d) $\dfrac{5m - n}{2m + n} - \dfrac{4m^2 - 4mn + n^2}{4m^2 - n^2} \div \dfrac{6m^2 - mn - n^2}{3m + 15n}$

10. K Simplify. State any restrictions on the variables.

a) $\dfrac{3m + 2}{2} + \dfrac{4m + 5}{5}$

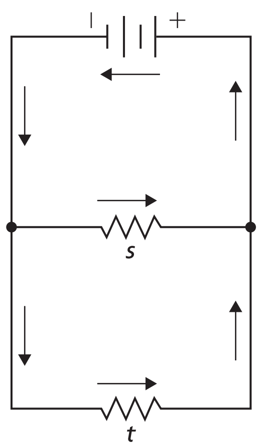
b) $\dfrac{5}{x^2} - \dfrac{3}{4x^3}$

c) $\dfrac{2}{y + 1} - \dfrac{3}{y - 2}$

d) $\dfrac{2x}{x^2 + x - 6} + \dfrac{5}{x^2 + 2x - 8}$

11. T When two resistors, s and t, are connected in parallel, their combined resistance, R, is given by $\dfrac{1}{R} = \dfrac{1}{s} + \dfrac{1}{t}$.

If s is increased by 1 unit and t is decreased by 1 unit, what is the change in R?



12. Fred drove his car a distance of $2x$ km in 3 h. Later, he drove a distance of
A $x + 100$ km in 2 h. Use the equation speed $= \dfrac{\text{distance}}{\text{time}}$.

a) Write a simplified expression for the difference between the first speed and the second speed.

b) Determine the values of x for which the speed was greater for the second trip.

13. Matthew is attending a very loud concert by The Discarded. To avoid permanent ear damage, he decides to move farther from the stage. Sound intensity is given by the formula $I = \dfrac{k}{d^2}$, where k is a constant and d is the distance in metres from the listener to the source of the sound.

Determine an expression for the decrease in sound intensity if Matthew moves x metres farther from the stage.

14. **a)** For two rational numbers in simplified form, the lowest common
C denominator is always one of the following:

i) one of the denominators

ii) the product of the denominators

iii) none of the above

Give an example of each of these.

b) Explain how you would determine the LCD of two simplified rational functions with different quadratic denominators. Illustrate with examples.

Extending

15. In question 11, you encountered an equation of the form $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{z}$, which can be written as $\dfrac{1}{x} = \dfrac{1}{z} - \dfrac{1}{y}$. Suppose you want to determine natural-number solutions of this equation; for example, $\dfrac{1}{6} = \dfrac{1}{2} - \dfrac{1}{3}$ and $\dfrac{1}{20} = \dfrac{1}{4} - \dfrac{1}{5}$.

a) Show that the difference between reciprocals of consecutive positive integers is the reciprocal of their product,

$$\frac{1}{n(n + 1)} = \frac{1}{n} - \frac{1}{n + 1}$$

b) State two more solutions of the equation $\dfrac{1}{x} = \dfrac{1}{z} - \dfrac{1}{y}$.

16. A Pythagorean triple is a triple of natural numbers satisfying the equation $a^2 + b^2 = c^2$. One way to produce a Pythagorean triple is to add the reciprocals of any two consecutive even or odd numbers. For example, $\dfrac{1}{5} + \dfrac{1}{7} = \dfrac{12}{35}$.

Now, $12^2 + 35^2 = 1369$. This is a triple if 1369 is a square, which it is: $1369 = 37^2$. So 12, 35, 37 is a triple.

a) Show that this method always produces a triple.

b) Determine a triple using the method.

2 Chapter Review

FREQUENTLY ASKED Questions

Q: What is a rational function and how do you determine its simplified form?

A: A rational function is a function that can be expressed as a quotient of two polynomials.

The domain of a rational function is the set of all real numbers, except the zeros of the denominator.

To simplify, divide out common factors of the numerator and denominator.

EXAMPLE

$$f(x) = \frac{4\cancel{(x+1)}^{1}(x+2)}{2\cancel{(x+1)}_{1}(x+3)} = \frac{2(x+2)}{(x+3)};\ x \neq -1, -3$$

Study Aid

- See Lesson 2.4, Examples 1 to 5.
- Try Chapter Review Questions 9, 10, and 11.

Q: How do we add, subtract, multiply, and divide rational expressions?

A: Rules for adding, subtracting, multiplying, and dividing rational expressions are the same as those for rational numbers.

EXAMPLE

$$\frac{2x^2}{(x-1)^2} \div \frac{4x}{x^2-1} + \frac{7}{2x-2}$$

$$= \frac{2x^2}{(x-1)(x-1)} \div \frac{4x}{(x+1)(x-1)} + \frac{7}{2(x-1)}$$

$$= \frac{2x^2}{(x-1)(x-1)} \times \frac{(x-1)(x+1)}{4x} + \frac{7}{2(x-1)}$$

$$= \frac{x(x+1)}{2(x-1)} + \frac{7}{2(x-1)}$$

$$= \frac{x^2+x+7}{2(x-1)};\ x \neq 1, -1, 0$$

Study Aid

- See Lesson 2.6, Examples 1 to 4 for multiplication and division.
- See Lesson 2.7, Examples 1 to 4 for addition and subtraction.
- Try Chapter Review Questions 12 to 17.

Q: Why are there sometimes restrictions on the variables in a rational expression, and how do you determine these restrictions?

A: The restrictions occur because division by zero is undefined. To determine restrictions, set all denominators equal to zero before simplifying and solve, usually by factoring.

In the preceding example, set

$(x-1)^2 = 0,\quad x^2 - 1 = 0,\quad 2x - 2 = 0,\quad$ and $\quad 4x = 0$

Solve by factoring:
$(x-1)^2 = 0,\quad (x-1)(x+1) = 0,\quad 2(x-1) = 0,\quad$ and $\quad 4x = 0$

Solving gives the restrictions $x \neq 1, -1, 0$.

Study Aid

- See Lessons 2.4, 2.6, and 2.7, all Examples.
- Try Chapter Review Questions 9 to 17.

PRACTICE Questions

Lesson 2.1

1. Simplify.

a) $(7x^2 - 2x + 1) + (9x^2 - 4x + 5) - (4x^2 + 6x - 7)$

b) $(7a^2 - 4ab + 9b^2) - (-a^2 + 2ab + 6b^2)$

2. Determine two non-equivalent polynomials $f(x)$ and $g(x)$, such that $f(0) = g(0)$ and $f(1) = g(1)$.

3. Ms. Flanagan has three daughters: Astrid, Beatrice, and Cassandra. Today, January 1, their ages are, respectively,

$$A(n) = -(n + 30) + (2n + 5)$$

$$B(n) = (7 - n) - (32 - 2n)$$

$$C(n) = (n - 26) - (n + 4) + (n - 3)$$

All ages are expressed in years, and n represents Ms. Flanagan's age.

a) Are the daughters triplets? Explain.

b) Are any of them twins? Explain.

c) How old was Ms. Flanagan when Cassandra was born?

Lesson 2.2

4. Expand and simplify.

a) $-3(7x - 5)(4x - 7)$

b) $-(y^2 - 4y + 7)(3y^2 - 5y - 3)$

c) $2(a + b)^3$

d) $3(x^2 - 2)^2(2x - 3)^2$

5. The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. Determine the volume of the cone in simplified form if the radius is increased by x and the height is increased by $2x$.

Lesson 2.3

6. Simplify.

a) $(2x^4 - 3x^2 - 6) + (6x^4 - x^3 + 4x^2 + 5)$

b) $(x^2 - 4)(2x^2 + 5x - 2)$

c) $-7x(x^2 + x - 1) - 3x(2x^2 - 5x + 6)$

d) $-2x^2(3x^3 - 7x + 2) - x^3(5x^3 + 2x - 8)$

e) $-2x[5x - (2x - 7)] + 6x[3x - (1 + 2x)]$

f) $(x + 2)^2(x - 1)^2 - (x - 4)^2(x + 4)^2$

g) $(x^2 + 5x - 3)^2$

7. Factor.

a) $12m^2n^3 + 18m^3n^2$

b) $x^2 - 9x + 20$

c) $3x^2 + 24x + 45$

d) $50x^2 - 72$

e) $9x^2 - 6x + 1$

f) $10a^2 + a - 3$

8. Factor.

a) $2x^2y^4 - 6x^5y^3 + 8x^3y$

b) $2x(x + 4) + 3(x + 4)$

c) $x^2 - 3x - 10$

d) $15x^2 - 53x + 42$

e) $a^4 - 16$

f) $(m - n)^2 - (2m + 3n)^2$

Lesson 2.4

9. Simplify. State any restrictions on the variables.

a) $\dfrac{10a^2b + 15bc^2}{-5b}$

b) $\dfrac{30x^2y^3 - 20x^2z^2 + 50x^2}{10x^2}$

c) $\dfrac{xy - xyz}{xy}$

d) $\dfrac{16mnr - 24mnp + 40kmn}{8mn}$

10. Simplify. State any restrictions on the variables.

a) $8xy^2 + 12x^2y - \dfrac{6x^3}{2xy}$

b) $\dfrac{7a - 14b}{2(a - 2b)}$

c) $\dfrac{m + 3}{m^2 + 10m + 21}$

d) $\dfrac{4x^2 - 4x - 3}{4x^2 - 9}$

e) $\dfrac{3x^2 - 21x}{7x^2 - 28x + 21}$

f) $\dfrac{3x^2 - 2xy - y^2}{3x^2 + 4xy + y^2}$

11. If two rational functions have the same restrictions, are they equivalent? Explain and illustrate with an example.

Lessons 2.6 and 2.7

12. Simplify. State any restrictions on the variables.

a) $\frac{6x}{8y} \times \frac{2y^2}{3x}$

b) $\frac{10m^2}{3n} \times \frac{6mn}{20m^2}$

c) $\frac{2ab}{5bc} \div \frac{6ac}{10b}$

d) $\frac{5p}{8pq} \div \frac{3p}{12q}$

13. Simplify. State any restrictions on the variables.

a) $\frac{x^2}{2xy} \times \frac{x}{2y^2} \div \frac{(3x)^2}{xy^2}$

b) $\frac{x^2 - 5x + 6}{x^2 - 1} \times \frac{x^2 - 4x - 5}{x^2 - 4} \div \frac{x - 5}{x^2 + 3x + 2}$

c) $\frac{1 - x^2}{1 + y} \times \frac{1 - y^2}{x + x^2} \div \frac{y^3 - y}{x^2}$

d) $\frac{x^2 - y^2}{4x^2 - y^2} \times \frac{4x^2 + 8xy + 3y^2}{x + y} \div \frac{2x + 3y}{2x - y}$

14. Simplify. State any restrictions on the variables.

a) $\frac{4}{5x} - \frac{2}{3x}$

b) $\frac{5}{x + 1} - \frac{2}{x - 1}$

c) $\frac{1}{x^2 + 3x - 4} + \frac{1}{x^2 + x - 12}$

d) $\frac{1}{x^2 - 5x + 6} - \frac{1}{x^2 - 9}$

15. Simplify and state any restrictions on the variables.

a) $\frac{1}{2x} - \frac{7}{3x^2} + \frac{4}{x^3}$

b) $\frac{3x}{x + 2} + \frac{4x}{x - 6}$

c) $\frac{6x}{x^2 - 5x + 6} - \frac{3x}{x^2 + x - 12}$

d) $\frac{2(x - 2)^2}{x^2 + 6x + 5} \times \frac{3x + 15}{(2 - x)^2}$

e) $\frac{(x - 2y)^2}{x^2 - y^2} \div \frac{(x - 2y)(x + 3y)}{(x + y)^2}$

f) $\frac{2b - 5}{b^2 - 2b - 15} + \frac{3b}{b^2 + b - 30} \times \frac{b^2 + 8b + 12}{b + 3}$

16. Fred's final mark in an online course was determined entirely by two exams. The mid-term exam was out of x marks and was worth 25% of his final mark. The final exam was out of $2x$ marks and was worth 75% of his final mark. Fred scored 40 marks on the first exam and 60 marks on the second exam. Determine the value of x if Fred earned a final mark of 50% in the course.

17. Sam plays a game in which he selects three different numbers from 1 to n $(n > 3)$. After he selects his numbers, four different winning numbers from 1 to n are chosen, one at a time. Sam wins if all three of his numbers are among the four winning numbers.

The first number chosen is one of Sam's! His probability of winning is now given by

$$P(n) = \frac{24}{n^3 - 3n^2 + 2n} \div \frac{3}{n}$$

a) Simplify $P(n)$ and state the restrictions on n.

b) What would Sam's probability of winning be if

i) $n = 5$?

ii) $n = 4$?

2 Chapter Self-Test

1. Simplify.
   a) $(-x^2 + 2x + 7) + (2x^2 - 7x - 7)$
   b) $(2m^2 - mn + 4n^2) - (5m^2 - n^2) + (7m^2 - 2mn)$

2. Expand and simplify.
   a) $2(12a - 5)(3a - 7)$
   b) $(2x^2y - 3xy^2)(4xy^2 + 5x^2y)$
   c) $(4x - 1)(5x + 2)(x - 3)$
   d) $(3p^2 + p - 2)^2$

3. Is there a value of a such that $f(x) = 9x^2 + 4$ and $g(x) = (3x - a)^2$ are equivalent? Explain.

4. If Bonnie is away from Clyde for n consecutive days, then the amount of heartache Clyde feels is given by $h(n) = (2n + 1)^3$.
   a) If Bonnie is absent, by how much does Clyde's pain increase between day n and day $n + 1$?
   b) How much more pain will he feel on day 6 than on day 5?

5. Factor.
   a) $3m(m - 1) + 2m(1 - m)$
   b) $x^2 - 27x + 72$
   c) $15x^2 - 7xy - 2y^2$
   d) $(2x - y + 1)^2 - (x - y - 2)^2$
   e) $5xy - 10x - 3y + 6$
   f) $p^2 - m^2 + 6m - 9$

6. Use factoring to determine the x-intercepts of the curve $y = x^3 - 4x^2 - x + 4$.

7. Simplify. State any restrictions on the variables.
   a) $\dfrac{4a^2b}{5ab^3} \div \dfrac{6a^2b}{35ab}$
   b) $\dfrac{x - 2}{x^2 - x - 12} \times \dfrac{2x - 8}{x^2 - 4x + 4}$
   c) $\dfrac{5}{t^2 - 7t - 18} + \dfrac{6}{t + 2}$
   d) $\dfrac{4x}{6x^2 + 13x + 6} - \dfrac{3x}{4x^2 - 9}$

8. Mauro found that two rational functions each simplified to $f(x) = \dfrac{2}{x + 1}$.

   Are Mauro's two rational functions equivalent? Explain.

9. Roman thinks that he has found a simple method for determining the sum of the reciprocals of any three consecutive natural numbers. He writes, for example,

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}, \quad \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{74}{120}, \quad \text{or} \quad \frac{37}{60}$$

   Roman conjectures that before simplification, the numerator of the sum is three times the product of the first and third denominators, plus 2. Also, the denominator of the sum is the product of the three denominators. Is Roman's conjecture true?

2 Chapter Task

The Algebraic Dominos Challenge

The game shown at the right consists of eight pairs of coloured squares called dominos.

1*R*	2*R*	2*P*	4*P*
1*P*	3*R*	5*R*	4*R*
6*P*	3*P*	5*P*	7*R*
6*R*	8*P*	8*R*	7*P*

Rules:

1. Write a polynomial in each square marked *P* and a rational function in each square marked *R*.
2. The expressions you write must satisfy each of these conditions:
   - Polynomials and numerators and denominators of each rational function must be quadratics without a constant common factor.
   - Restrictions on the variable of each rational function must be stated in its square.
   - When two polynomials are side by side, then one or both of the polynomials must be perfect squares.
   - When a polynomial and a *different-coloured* rational expression are side by side, their product must simplify.
   - When two rational expressions are side by side, their product must simplify.
   - When a polynomial is on top of another polynomial, their quotient must simplify.
   - When a polynomial is on top of a *different-coloured* rational expression (or vice versa), their quotient must simplify.
   - When a rational expression is on top of a rational expression, their quotient must simplify.
3. After you have completed the table, simplify the products and quotients wherever possible. You get one point for every *different* linear factor that remains in your table.
4. Count the linear factors and write your score next to your table.

? How can you maximize your score?

A. What form for the polynomials, including numerators and denominators, will make filling the table and counting your score as easy as possible?

B. Why should you avoid reusing a factor unless it is necessary?

C. Play the game by completing the table.

D. Tally your score.

E. Check your answers. What could you do to increase your score?

F. List some strategies you can use to maximize your score.

Task | *Checklist*

- ✔ Does each square contain a polynomial and rational function of the right type?
- ✔ Are all of the rules satisfied?
- ✔ Did you check to see if you could make changes to improve your score?

Chapter 3

Quadratic Functions

GOALS

You will be able to

- Graph and analyze the properties of quadratic functions
- Determine the zeros of quadratic functions
- Calculate the maximum or minimum values of quadratic functions
- Solve problems involving quadratic functions

? What role does the parabola play in the construction of the bridge in the photograph?

3 Getting Started

SKILLS AND CONCEPTS You Need

1. Given $f(x) = -3x^2 + 4x - 1$, evaluate the following.

a) $f(1)$ **b)** $f(-2)$ **c)** $f\left(\frac{1}{3}\right)$ **d)** $f(0)$ **e)** $f(k)$ **f)** $f(-k)$

Study | *Aid*

- For help, see Essential Skills Appendix.

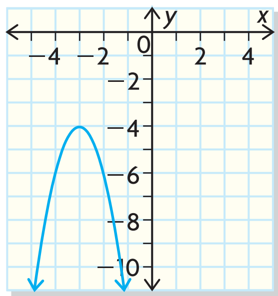
Question	Appendix
1	A-7
2	A-8
3, 4, 7	A-12
5, 6	A-9, A-10

2. Express each function in standard form, $f(x) = ax^2 + bx + c$.

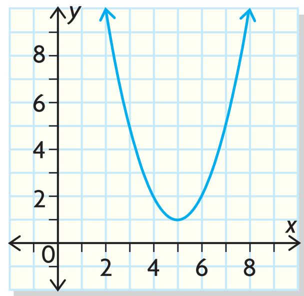
a) $f(x) = (x - 3)(x + 5)$ **b)** $f(x) = 2x(x + 6)$ **c)** $f(x) = -3(x + 2)^2 + 3$ **d)** $f(x) = (x - 1)^2$

3. State the **vertex**, **axis of symmetry**, **domain**, and **range** of each parabola.

a)



b)



4. State the vertex, axis of symmetry, and **direction of opening** for each parabola.

a) $y = x^2 + 4$ **b)** $y = 3(x - 4)^2 + 1$ **c)** $y = -0.5(x + 7)^2 - 3$ **d)** $y = -3(x + 2)(x - 5)$

5. Solve each equation. Answer to two decimal places if necessary.

a) $x^2 - 11x + 24 = 0$ **b)** $x^2 - 6x + 3 = 0$ **c)** $3x^2 - 2x - 5 = 0$ **d)** $3x^2 + 2x = x^2 + 9x - 3$

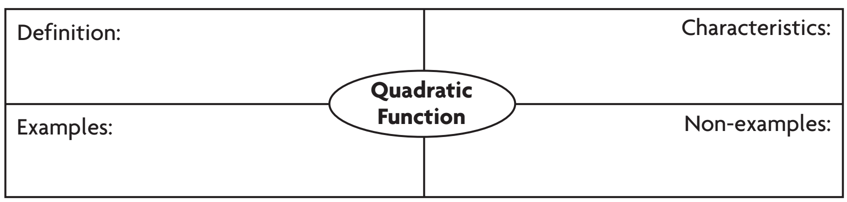
6. Determine the x-intercepts of each function. Answer to two decimal places if necessary.

a) $f(x) = x^2 - 9$ **b)** $f(x) = x^2 - 8x - 18$ **c)** $f(x) = -3x^2 + 10x - 8$ **d)** $f(x) = 6x - 2x^2$

7. Sketch the graph of each function.

a) $f(x) = x^2 + 3$ **b)** $f(x) = 2x^2 - 4$ **c)** $f(x) = -(x - 2)(x + 8)$ **d)** $f(x) = -(x + 2)^2 + 3$

8. Complete the chart by writing what you know about quadratic functions.



APPLYING What You Know

Building Rectangles

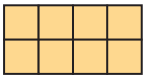
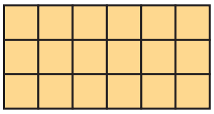
A set of rectangles can be formed out of 1 cm squares on centimetre grid paper. Martina draws the first rectangle with dimensions 1 cm × 2 cm, the second 2 cm × 4 cm, and the third 3 cm × 6 cm.

YOU WILL NEED

- centimetre grid paper
- graph paper

? If this pattern continues, what function can be used to model the relationship between width and area?

A. Use centimetre grid paper to draw the next four rectangles in the pattern. Use your diagrams to extend and complete the table.

Shape	Width (cm)	Length (cm)	Perimeter (cm)	Area (cm)^2
	1	2	6	2
	2	4	12	8
	3	6	18	

B. Calculate the first differences for the Perimeter and Area columns.
Is the relation between width and perimeter linear or nonlinear? How do you know?

C. Is the relation between width and area linear or nonlinear? How do you know?

D. Determine the second differences for the Area column. What do they tell you about the relationship between width and area?

E. Create a scatter plot of area versus width. Draw a curve of good fit. Does the shape of your graph support your answer to part D? Explain.

F. What is the relationship between length and width for each rectangle?

G. Write the function that models the relationship between width and area.

3.1 Properties of Quadratic Functions

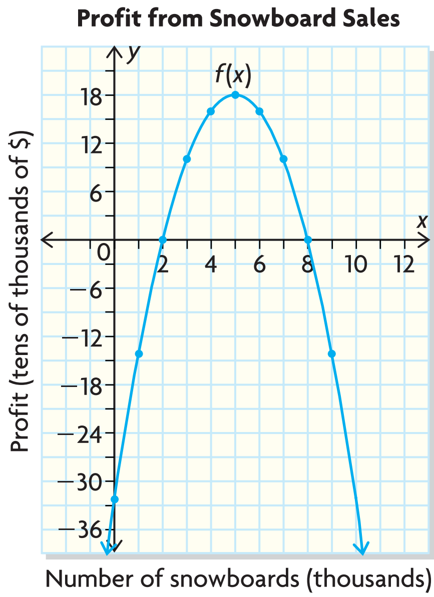
GOAL

Represent and interpret quadratic functions in a number of different forms.

LEARN ABOUT the Math

Francisco owns a business that sells snowboards. His accountants have presented him with data on the business' profit in a table and a graph.

Snowboards Sold, *x* (1000s)	0	1	2	3	4	5	6	7	8	9
Profit, *f*(*x*) ($10 000s)	−32	−14	0	10	16	18	16	10	0	−14



? What function models Francisco's profit?

EXAMPLE 1 Selecting a strategy to describe the algebraic model

Develop an algebraic expression for the function that models Francisco's profit from selling snowboards.

Kelly's Solution: Using the Vertex Form of the Quadratic Function

Snowboards Sold (1000s)	Profit ($10 000s)	First Differences	Second Differences
0	−32		
1	−14	18	−4
2	0	14	−4
3	10	10	−4
4	16	6	−4
5	18	2	−4
6	16	−2	−4
7	10	−6	−4
8	0	−10	−4
9	−14	−14	

This function looks quadratic, since its graph appears to be a parabola. To make sure, I checked the first and second differences. Since the first differences are not constant, the function is nonlinear. The second differences are all equal and negative. So the function is quadratic. This confirms that the graph is a parabola that opens downward.

From the graph, the vertex is $(5, 18)$.
The parabola also passes through $(2, 0)$.

I could determine the quadratic function model if I knew the vertex and at least one other point on the graph.

$$f(x) = a(x - h)^2 + k$$
$$= a(x - 5)^2 + 18$$

I used the **vertex form** of the quadratic function and substituted the coordinates of the vertex from the graph.

$$0 = a(2 - 5)^2 + 18$$
$$0 = 9a + 18$$
$$-18 = 9a$$
$$-2 = a$$

$f(2) = 0$, so I substituted $(2, 0)$ into the function. Once I did that, I solved for a.

The function $f(x) = -2(x - 5)^2 + 18$ models Francisco's profit.

Jack's Solution: Using the Factored Form of the Quadratic Function

The graph is a parabola, opening down with axis of symmetry $x = 5$.

The graph looks like a parabola, so it has to be a quadratic function. The graph is symmetric about the line $x = 5$.

The x-intercepts are the points $(2, 0)$ and $(8, 0)$.

I could find the **factored form** of the quadratic function if I knew the x-intercepts, or zeros. I read these from the graph and used the table of values to check.

$$f(x) = a(x - r)(x - s)$$

I took the factored form of a quadratic function and substituted the values of the x-intercepts for r and s.

$$f(x) = a(x - 2)(x - 8)$$
$$10 = a(3 - 2)(3 - 8)$$
$$10 = a(1)(-5)$$
$$10 = -5a$$
$$-2 = a$$

I then chose the point (3, 10) from the table of values and substituted its coordinates into $f(x)$ to help me find the value of a.

The function $f(x) = -2(x - 2)(x - 8)$ models Francisco's profit.

Reflecting

A. What information do you need to write the vertex form of the quadratic function?

B. What information do you need to write the factored form of the quadratic function?

C. Use the graph to state the domain and range of the function that models Francisco's profit. Explain.

D. Will both of the models for Francisco's profit lead to the same function when expressed in **standard form**?

APPLY the Math

EXAMPLE 2 Determining the properties of a quadratic function

A construction worker repairing a window tosses a tool to his partner across the street. The height of the tool above the ground is modelled by the quadratic function $h(t) = -5t^2 + 20t + 25$, where $h(t)$ is height in metres and t is the time in seconds after the toss.

a) How high above the ground is the window?

b) If his partner misses the tool, when will it hit the ground?

c) If the path of the tool's height were graphed, where would the axis of symmetry be?

d) Determine the domain and range of the function in this situation.

André's Solution

a) $h(0) = -5(0)^2 + 20(0) + 25$

$= 25$

The height of the window must be the value of the function at $t = 0$ s.

The window is 25 m above the ground.

b) $0 = -5t^2 + 20t + 25$

If the partner missed the tool, it would hit the ground. The height at the ground is zero. I set $h(t)$ equal to zero. Then I factored the quadratic.

$0 = -5(t^2 - 4t - 5)$

$0 = -5(t + 1)(t - 5)$

$t = -1 \quad \text{or} \quad t = 5$

I found two values for t, but the negative answer is not possible, since time must be positive.

The tool will hit the ground after 5 s.

c) $t = \dfrac{-1 + 5}{2}$

The axis of symmetry passes through the midpoint of the two zeros of the function. I added the zeros together and divided by two to find that t-value. The axis of symmetry is a vertical line, so its equation is $t = 2$.

$t = \dfrac{4}{2}$

$t = 2$

The axis of symmetry is $t = 2$.

d) Domain $= \{t \in \mathbf{R} \mid 0 \le t \le 5\}$

In this situation, t must be 0 or greater, and the tool will stop when it hits the ground.

$h(2) = -5(2)^2 + 20(2) + 25$

$= -20 + 40 + 25$

$= 45$

The least possible value of $h(t)$ is zero. I found the value of h at $t = 2$, since the greatest value is the y-coordinate of the vertex, which is always on the axis of symmetry.

Range $= \{h \in \mathbf{R} \mid 0 \le h \le 45\}$

EXAMPLE 3 Graphing a quadratic function from the vertex form

Given $f(x) = 2(x - 1)^2 - 5$, state the vertex, axis of symmetry, direction of opening, y-intercept, domain, and range. Graph the function.

Sacha's Solution

Vertex: $(1, -5)$

The x-coordinate of the vertex is 1 and the y-coordinate is -5.

Axis of symmetry: $x = 1$

The axis of symmetry is a vertical line through the vertex at $(1, -5)$.

Direction of opening: up

Since a is positive, the parabola opens up.

$$\begin{aligned} f(0) &= 2(0 - 1)^2 - 5 \\ &= 2(-1)^2 - 5 \\ &= 2 - 5 \\ &= -3 \end{aligned}$$

I substituted $x = 0$ to calculate the y-intercept and solved the equation.

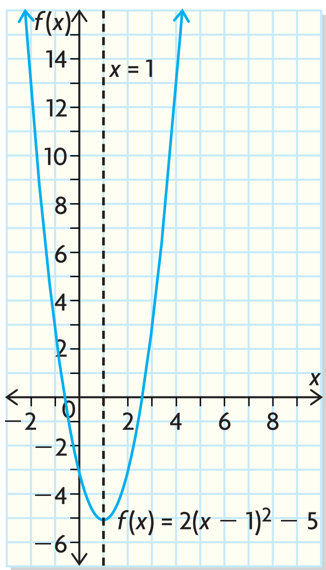
y-intercept: -3

Domain: $\{x \in \mathbf{R}\}$

There are no restrictions on the values for x.

Range: $\{y \in \mathbf{R} \mid y \geq -5\}$

Because the vertex has a y-value of -5 and the parabola opens up, the y-values have to be greater than or equal to -5.



To graph the function, I plotted the vertex and the axis of symmetry.

I found the values of $f(x)$ when $x = 2$, 3, and 4.

$$f(2) = -3$$

$$f(3) = 3$$

$$f(4) = 13$$

The values had to be the same for 0, -1, and -2 because the graph is symmetric about the line $x = 1$.
I plotted the points and joined them with a smooth curve.

In Summary

Key Ideas

- Graphs of quadratic functions with no domain restrictions are parabolas.
- Quadratic functions have constant nonzero second differences. If the second differences are positive, the parabola opens up and the coefficient of x^2 is positive. If the second differences are negative, the parabola opens down and the coefficient of x^2 is negative.

Need to Know

- Quadratic functions can be represented by equations in function notation, by tables of values, or by graphs.
- Quadratic functions have a degree of 2.
- Quadratic functions can be expressed in different algebraic forms:
  - standard form: $f(x) = ax^2 + bx + c, a \neq 0$
  - factored form: $f(x) = a(x - r)(x - s), a \neq 0$
  - vertex form: $f(x) = a(x - h)^2 + k, a \neq 0$

CHECK Your understanding

1. Determine whether each function is linear or quadratic. Give a reason for your answer.

a)

x	y
−2	15
−1	11
0	7
1	3
2	−1

b)

x	y
−2	1
−1	3
0	6
1	10
2	15

c)

x	y
−2	4
−1	8
0	12
1	16
2	20

d)

x	y
−2	7
−1	4
0	3
1	4
2	7

2. State whether each parabola opens up or down.

a) $f(x) = 3x^2$

b) $f(x) = -2(x - 3)(x + 1)$

c) $f(x) = -(x + 5)^2 - 1$

d) $f(x) = \frac{2}{3}x^2 - 2x - 1$

3. Given $f(x) = -3(x - 2)(x + 6)$, state

a) the zeros

b) the direction of opening

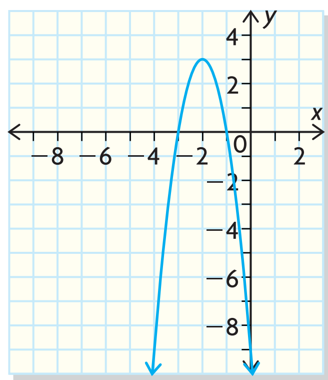
c) the equation of the axis of symmetry

4. Given the parabola at the right, state

a) the vertex

b) the equation of the axis of symmetry

c) the domain and range



PRACTISING

5. Graph each function. State the direction of opening, the vertex, and the equation of the axis of symmetry.

a) $f(x) = x^2 - 3$

b) $f(x) = -(x + 3)^2 - 4$

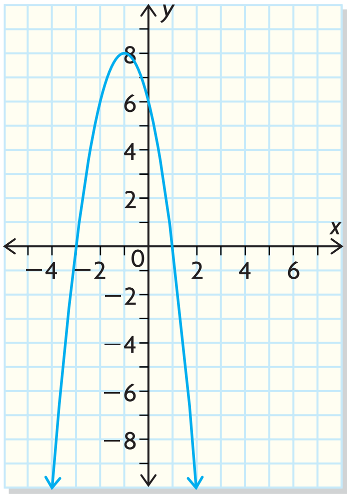
c) $f(x) = 2(x - 4)(x + 2)$

d) $f(x) = -\frac{1}{2}x^2 + 4$

6. Express each quadratic function in standard form. State the y-intercept of each.

a) $f(x) = -3(x - 1)^2 + 6$

b) $f(x) = 4(x - 3)(x + 7)$



7. Examine the parabola at the left.

K **a)** State the direction of opening.

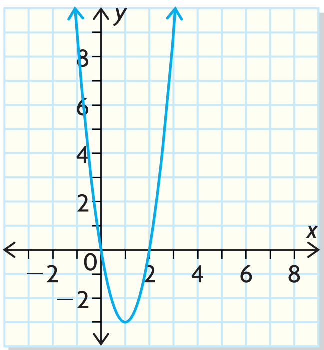
b) Name the coordinates of the vertex.

c) List the values of the x-intercepts.

d) State the domain and range of the function.

e) If you calculated the second differences, what would their sign be? How do you know?

f) Determine the algebraic model for this quadratic function.



8. Examine the parabola at the left.

a) State the direction of opening.

b) Find the coordinates of the vertex.

c) What is the equation of the axis of symmetry?

d) State the domain and range of the function.

e) If you calculated the second differences, what would their sign be? Explain.

9. Each pair of points (x, y) are the same distance from the vertex of their parabola. Determine the equation of the axis of symmetry of each parabola.

a) $(-2, 2), (2, 2)$

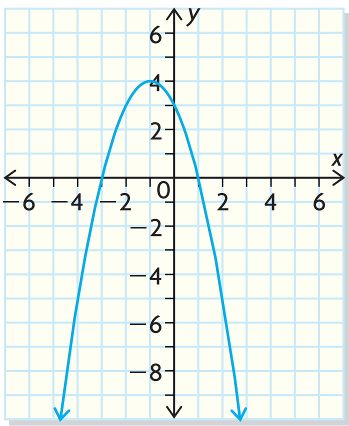
b) $(-9, 1), (-5, 1)$

c) $(6, 3), (18, 3)$

d) $(-5, 7), (1, 7)$

e) $(-6, -1), (3, -1)$

f) $\left(-\frac{11}{8}, 0\right), \left(\frac{3}{4}, 0\right)$



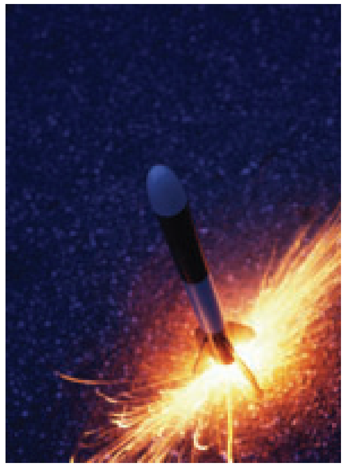
10. Examine the parabola shown at the left.

a) Copy and complete this table.

x	−2	−1	0	1	2
***f*(*x*)**					

b) Calculate the second differences of the function. How could you have predicted their signs?

c) Determine the equation of the function.



11. The height of a rocket above the ground is modelled by the quadratic function $h(t) = -4t^2 + 32t$, where $h(t)$ is the height in metres t seconds after the rocket was launched.

A

a) Graph the quadratic function.

b) How long will the rocket be in the air? How do you know?

c) How high will the rocket be after 3 s?

d) What is the maximum height that the rocket will reach?

12. A quadratic function has these characteristics:

T

- $x = -1$ is the equation of the axis of symmetry.
- $x = 3$ is the x-intercept.
- $y = 32$ is the maximum value.

Determine the y-intercept of this parabola.

13. Describe two ways in which the functions $f(x) = 2x^2 - 4x$ and $g(x) = -(x - 1)^2 + 2$ are alike, and two ways in which they are different.

C

Extending

14. The first differences and second differences of a quadratic function with domain ranging from $x = -2$ to $x = 3$ are given. If $f(-2) = 19$, copy the table and complete the second row by determining the missing values of the function.

x	−2	−1	0	1	2	3
***f*(*x*)**	19					
First Differences		−10	−6	−2	2	6
Second Differences		4	4	4	4	

15. A company's profit, in thousands of dollars, on sales of computers is modelled by the function $P(x) = -2(x - 3)^2 + 50$, where x is in thousands of computers sold. The company's profit, in thousands of dollars, on sales of stereo systems is modelled by the function $P(x) = -(x - 2)(x - 7)$, where x is in thousands of stereo systems sold. Calculate the maximum profit the business can earn.

16. Jim has a difficult golf shot to make. His ball is 100 m from the hole. He wants the ball to land 5 m in front of the hole, so it can roll to the hole. A 20 m tree is between his ball and the hole, 40 m from the hole and 60 m from Jim's ball. With the base of the tree as the origin, write an algebraic expression to model the height of the ball if it just clears the top of the tree.

3.2 Determining Maximum and Minimum Values of a Quadratic Function

GOAL

Use a variety of strategies to determine the maximum or minimum value of a quadratic function.

LEARN ABOUT the Math

A golfer attempts to hit a golf ball over a gorge from a platform above the ground. The function that models the height of the ball is $h(t) = -5t^2 + 40t + 100$, where $h(t)$ is the height in metres at time t seconds after contact. There are power lines 185 m above the ground.

? Will the golf ball hit the power lines?

EXAMPLE 1 Selecting a strategy to find the vertex

Using the function for the golf ball's height, determine whether the ball will hit the power line.

Jonah's Solution: Completing the Square

$$h(t) = -5t^2 + 40t + 100$$

I needed to find the maximum height of the ball to compare it to the height of the power lines.

a is negative. The graph of $h(t)$ is a parabola that opens down. Its maximum value occurs at the vertex.

$$h(t) = -5(t^2 - 8t) + 100$$

I put the function into vertex form by **completing the square**.

I factored -5 from the t^2 and t terms.

$$= -5(t^2 - 8t + 16 - 16) + 100$$

I divided the coefficient of t in half, then squared it to create a perfect-square trinomial.

By adding 16, I changed the value of the expression. To make up for this, I subtracted 16.

$$= -5(t^2 - 8t + 16) + 80 + 100$$

I grouped the first 3 terms that formed the perfect square and moved the subtracted value of 16 outside the brackets by multiplying by -5.

$$= -5(t - 4)^2 + 180$$

I factored the perfect square and collected like terms.

The vertex is (4, 180). The maximum height will be 180 m after 4 s.

Since the power lines are 185 m above the ground, the ball will not hit them.

Since the vertex is at the maximum height, the ball goes up only 180 m.

Sophia's Solution: Factoring to Determine the Zeros

$$h(t) = -5t^2 + 40t + 100$$

The maximum height of the golf ball is at the vertex of the parabola.

The vertex is located on the axis of symmetry, which is always in the middle of the two zeros of the function. To find the zeros, I factored the quadratic.

$$h(t) = -5(t^2 - 8t - 20)$$

$$h(t) = -5(t - 10)(t + 2)$$

$$0 = -5(t - 10)(t + 2)$$

I divided -5 out as a common factor. Inside the brackets was a simple trinomial I could factor.

$$t = 10 \quad \text{or} \quad t = -2$$

The zeros are the values that make $h(t) = 0$. I found them by setting each factor equal to 0 and solving the resulting equations.

For the axis of symmetry,

$$t = \frac{10 + (-2)}{2}$$

$$t = \frac{8}{2}$$

$$t = 4$$

I added the zeros and divided the result by 2 to locate the axis of symmetry. This was also the x-coordinate, or in this case, the t-coordinate of the vertex.

The t-coordinate of the vertex is 4.

$$h(4) = -5(4 - 10)(4 + 2)$$

$$= -5(-6)(6)$$

$$= 180$$

To find the y-value, or height h, I substituted $t = 4$ into the factored form of the equation. Alternatively, I could have substituted into the function in standard form.

The vertex is (4, 180). The maximum height will be 180 m, after 4 s. Since the power lines are 185 m above the ground, the ball will not hit them.

Reflecting

A. How can you tell from the algebraic form of a quadratic function whether the function has a maximum or a minimum value?

B. Compare the two methods for determining the vertex of a quadratic function. How are they the same? How are they different?

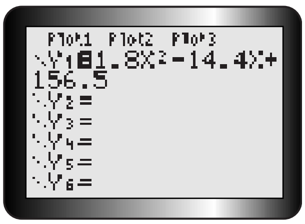
C. Not all quadratic functions have zeros. Which method allows you to find the vertex without finding the zeros? Explain.

APPLY the Math

EXAMPLE 2 | Using the graphing calculator as a strategy to determine the minimum value

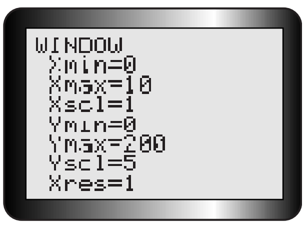
The cost, $c(x)$, in dollars per hour of running a certain steamboat is modelled by the quadratic function $c(x) = 1.8x^2 - 14.4x + 156.5$, where x is the speed in kilometres per hour. At what speed should the boat travel to achieve the minimum cost?

Rita's Solution



This parabola opens up. Therefore, the minimum value will be at the vertex.

I used a graphing calculator because the numbers in the question were decimals and it would not have been easy to complete the square or factor.

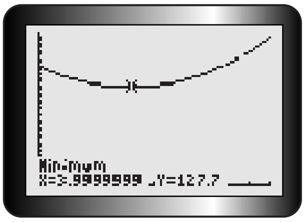


I chose a **WINDOW** that would display the graph. I chose both the *x*- and *y*-values to have a minimum of 0, since neither cost nor speed could be negative.

I picked a maximum *x* of 10 and an *x* scale of 1. I estimated the corresponding maximum *y* from the function.

Tech *Support*

For help using the graphing calculator to determine the minimum value of a function, see Technical Appendix, B-9.



I used the minimum operation to locate the vertex.

The minimum cost to operate the steamboat is \$127.70/h, when the boat is travelling at about 4 km/h.

The vertex is (4,127.70).

EXAMPLE 3 Solving a problem to determine when the maximum value occurs

The demand function for a new magazine is $p(x) = -6x + 40$, where $p(x)$ represents the selling price, in thousands of dollars, of the magazine and x is the number sold, in thousands. The cost function is $C(x) = 4x + 48$. Calculate the maximum profit and the number of magazines sold that will produce the maximum profit.

Levi's Solution

$$\begin{aligned}\text{Revenue} &= \text{Demand} \times \text{Number sold}\\ &= [p(x)](x)\end{aligned}$$

I found the revenue function by multiplying the demand function by the number of magazines sold.

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$\begin{aligned}P(x) &= [p(x)](x) - C(x)\\ &= (-6x + 40)(x) - (4x + 48)\\ &= -6x^2 + 40x - 4x - 48\\ &= -6x^2 + 36x - 48\end{aligned}$$

To find the profit function, I subtracted the cost function from the revenue function and simplified.

The coefficient of x^2 is negative, so the parabola opens down with its maximum value at the vertex. Instead of completing the square, I determined two points symmetrically opposite each other.

$$P(x) = -6x(x - 6) - 48$$

I started by factoring the common factor $-6x$ from $-6x^2$ and $36x$.

Communication *Tip*

The demand function $p(x)$ is the relation between the price of an item and the number of items sold, x. The cost function $C(x)$ is the total cost of making x items. Revenue is the money brought in by selling x items. Revenue is the product of the demand function and the number sold. Profit is the difference between revenue and cost.

$$-6x(x-6)=0$$

$$x=0 \quad \text{or} \quad x=6$$

Points on the graph of the profit function are $(0, -48)$ and $(6, -48)$.

I knew that the x-intercepts of the graph of $y = -6x(x-6)$ would help me find the two points I needed on the graph of the profit function, since both functions have the same axis of symmetry.

The axis of symmetry is $x = \frac{0+6}{2}$ or $x = 3$. So the x-coordinate of the vertex is 3.

I found the axis of symmetry, which gave me the x-coordinate of the vertex.

$$\begin{aligned} P(3) &= -6(3)^2 + 36(3) - 48 \\ &= -54 + 108 - 48 \\ &= 6 \end{aligned}$$

I substituted $x = 3$ into the function to determine the profit. I remembered that x is in *thousands* of magazines sold, and $P(x)$ is in *thousands* of dollars.

The maximum profit is \$6000, when 3000 magazines are sold.

In Summary

Key Idea

- The maximum or minimum value of a quadratic function is the y-coordinate of the vertex.

Need to Know

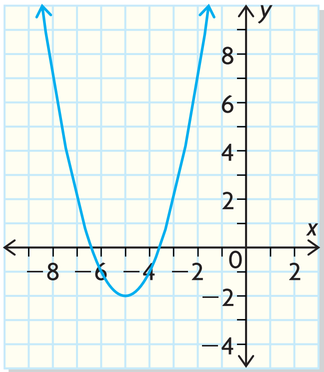
- If $a > 0$ in standard form, factored form, or vertex form, then the parabola opens up. The quadratic has a minimum value.
- If $a < 0$ in standard form, factored form, or vertex form, then the parabola opens down. The quadratic has a maximum value.
- The vertex can be found from the standard form $f(x) = ax^2 + bx + c$ algebraically in several ways:
  - by completing the square to put the quadratic in vertex form $f(x) = a(x-h)^2 + k$
  - by expressing the quadratic in factored form $f(x) = a(x-r)(x-s)$, if possible, and averaging the zeros at r and s to locate the axis of symmetry. This will give the x-coordinate of the vertex
  - by factoring out the common factor from $ax^2 + bx$ to determine two points on the parabola that are symmetrically opposite each other, and averaging the x-coordinates of these points to determine the x-coordinate of the vertex
  - by using a graphing calculator

CHECK Your Understanding

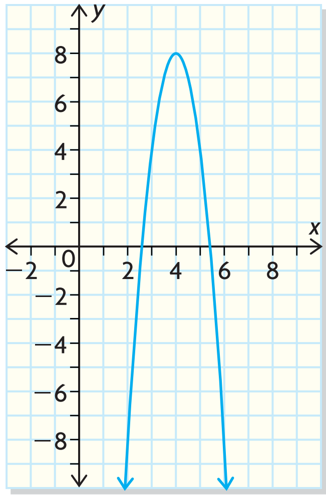
1. Which of the following quadratic functions will have a maximum value? Explain how you know.
   - **a)** $y = -x^2 + 7x$
   - **b)** $f(x) = 3(x - 1)^2 - 4$
   - **c)** $f(x) = -4(x + 2)(x - 3)$
   - **d)** $g(x) = 4x^2 + 3x - 5$

2. State the vertex of each parabola and indicate the maximum or minimum value of the function.

   a)



   b)



3. Determine the maximum or minimum value for each.
   - **a)** $y = -4(x + 1)^2 + 6$
   - **b)** $f(x) = (x - 5)^2$
   - **c)** $f(x) = -2x(x - 4)$
   - **d)** $g(x) = 2x^2 - 7$

PRACTISING

4. Determine the maximum or minimum value. Use at least two different methods. **K**
   - **a)** $y = x^2 - 4x - 1$
   - **b)** $f(x) = x^2 - 8x + 12$
   - **c)** $y = 2x^2 + 12x$
   - **d)** $y = -3x^2 - 12x + 15$
   - **e)** $y = 3x(x - 2) + 5$
   - **f)** $g(x) = -2(x + 1)^2 - 5$

5. Each function is the demand function of some item, where x is the number of items sold, in thousands. Determine
   - **i)** the revenue function
   - **ii)** the maximum revenue in thousands of dollars

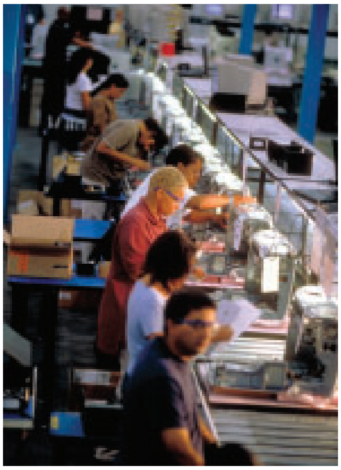
   - **a)** $p(x) = -x + 5$
   - **b)** $p(x) = -4x + 12$
   - **c)** $p(x) = -0.6x + 15$
   - **d)** $p(x) = -1.2x + 4.8$

6. Use a graphing calculator to determine the maximum or minimum value. Round to two decimal places where necessary.
   - **a)** $f(x) = 2x^2 - 6.5x + 3.2$
   - **b)** $f(x) = -3.6x^2 + 4.8x$

7. For each pair of revenue and cost functions, determine
   i) the profit function
   ii) the value of x that maximizes profit

   a) $R(x) = -x^2 + 24x$, $C(x) = 12x + 28$
   b) $R(x) = -2x^2 + 32x$, $C(x) = 14x + 45$
   c) $R(x) = -3x^2 + 26x$, $C(x) = 8x + 18$
   d) $R(x) = -2x^2 + 25x$, $C(x) = 3x + 17$

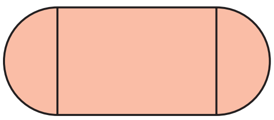
8. The height of a ball thrown vertically upward from a rooftop is modelled by $h(t) = -5t^2 + 20t + 50$, where $h(t)$ is the ball's height above the ground, in metres, at time t seconds after the throw.
   a) Determine the maximum height of the ball.
   b) How long does it take for the ball to reach its maximum height?
   c) How high is the rooftop?



9. The cost function in a computer manufacturing plant is $C(x) = 0.28x^2 - 0.7x + 1$, where $C(x)$ is the cost per hour in millions of dollars and x is the number of items produced per hour in thousands. Determine the minimum production cost.

10. Show that the value of $3x^2 - 6x + 5$ cannot be less than 1.

11. **A** The profit $P(x)$ of a cosmetics company, in thousands of dollars, is given by $P(x) = -5x^2 + 400x - 2550$, where x is the amount spent on advertising, in thousands of dollars.
   a) Determine the maximum profit the company can make.
   b) Determine the amount spent on advertising that will result in the maximum profit.
   c) What amount must be spent on advertising to obtain a profit of at least \$4 000 000?



12. **T** A high school is planning to build a new playing field surrounded by a running track. The track coach wants two laps around the track to be 1000 m. The football coach wants the rectangular infield area to be as large as possible. Can both coaches be satisfied? Explain your answer.

13. **C** Compare the methods for finding the minimum value of the quadratic function $f(x) = 3x^2 - 7x + 2$. Which method would you choose for this particular function? Give a reason for your answer.

Extending

14. A rock is thrown straight up in the air from an initial height h_0, in metres, with initial velocity v_0, in metres per second. The height in metres above the ground after t seconds is given by $h(t) = -4.9t^2 + v_0t + h_0$. Find an expression for the time it takes the rock to reach its maximum height.

15. A ticket to a school dance is \$8. Usually, 300 students attend. The dance committee knows that for every \$1 increase in the price of a ticket, 30 fewer students attend the dance. What ticket price will maximize the revenue?

3.3 The Inverse of a Quadratic Function

GOAL

Determine the inverse of a quadratic function, given different representations.

YOU WILL NEED

- graph paper
- ruler
- graphing calculator

INVESTIGATE the Math

Suzanne needs to make a box in the shape of a cube. She has 864 cm^2 of cardboard to use. She wants to use all of the material provided.

? How long will each side of Suzanne's box be?

A. Copy and complete this table.

Cube Side Length (cm)	1	2	3	4	5	6	7	8	9	10
Area of Each Face (cm^2)	1	4								
Surface Area (cm^2)	6	24								

B. Draw a graph of surface area versus side length. What type of function is this? Explain how you know.

C. Determine the equation that represents the cube's surface area as a function of its side length. Use function notation and state the domain and range.

D. How would you calculate the inverse of this function to describe the side length of the cube if you know its surface area?

E. Make a table of values for the inverse of the surface area function.

F. Draw a graph of the inverse. Compare the graph of the inverse with the original graph. Is the inverse a function? Explain.

G. State the domain and range of the inverse.

H. Write the equation that represents the cube's side length for a given surface area.

I. Use your equation from part H to determine the largest cube Suzanne will be able to construct.

Reflecting

J. How are the surface area function and its inverse related

a) in the table of values?

b) in their graphs?

c) in their domains and ranges?

K. How is any quadratic function related to its inverse

a) in their domains and ranges?

b) in their equations?

APPLY the Math

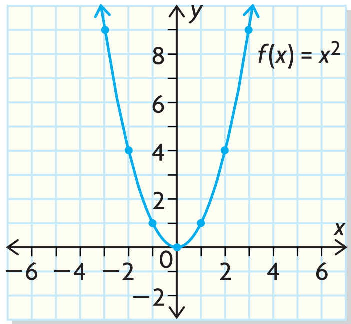
EXAMPLE 1 Determining the domain and range of the inverse of a quadratic function

Given the graph of $f(x) = x^2$,

a) graph the inverse relation

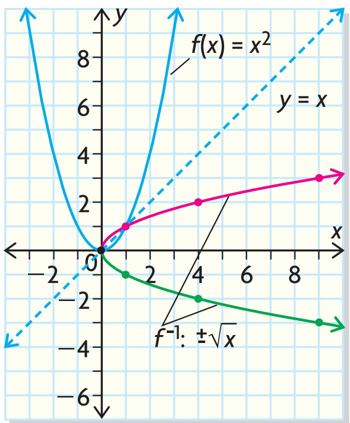
b) state the domain and range of $f(x) = x^2$ and the inverse relation

c) determine whether the inverse of $f(x) = x^2$ is also a function. Give a reason for your answer.



Paul's Solution

a)



To graph the inverse of $f(x) = x^2$, I took the coordinates of each point on the original graph and switched the x- and y-coordinates. For example, (2, 4) became (4, 2). I had to do this because the input value becomes the output value in the inverse, and vice versa.

The graph of the inverse is a reflection of the original function about the line $y = x$.

b) The domain of $f(x) = x^2$ is $\{x \in \mathbf{R}\}$. The range of $f(x) = x^2$ is $\{y \in \mathbf{R} \mid y \geq 0\}$. Therefore, the domain of f^{-1} is $\{x \in \mathbf{R} \mid x \geq 0\}$, and the range is $\{y \in \mathbf{R}\}$.

The domain of the inverse is the range of the original function. The range of the inverse is the domain of the original function.

c) The inverse of $f(x) = x^2$ is not a function.

I knew this because the inverse fails the vertical-line test: For each number in the domain except 0, there are two values in the range.

EXAMPLE 2 Determining the equation of the inverse of a quadratic function

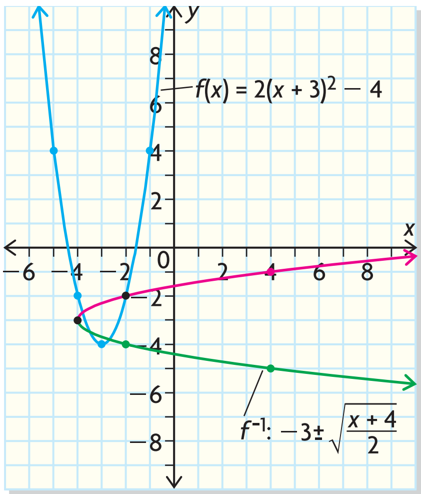
Given the quadratic function $f(x) = 2(x + 3)^2 - 4$,

a) graph $f(x)$ and its inverse

b) determine the equation of the inverse

Prashant's Solution

a)



I graphed $f(x)$ by plotting the vertex, $(-3, -4)$. The parabola opens up because the value of a is positive. I found $f(-2) = -2$ and $f(-1) = 4$, which are also the same values of $f(-4)$ and $f(-5)$, respectively.

To graph the inverse, I interchanged the x- and y-coordinates of the points on the original function.

b)

$$f(x) = 2(x + 3)^2 - 4$$
$$y = 2(x + 3)^2 - 4$$
$$x = 2(y + 3)^2 - 4$$

First I wrote the equation with y replacing $f(x)$, because y represents the output value in the function. To find the equation of the inverse, I interchanged x and y in the original function.

$$x + 4 = 2(y + 3)^2$$
$$\frac{x + 4}{2} = (y + 3)^2$$
$$\pm\sqrt{\frac{x + 4}{2}} = y + 3$$
$$-3 \pm \sqrt{\frac{x + 4}{2}} = y$$

I then rearranged the equation and solved for y by using the inverse of the operations given in the original function. I could tell from the graph of the inverse that there were two parts to the inverse function, an upper branch and a lower branch. The upper branch came from taking the positive square root of both sides, the lower from taking the negative square root.

For $f(x)$ restricted to $x \geq -3$,

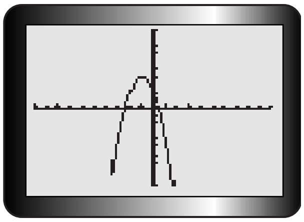
$$f^{-1}(x) = -3 + \sqrt{\frac{x + 4}{2}}$$

I couldn't write $f^{-1}(x)$ for y, since the inverse is not a function. But if I restricted the original domain to $x \geq -3$, then I would use only one branch of the inverse, and I could write it in function notation.

EXAMPLE 3 | **Using a graphing calculator as a strategy to graph a quadratic function and its inverse**

Using a graphing calculator, graph $f(x) = -2(x + 1)^2 + 4$ and its inverse.

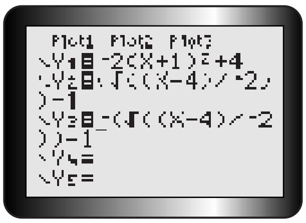
Bonnie's Solution



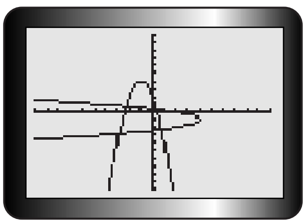
I entered the function into the equation editor at **Y1**. I used **WINDOW** settings $-10 \le x \le 10$, $-10 \le y \le 10$, because I knew the vertex was located at $(-1, 4)$ and the parabola opened down.

$$
\begin{aligned}
f(x) &= -2(x + 1)^2 + 4 \\
y &= -2(x + 1)^2 + 4 \\
x &= -2(y + 1)^2 + 4 \\
x - 4 &= -2(y + 1)^2 \\
\frac{x - 4}{-2} &= (y + 1)^2 \\
\pm\sqrt{\frac{x - 4}{-2}} &= y + 1 \\
\pm\sqrt{\frac{x - 4}{-2}} - 1 &= y
\end{aligned}
$$

To graph the inverse, I needed to find the equation of the inverse. I switched x and y and used inverse operations with the original function.



I entered both parts of the inverse separately. I entered $y = \sqrt{\frac{x - 4}{-2}} - 1$ into Y2 and $y = -\sqrt{\frac{x - 4}{-2}} - 1$ into Y3.



The calculator displayed the function and its inverse.

EXAMPLE 4 Solving a problem by using the inverse of a quadratic function

The rate of change in the surface area of a cell culture can be modelled by the function $S(t) = -0.005(t - 6)^2 + 0.18$, where $S(t)$ is the rate of change in the surface area in square millimetres per hour at time t in hours, and $0 \leq t \leq 12$.

a) State the domain and range of $S(t)$.

b) Determine the model that describes time in terms of the surface area.

c) Determine the domain and range of the new model.

Thomas' Solution

a) Domain $= \{t \in \mathbf{R} \mid 0 \leq t \leq 12\}$

The domain is given in the problem as part of the model.

Range $= \{S \in \mathbf{R} \mid 0 \leq S \leq 0.18\}$

This function is a parabola that opens down. The vertex is (6, 0.18), so the maximum value is 0.18. The surface area also cannot be negative, so 0 is the minimum value.

$$S = -0.005(t - 6)^2 + 0.18$$
$$S - 0.18 = -0.005(t - 6)^2$$
$$\frac{S - 0.18}{-0.005} = (t - 6)^2$$
$$\pm\sqrt{\frac{S - 0.18}{-0.005}} = t - 6$$
$$t = 6 \pm \sqrt{\frac{S - 0.18}{-0.005}}$$
$$t = 6 \pm \sqrt{\frac{-S + 0.18}{0.005}}$$

To find the inverse of the original function, I solved the given equation for t by using the inverse operations.

I did not interchange S and t in this case because S always means surface area and t always means time.

The domain and range of the new model:
Domain $= \{S \in \mathbf{R} \mid 0 \leq S \leq 0.18\}$

The value under the square root sign has to be positive, so the greatest value S can have is 0.18. For values greater than 0.18, the numerator would be positive, so the value under the square root would be negative.

Surface area cannot be less than zero, so S must be at least 0.

Range $= \{t \in \mathbf{R} \mid 0 \leq t \leq 12\}$

If $S = 0.18$, then the value of t is 6. If $S = 0$, then $t = 6 \pm 6$, so $t = 0$ or $t = 12$.

Therefore, the range values are between 0 and 12.

In Summary

Key Ideas

- The inverse of the original function undoes what the original function has done. It can be used to determine which values of the original dependent variable produce given values for the original independent variable.
- The inverse of a quadratic function defined over all the real numbers is not a function. It is a parabolic relation that opens either to the left or to the right. If the original quadratic opens up $(a > 0)$, the inverse opens to the right. If the original quadratic opens down $(a < 0)$, the inverse opens to the left.

Need to Know

- The equation of the inverse of a quadratic can be found by interchanging x and y in vertex form and solving for y.
- In the equation of the inverse of a quadratic function, the positive square root function represents the upper branch of the parabola, while the negative root represents the lower branch.
- The inverse of a quadratic function can be a function if the domain of the original function is restricted.

CHECK Your Understanding

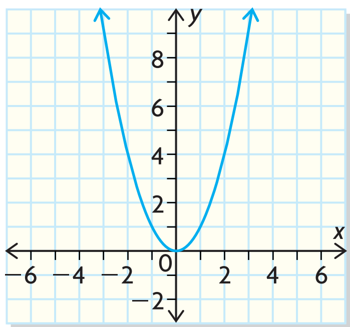
1. Each set of ordered pairs defines a parabola. Graph the relation and its inverse.

a) $\{(0, 0), (1, 3), (2, 12), (3, 27)\}$

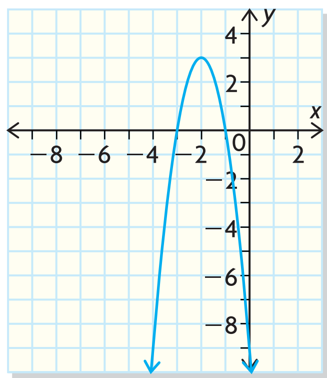
b) $\{(-3, -4), (-2, 1), (-1, 4), (0, 5), (1, 4), (2, 1), (3, -4)\}$

2. Given the graph of $f(x)$, graph the inverse relation.

a)



b)



3. Given $f(x) = 2x^2 - 1$, determine the equation of the inverse.

PRACTISING

4. Given $f(x) = 7 - 2(x - 1)^2$, $x \geq 1$, determine
   a) $f(3)$ b) $f^{-1}(x)$ c) $f^{-1}(5)$ d) $f^{-1}(2a + 7)$

5. a) Sketch the graph of $f(x) = 3(x - 2)^2 - 2$.
   b) Sketch the graph of its inverse on the same axes.

6. a) Graph $g(x) = -\sqrt{x}$ for $x \geq 0$.
   b) Graph its inverse on the same axes.
   c) State the domain and range of $g^{-1}(x)$.
   d) Determine the equation for $g^{-1}(x)$.

7. K Given $f(x) = -(x + 1)^2 - 3$ for $x \geq -1$, determine the equation for $f^{-1}(x)$. Graph the function and its inverse on the same axes.

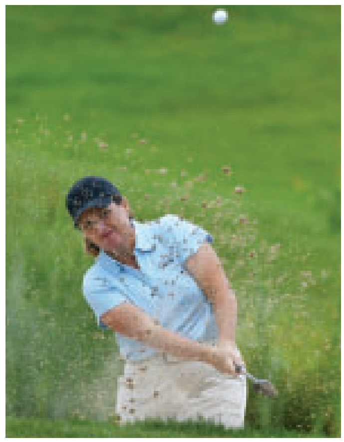
8. Given $f(x) = \frac{1}{2}(x - 5)^2 + 3$, find the equation for $f^{-1}(x)$ for the part of the function where $x \leq 5$. Use a graphing calculator to graph $f^{-1}(x)$.

9. For $-2 < x < 3$ and $f(x) = 3x^2 - 6x$, determine
   a) the domain and range of $f(x)$
   b) the equation of $f^{-1}(x)$ if $f(x)$ is further restricted to $1 < x < 3$

10. A The height of a ball thrown from a balcony can be modelled by the function $h(t) = -5t^2 + 10t + 35$, where $h(t)$ is the height above the ground, in metres, at time t seconds after it is thrown.
    a) Write $h(t)$ in vertex form.
    b) Determine the domain and range of $h(t)$.
    c) Determine the model that describes time in terms of the height.
    d) What are the domain and range of the new model?

11. The height of a golf ball after Lori Kane hits it is shown in the table.

Time (s)	0	0.5	1	1.5	2	2.5
Height (m)	0	12.375	22.5	30.375	36.0	39.375

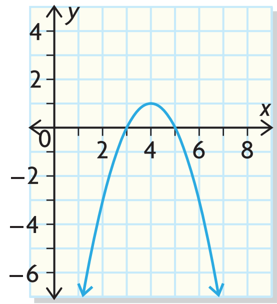


    a) Use first and second differences to extend the table.
    b) Graph the data and a curve of good fit for the relationship.
    c) Graph the inverse relation and its curve of good fit.
    d) Is the inverse a function? Explain.

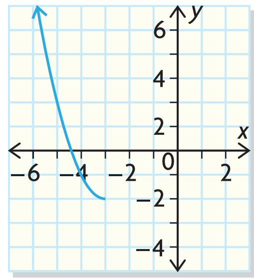
12. T Consider $f(x) = -2x^2 + 3x - 1$.
    a) Determine the vertex of the parabola.
    b) Graph $f(x)$.
    c) Graph $f^{-1}(x)$ for $y \geq 0.75$.
    d) Determine the domain and range of $f^{-1}(x)$ for $y \geq 0.75$.
    e) Why were the values of x restricted in parts (c) and (d)?

13. Each graph shows a function f that is a parabola or a branch of a parabola.

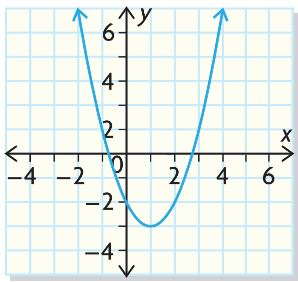
a)



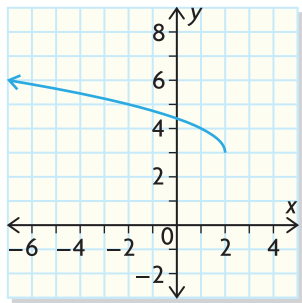
c)



b)



d)



i) Determine $f(x)$.

ii) Graph f^{-1}.

iii) State restrictions on the domain or range of f to make its inverse a function.

iv) Determine the equation(s) for f^{-1}.

14. What must happen for the inverse of a quadratic function defined over all the real numbers also to be a function?

15. C **a)** If you are given a quadratic function in standard form, explain how you could determine the equation of its inverse.

b) If the domain of the quadratic function is $\{x \in \mathbf{R}\}$, will its inverse be a function? Explain.

Extending

16. A meat department manager discovers that she can sell $m(x)$ kilograms of ground beef in a week, where $m(x) = 14\,700 - 3040x$, if she sells it at x dollars per kilogram. She pays her supplier \$3.21/kg for the beef.

a) Determine an algebraic expression for $P(x)$, where $P(x)$ represents the total profit in dollars for 1 week.

b) Find the equation for the inverse relation. Interpret its meaning.

c) Write an expression in function notation to represent the price that will earn \$1900 in profit. Evaluate and explain.

d) Determine the price that will maximize profit.

e) The supply cost drops to \$3.10/kg. What price should the manager set? How much profit will be earned at this price?

17. You are given the relation $x = 4 - 4y + y^2$.

a) Graph the relation.

b) Determine the domain and range of the relation.

c) Determine the equation of the inverse.

d) Is the inverse a function? Explain.

3.4 Operations with Radicals

GOAL

Simplify and perform operations on mixed and entire radicals.

INVESTIGATE the Math

The distance, $s(t)$, in millimetres of a particle from a certain point at any time, t, is given by $s(t) = 10\sqrt{4 + t}$. Don needs to find the exact distance between the point and the particle after 20 s. His answer must be in simplest form and he is not permitted to use a decimal.

? What is the exact value of $s(20)$, the distance between the particle and the given point at 20 s?

A. Copy and complete these products of **radicals**:

$\sqrt{25} \times \sqrt{4} = 5 \times 2 = 10$ $\qquad$ $\sqrt{25 \times 4} = \sqrt{100} = 10$

$\sqrt{16} \times \sqrt{9} =$ $\qquad$ $\sqrt{16 \times 9} = \sqrt{} =$

$\sqrt{4} \times \sqrt{36} =$ $\qquad$ $\sqrt{4 \times 36} = \sqrt{} =$

$\sqrt{100} \times \sqrt{9} =$ $\qquad$ $\sqrt{100 \times 9} = \sqrt{} =$

radical

a square, cube, or higher root, such as $\sqrt{4} = 2$ or $\sqrt[3]{27} = 3$; $\sqrt{}$ is called the radical symbol

B. Compare the results in each pair of products.

C. Consider $\sqrt{4} \times \sqrt{6}$. From the preceding results, express this product as

a) an **entire radical** $\qquad$ **b)** a **mixed radical**

Use your calculator to verify that your products are equivalent.

D. Determine $s(20)$. Use what you observed in parts A to C to simplify the expression so that your answer uses the smallest possible radical.

entire radical

a radical with coefficient 1; for example, $\sqrt{12}$

mixed radical

a radical with coefficient other than 1; for example, $2\sqrt{3}$

Reflecting

E. Determine $s(20)$ as a decimal. Why would the decimal answer for $\sqrt{24}$ not be considered exact?

F. To express $\sqrt{24}$ as a mixed radical, explain why using the factors $\sqrt{4} \times \sqrt{6}$ is a better choice than using $\sqrt{3} \times \sqrt{8}$.

G. If a and b are positive whole numbers, describe how $\sqrt{ab}$ is related to $\sqrt{a} \times \sqrt{b}$.

H. If $a > 0$, why is $b\sqrt{a}$ a simpler form of $\sqrt{ab^2}$?

APPLY the Math

EXAMPLE 1 Simplifying radicals by using a strategy involving perfect-square factors

Express each of the following as a mixed radical in lowest terms.

a) $\sqrt{72}$ **b)** $5\sqrt{27}$

Jasmine's Solution

a) $\sqrt{72} = \sqrt{36} \times \sqrt{2}$
$= 6\sqrt{2}$

I needed to find a perfect square number that divides evenly into 72. I could have chosen 4 or 9, but to put the mixed radical in lowest terms, I had to choose the greatest perfect square, which was 36. Once I expressed $\sqrt{72}$ as $\sqrt{36} \times \sqrt{2}$, I evaluated $\sqrt{36}$.

b) $5\sqrt{27} = 5 \times \sqrt{9} \times \sqrt{3}$
$= 5 \times 3 \times \sqrt{3}$
$= 15\sqrt{3}$

I found the largest perfect square that would divide evenly into 27; it was 9. I evaluated the square root of 9 and multiplied it by the coefficient 5.

EXAMPLE 2 Changing mixed radicals to entire radicals

Express each of the following as entire radicals.

a) $4\sqrt{5}$ **b)** $-6\sqrt{3}$

Sami's Solution

a) $4\sqrt{5} = 4 \times \sqrt{5}$
$= \sqrt{16} \times \sqrt{5}$
$= \sqrt{80}$

To create an entire radical, I had to change 4 into a square root. I expressed 4 as the square root of 16. Then I was able to multiply the numbers under the radical signs.

b) $-6\sqrt{3} = (-6) \times \sqrt{3}$
$= (-1) \times 6 \times \sqrt{3}$
$= (-1) \times \sqrt{36} \times \sqrt{3}$
$= -\sqrt{108}$

I knew that the negative sign would not go under the radical, since squares of real numbers are always positive. So I wrote -6 as the product of -1 and 6. I expressed 6 as $\sqrt{36}$ so that I could multiply the radical parts together to make an entire radical.

EXAMPLE 3 Multiplying radicals

Simplify.

a) $\sqrt{5} \times \sqrt{11}$

b) $-4\sqrt{6} \times 2\sqrt{6}$

Caleb's Solution

a) $\sqrt{5} \times \sqrt{11} = \sqrt{55}$

I multiplied the numbers under the radical signs together. 55 was not divisible by a perfect square, so my answer was in lowest terms.

b) $-4\sqrt{6} \times 2\sqrt{6} = (-4) \times 2 \times \sqrt{6} \times \sqrt{6}$

$= (-8)\sqrt{36}$

A mixed radical is the product of the integer and the radical, so I grouped together the integer products and the radical products.

$= (-8) \times 6$

$= -48$

Since 36 is a perfect square, I was able to simplify.

EXAMPLE 4 Adding radicals

In Don's research, he may also have to add expressions that contain radicals. Can he add radicals that are **like radicals**? What about other radicals?

like radicals

radicals that have the same number under the radical symbol, such as $3\sqrt{6}$ and $-2\sqrt{6}$

Marta's Solution

$\sqrt{3} \doteq 1.732$

$\sqrt{5} \doteq 2.236$

$\sqrt{3} + \sqrt{5} = 1.732 + 2.236$

$= 3.968$

$\sqrt{8} \doteq 2.828$

So $\sqrt{3} + \sqrt{5} \neq \sqrt{8}$.

I used my calculator to evaluate two radicals that were not like each other: $\sqrt{3}$ and $\sqrt{5}$. I rounded each value to 3 decimal places and then performed the addition. When I calculated $\sqrt{8}$, I found that it was not equal to $\sqrt{3} + \sqrt{5}$.

It looks like I cannot add radicals together if the numbers under the radical signs are different.

$3\sqrt{2} = \sqrt{2} + \sqrt{2} + \sqrt{2}$

$3\sqrt{2} + \sqrt{2} = \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2}$

$= 4\sqrt{2}$

Then I tried two like radicals. I used $3\sqrt{2}$ and $\sqrt{2}$. I expressed $3\sqrt{2}$ as the sum of three $\sqrt{2}$s. When I added $\sqrt{2}$ to this sum, I had 4 of them altogether, or $4\sqrt{2}$.

Also,

$3\sqrt{2} + \sqrt{2} = \sqrt{2}(3 + 1)$

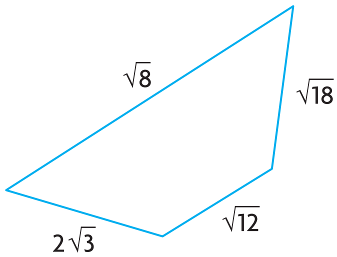
$= \sqrt{2} \times 4$

$= 4\sqrt{2}$

It makes sense that I can add like radicals by adding the integers in front of the radicals together.

EXAMPLE 5 Solving a problem involving radicals

Calculate the perimeter. Leave your answer in simplest radical form.



Robert's Solution

$$\begin{aligned}
P &= \sqrt{8} + 2\sqrt{3} + \sqrt{12} + \sqrt{18} \\
&= \sqrt{4} \times \sqrt{2} + 2\sqrt{3} + \sqrt{4} \times \sqrt{3} + \sqrt{9} \times \sqrt{2} \\
&= 2\sqrt{2} + 2\sqrt{3} + 2\sqrt{3} + 3\sqrt{2} \\
&= 2\sqrt{2} + 3\sqrt{2} + 2\sqrt{3} + 2\sqrt{3} \\
&= 5\sqrt{2} + 4\sqrt{3}
\end{aligned}$$

To find the perimeter I needed to add up the sides. I can add only like radicals. I factored the numbers I could by using perfect squares to see if any of these are like radicals.

I grouped, and then added the like radicals together.

EXAMPLE 6 Multiplying binomial radical expressions

Simplify $\left(3 - \sqrt{6}\right)\left(2 + \sqrt{24}\right)$.

Barak's Solution

$$\begin{aligned}
&\left(3 - \sqrt{6}\right)\left(2 + \sqrt{24}\right) \\
&= 6 + 3\sqrt{24} - 2\sqrt{6} - \sqrt{144} \\
&= 6 + 3(\sqrt{4} \times \sqrt{6}) - 2\sqrt{6} - 12 \\
&= 6 + 3(2\sqrt{6}) - 2\sqrt{6} - 12 \\
&= 6 - 12 + 6\sqrt{6} - 2\sqrt{6} \\
&= -6 + 4\sqrt{6}
\end{aligned}$$

I simplified this expression by first expanding the quantities in brackets.

After I multiplied the terms, I noticed that some of them could be simplified further. I factored $\sqrt{24}$ by using a perfect square. I then evaluated $\sqrt{144}$ and simplified.

I collected and combined like radicals.

In Summary

Key Idea

- Entire radicals can sometimes be simplified by expressing them as the product of two radicals, one of which contains a perfect square. This results in a mixed radical.
  - $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for $a \geq 0$, $b \geq 0$
  - $c\sqrt{a} \times d\sqrt{b} = cd\sqrt{ab}$ for $a \geq 0$, $b \geq 0$

Need to Know

- The only radicals that can be added or subtracted into a single term are like radicals.
- An answer containing a radical is an exact answer. An answer containing a decimal is an approximate answer.
- A mixed radical is in simplest form when the smallest possible number is written under the radical sign.

CHECK YOUR Understanding

1. Express each of these as mixed radicals in simplest form.

a) $\sqrt{27}$
b) $\sqrt{50}$
c) $\sqrt{98}$
d) $\sqrt{32}$

2. Simplify.

a) $\sqrt{5} \times \sqrt{7}$
b) $\sqrt{11} \times \sqrt{6}$
c) $2\sqrt{3} \times 5\sqrt{2}$
d) $-4\sqrt{3} \times 8\sqrt{13}$

3. Simplify.

a) $4\sqrt{5} + 3\sqrt{5}$
b) $9\sqrt{7} - 4\sqrt{7}$
c) $3\sqrt{3} + 8\sqrt{2} - 4\sqrt{3} + 11\sqrt{2}$
d) $\sqrt{8} - \sqrt{18}$

PRACTISING

4. Express as a mixed radical in simplest form.

a) $3\sqrt{12}$
b) $-5\sqrt{125}$
c) $10\sqrt{40}$
d) $-\frac{1}{2}\sqrt{60}$
e) $\frac{2}{3}\sqrt{45}$
f) $-\frac{9}{10}\sqrt{1200}$

5. Simplify.

a) $\sqrt{3}(2 - \sqrt{5})$
b) $2\sqrt{2}(\sqrt{7} + 3\sqrt{3})$
c) $(4\sqrt{2})^2$
d) $(-2\sqrt{3})^3$
e) $4\sqrt{3} \times 3\sqrt{6}$
f) $-7\sqrt{2} \times 5\sqrt{8}$

6. Simplify.
   a) $\sqrt{8} - \sqrt{32}$
   b) $\sqrt{12} + \sqrt{18} - \sqrt{27} + \sqrt{50}$
   c) $3\sqrt{98} - 5\sqrt{72}$
   d) $-4\sqrt{200} + 5\sqrt{242}$
   e) $-5\sqrt{45} + \sqrt{52} + 3\sqrt{125}$
   f) $7\sqrt{12} - 3\sqrt{28} + \frac{1}{2}\sqrt{48} + \frac{2}{3}\sqrt{63}$

7. Simplify.
   K a) $(6 - \sqrt{5})(3 + 2\sqrt{10})$
   b) $(2 + 3\sqrt{3})^2$
   c) $(\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5})$
   d) $(3\sqrt{3} + 4\sqrt{2})(\sqrt{3} - 2\sqrt{2})$
   e) $(2\sqrt{5} - 3\sqrt{7})^2$
   f) $(1 - \sqrt{3})(2 + \sqrt{6})(5 + \sqrt{2})$

For questions 8 to 12, calculate the exact values and express your answers is simplest radical form.

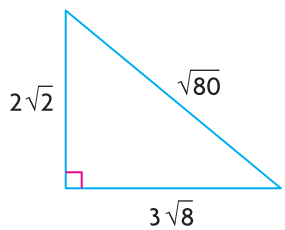
8. Calculate the length of the diagonal of a square with side length 4 cm.

9. A square has an area of 450 cm^2. Calculate the side length.

10. Determine the length of the diagonal of a rectangle with dimensions 3 cm $\times$ 9 cm.

11. Determine the length of the line segment from $A(-2, 7)$ to $B(4, 1)$.
A

12. Calculate the perimeter and area of this triangle.



13. If $a > 0$ and $b > 0$, which is greatest, $\left(\sqrt{a} + \sqrt{b}\right)^2$ or $\sqrt{a^2} + \sqrt{b^2}$?
T

14. Give three mixed radicals that are equivalent to $\sqrt{200}$. Which answer is in
C simplest radical form? Explain how you know.

Extending

15. Express each radical in simplest radical form.
   a) $\sqrt{a^3}$
   b) $\sqrt{x^5y^6}$
   c) $5\sqrt{n^7} - 2n\sqrt{n^5}$
   d) $(\sqrt{p} + 2\sqrt{q})(\sqrt{q} - \sqrt{p})$

16. Simplify $\sqrt{\sqrt{\sqrt{4096}}}$.

17. Solve $(\sqrt{2})^x = 256$ for x.

3 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How can you tell if a function is quadratic from its table of values? its graph? its equation?

A:
- If the second differences are constant, then the function is quadratic.
- If the graph is a parabola opening either up or down, then the function is quadratic.
- If the equation is of degree 2, the function is quadratic.

Study *Aid*
- See Lesson 3.1, Example 2.
- Try Mid-Chapter Review Questions 1 and 2.

Q: How can you determine the maximum or minimum value of a quadratic function?

A1: If the equation is in standard form $f(x) = ax^2 + bx + c$, complete the square to find the vertex, which is the maximum point if the parabola opens down $(a < 0)$ and the minimum point if the parabola opens up $(a > 0)$.

A2: If the equation is in factored form $f(x) = a(x - r)(x - s)$, find the x-intercepts of the function. The maximum or minimum value occurs at the x-value that is the average of the two x-intercepts.

A3: If two points are known that are the same distance from the vertex and opposite each other on the graph, then the maximum or minimum value occurs at the x-value that is the average of the x-coordinates of the two points.

A4: If the equation is in standard form and the values of the coefficients are decimals, graph the function on a graphing calculator and determine the maximum or minimum value.

Study *Aid*
- See Lesson 3.2, Examples 1 to 3.
- Try Mid-Chapter Review Questions 5 and 8.

Q: How can you determine the equation of the inverse of a quadratic function?

A: Interchange the values of x and y in the equation that defines the function, and then solve the new equation for y. Remember that when you take the square root of an expression, there are two possible solutions, one positive and one negative.

Study *Aid*
- See Lesson 3.3, Examples 2 and 3.
- Try Mid-Chapter Review Questions 9, 10, and 12.

Q: How do you know if a radical can be simplified?

A: A radical can be simplified if the number under the radical sign is divisible by a perfect-square number other than 1.

Study *Aid*
- See Lesson 3.4, Example 1.
- Try Mid-Chapter Review Question 13.

PRACTICE Questions

Lesson 3.1

1. For each table, calculate the second differences. Determine whether the function is quadratic.

a)

x	y
-2	-8
-1	-2
0	0
1	-2
2	-8

b)

x	y
-2	0
-1	1
0	4
1	9
2	16

2. Graph each function.

a) $f(x) = -3(x - 2)^2 + 5$
b) $f(x) = 2(x + 4)(x - 6)$

3. For each function in question 2, state the vertex, the equation of the axis of symmetry, and the domain and range.

4. Express each function in question 2 in standard form.

Lesson 3.2

5. Determine the maximum or minimum value of each quadratic function.

a) $f(x) = x^2 - 6x + 2$
b) $f(x) = 2(x - 4)(x + 6)$
c) $f(x) = -2x^2 + 10x$
d) $f(x) = 3.2x^2 + 15x - 7$

6. The profit function for a business is given by the equation $P(x) = -4x^2 + 16x - 7$, where x is the number of items sold, in thousands, and $P(x)$ is dollars in thousands. Calculate the maximum profit and how many items must be sold to achieve it.

7. The cost per hour of running an assembly line in a manufacturing plant is a function of the number of items produced per hour. The cost function is $C(x) = 0.3x^2 - 1.2x + 2$, where $C(x)$ is the cost per hour in thousands of dollars, and x is the number of items produced per hour, in thousands. Determine the most economical production level.

8. The sum of two numbers is 16. What is the largest possible product between these numbers?

Lesson 3.3

9. a) Determine the equation of the inverse of the quadratic function $f(x) = x^2 - 4x + 3$.
b) List the domain and range of $f(x)$ and its inverse.
c) Sketch the graphs of $f(x)$ and its inverse.

10. The revenue for a business is modelled by the function $R(x) = -2.8(x - 10)^2 + 15$, where x is the number of items sold, in thousands, and $R(x)$ is the revenue in thousands of dollars. Express the number sold in terms of the revenue.

11. Almost all linear functions have an inverse that is a function, but quadratic functions do not. Explain why?

12. Graph $f(x) = -\sqrt{x + 3}$ and determine

a) the domain and range of $f(x)$
b) the equation of f^{-1}

Lesson 3.4

13. Express each radical in simplest form.

a) $\sqrt{48}$
b) $\sqrt{68}$
c) $\sqrt{180}$
d) $-3\sqrt{75}$
e) $5\sqrt{98}$
f) $-8\sqrt{12}$

14. Simplify.

a) $\sqrt{7} \times \sqrt{14}$
b) $3\sqrt{5} \times 2\sqrt{15}$
c) $\sqrt{12} + 2\sqrt{48} - 5\sqrt{27}$
d) $3\sqrt{28} - 2\sqrt{50} + \sqrt{63} - 3\sqrt{18}$
e) $(4 - \sqrt{3})(5 + 2\sqrt{3})$
f) $(3\sqrt{5} + 2\sqrt{10})(-2\sqrt{5} + 5\sqrt{10})$

Curious Math

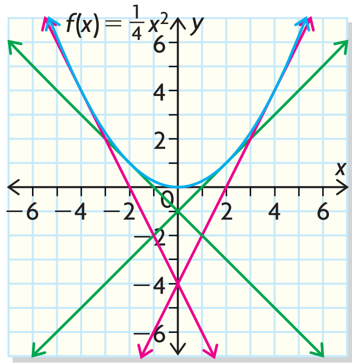
Investigating a Special Property of Parabolas

In a parabolic satellite reception dish, all of the signal rays are directed to a feed horn, which is positioned at a fixed point called the focus. You can locate this point by using math. Let the function $f(x) = \frac{1}{4}x^2$ model the shape of a satellite dish.

The graph of $f(x) = \frac{1}{4}x^2$ shows four tangent lines. (Each tangent line just touches the curve at one point.) You can use the tangent lines to determine the focus.

YOU WILL NEED

- ruler
- protractor



tangent lines

1. Mark these points on a copy of the graph: $(-4, 4)$, $(4, 4,)$, $(-2, 1)$, $(2, 1)$.
2. Draw a line parallel to the y-axis through each of the points.
3. Each vertical line represents a satellite signal ray, and the parabola represents the dish. Each signal ray will be reflected. With a protractor, measure the angle made between the ray and the tangent line.
4. The reflected ray will make the same angle with the tangent line. Use the protractor to draw the line representing the reflected ray.
5. Extend the lines representing the reflected ray until they cross the y-axis.
6. All reflected rays should pass through the same point on the y-axis, called the focus of the parabola. What point is the focus of this parabola?

3.5 Quadratic Function Models: Solving Quadratic Equations

YOU WILL NEED

- graphing calculator

GOAL

Solve problems involving quadratic functions in different ways.

LEARN ABOUT the Math

Anthony owns a business that sells parts for electronic game systems. The profit function for his business can be modelled by the equation $P(x) = -0.5x^2 + 8x - 24$, where x is the quantity sold, in thousands, and $P(x)$ is the profit in thousands of dollars.

? How many parts must Anthony sell in order for his business to break even?

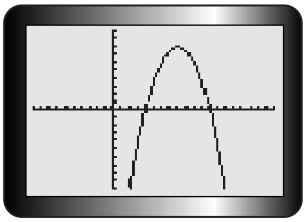
Tech *Support*

For help using the graphing calculator to graph functions and find their x-intercepts, see Technical Appendix, B-2 and B-8.

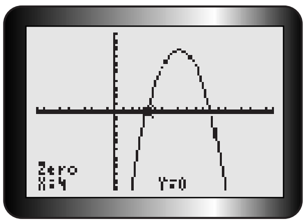
EXAMPLE 1 Selecting a strategy to solve a quadratic equation

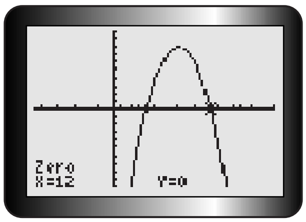
Determine the number of parts Anthony must sell to break even.

Derek's Solution: Using Graphing Technology



I graphed $P(x)$. Breaking even means that the profit is zero, so I looked for the x-intercepts of my function.





Anthony must sell 4000 parts or 12 000 parts.

I used the zero operation and found the x-intercepts from my graph. There were two possible values, at $x = 4$ and $x = 12$.

Because x is measured in thousands, I knew that the break-even values were 4000 and 12 000.

Tina's Solution: By Factoring

$$P(x) = 0$$
$$-0.5x^2 + 8x - 24 = 0$$
$$-0.5(x^2 - 16x + 48) = 0$$
$$-0.5(x - 4)(x - 12) = 0$$

If Anthony's business breaks even, the profit is zero.

I divided all the terms by the common factor -0.5. Inside the brackets was a simple trinomial that I could also factor.

$$x - 4 = 0 \quad \text{or} \quad x - 12 = 0$$
$$x = 4 \quad \text{or} \quad x = 12$$

I found the values of *x* that would give me zero in each bracket.

Anthony's business must sell 4000 parts or 12 000 parts to break even.

Since *x* is measured in thousands, my answer was 4000 or 12 000 parts.

Tracey's Solution: Using the Quadratic Formula

$$P(x) = 0$$
$$-0.5x^2 + 8x - 24 = 0$$

I needed to find the values of *x* that would make the profit function equal to zero. To solve the equation, I used the **quadratic formula**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-8 \pm \sqrt{(8)^2 - 4(-0.5)(-24)}}{2(-0.5)}$$
$$= \frac{-8 \pm \sqrt{64 - 48}}{-1}$$
$$= \frac{-8 \pm \sqrt{16}}{-1}$$

I substituted the values $a = -0.5$, $b = 8$, and $c = -24$ from the equation into the quadratic formula and simplified.

$$x = \frac{-8 + 4}{-1} \quad \text{or} \quad x = \frac{-8 - 4}{-1}$$
$$x = 4 \quad \text{or} \quad x = 12$$

Anthony must sell 4000 parts or 12 000 parts to break even.

My answer would have to be in thousands, since the number sold was in thousands.

Reflecting

A. How are the three methods for calculating the break-even points for Anthony's business the same? How are they different?

B. Will there always be two break-even points for a profit function? Why or why not?

C. If break-even points exist, which method may not work to determine where they are?

APPLY the Math

EXAMPLE 2 Solving a problem involving a quadratic equation

A water balloon is catapulted into the air from the top of a building. The height, $h(t)$, in metres, of the balloon after t seconds is $h(t) = -5t^2 + 30t + 10$.

a) What are the domain and range of this function?

b) When will the balloon reach a height of 30 m?

Brian's Solution

a) $h(t) = -5(t^2 - 6t) + 10$

The graph must be a parabola opening down because the value of *a* is negative.

$= -5(t^2 - 6t + 9 - 9) + 10$

$= -5(t^2 - 6t + 9) + 45 + 10$

$= -5(t - 3)^2 + 55$

To get the range, I found the vertex by completing the square. The vertex is (3, 55), so the maximum height is 55 m and the minimum height is 0.

Range $= \{h(t) \in \mathbf{R} \mid 0 \leq h(t) \leq 55\}$

$-5(t - 3)^2 + 55 = 0$

$-5(t - 3)^2 = -55$

$(t - 3)^2 = 11$

$t - 3 = \pm\sqrt{11}$

$t = 3 + \sqrt{11}$ or $t = 3 - \sqrt{11}$

$t = 3 + 3.32$ or $t = 3 - 3.32$

$t = 6.32$ or $t = -0.32$

The domain is the interval of time the balloon was in flight. It stops when it hits the ground. The time *t* must be greater than 0. I calculated the values of *t* that would make the height 0.

Domain $= \{t \in \mathbf{R} \mid 0 \leq t \leq 6.32\}$

One value of *t* is negative, so the domain must start at 0 and go to the positive value of *t* that I found.

b) $30 = -5t^2 + 30t + 10$

To know when the ball reached 30 m, I replaced $h(t)$ with 30 and solved for t.

$$0 = -5t^2 + 30t - 20$$

$$t = \frac{-30 \pm \sqrt{(30)^2 - 4(-5)(-20)}}{2(-5)}$$

I used the quadratic formula to solve for t.

$$= \frac{-30 \pm \sqrt{500}}{-10}$$

$$\doteq \frac{-30 \pm 22.36}{-10}$$

$$t \doteq \frac{-30 + 22.36}{-10} \quad \text{or} \quad t \doteq \frac{-30 - 22.36}{-10}$$

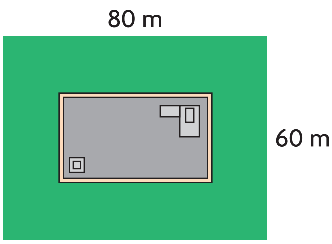
$$t \doteq 0.764 \quad \text{or} \quad t \doteq 5.236$$

The ball will reach a height of 30 m after 0.764 s or 5.236 s.

Both answers are possible in this question. The ball reaches 30 m going up and again coming down.

EXAMPLE 3 Representing and solving a problem by using a quadratic equation

A factory is to be built on a lot that measures 80 m by 60 m. A lawn of uniform width, equal to the area of the factory, must surround it. How wide is the strip of lawn, and what are the dimensions of the factory?



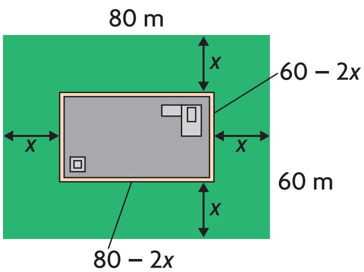
Communication *Tip*

Any solution of an equation that does not work in the context of a problem is said to be an inadmissible solution.

Rachael's Solution

Let the width of the lawn be x metres.

I chose a variable for the width of the lawn.



The dimensions of the factory are $(60 - 2x)$ m and $(80 - 2x)$ m.

Since the lawn was the same width all around, I had to subtract $2x$ from 60 and $2x$ from 80 to get the dimensions of the factory.

$$\begin{aligned}\text{Area of factory} &= \text{length} \times \text{width}\\ &= (60 - 2x)(80 - 2x)\\ &= 4800 - 120x - 160x + 4x^2\\ &= 4800 - 280x + 4x^2\end{aligned}$$

I wrote down and simplified an expression for the area of the factory.

$$\begin{aligned}\text{Area of lawn} &= \text{Area of lot} - \text{Area of factory}\\ &= 4800 - (4800 - 280x + 4x^2)\\ &= -4x^2 + 280x\end{aligned}$$

The area of the lawn is the difference between the area of the lot and the area of the factory.

$$-4x^2 + 280x = 4800 - 280x + 4x^2$$
$$-8x^2 + 560x - 4800 = 0$$
$$-8(x^2 - 70x + 600) = 0$$
$$-8(x - 60)(x - 10) = 0$$
$$x = 60 \quad \text{or} \quad x = 10$$

The area of the lawn is equal to the area of the factory, so I set the two expressions equal to each other. This equation was quadratic, so I rearranged it so that it was equal to zero and solved it by factoring.

But $x = 60$ is inadmissible in this problem, so $x = 10$.

I found two possible values of x. Since a width of 60 for the strip made the dimensions of the factory negative, the width of the lawn had to be 10 m.

$$60 - 2(10) = 40$$
$$80 - 2(10) = 60$$

The lawn is 10 m wide, and the dimensions of the factory are 60 m by 40 m.

I then substituted $x = 10$ into the expressions for the length and width of the factory to find its dimensions.

In Summary

Key Idea

- All quadratic equations can be expressed in the form $ax^2 + bx + c = 0$ by algebraic techniques. The equations can be solved in a number of ways.

Need to Know

- Quadratic equations can be solved by graphing the corresponding functions $f(x) = ax^2 + bx + c$ and locating the x-intercepts, or zeros, either by hand or with technology. These zeros are the solutions or roots of the equation $ax^2 + bx + c = 0$.
- Quadratic equations can also be solved
  - by factoring
  - with the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Depending on the problem and the degree of accuracy required, the solutions of a quadratic equation may be expressed exactly by using radicals or rational numbers, or approximately with decimals.

CHECK Your Understanding

1. Determine the roots of each equation by factoring.

a) $x^2 + 5x + 4 = 0$
b) $x^2 - 11x + 18 = 0$
c) $4x^2 - 9 = 0$
d) $2x^2 - 7x - 4 = 0$

2. Use the quadratic formula to determine each of the roots to two decimal places.

a) $x^2 - 4x - 9 = 0$
b) $3x^2 + 2x - 8 = 0$
c) $-2x^2 + 3x - 6 = 0$
d) $0.5x^2 - 2.2x - 4.7 = 0$

3. Use a graphing calculator to solve each equation.

a) $0 = -4x^2 - 5x - 1$
b) $0 = 2x^2 - 11x + 9$

PRACTISING

4. K **i)** For each equation, decide on a strategy to solve it and explain why you chose that strategy.

ii) Use your strategy to solve the equation. When appropriate, leave your answer in simplest radical form.

a) $2x^2 - 3x = x^2 + 7x$
b) $4x^2 + 6x + 1 = 0$
c) $x^2 + 4x - 3 = 0$
d) $(x + 3)^2 = -2x$
e) $3x^2 - 5x = 2x^2 + 4x + 10$
f) $2(x + 3)(x - 4) = 6x + 6$

5. Locate the x-intercepts of the graph of each function.

a) $f(x) = 3x^2 - 7x - 2$
b) $f(x) = -4x^2 + 25x - 21$

6. Determine the break-even quantities for each profit function, where x is the number sold, in thousands.

a) $P(x) = -x^2 + 12x + 28$

b) $P(x) = -2x^2 + 18x - 40$

c) $P(x) = -2x^2 + 22x - 17$

d) $P(x) = -0.5x^2 + 6x - 5$

7. **A** The flight of a ball hit from a tee that is 0.6 m tall can be modelled by the function $h(t) = -4.9t^2 + 6t + 0.6$, where $h(t)$ is the height in metres at time t seconds. How long will it take for the ball to hit the ground?

8. The population of a region can be modelled by the function $P(t) = 0.4t^2 + 10t + 50$, where $P(t)$ is the population in thousands and t is the time in years since the year 1995.

a) What was the population in 1995?

b) What will be the population in 2010?

c) In what year will the population be at least 450 000? Explain your answer.

9. A rectangle has an area of 330 m^2. One side is 7 m longer than the other. What are the dimensions of the rectangle?

10. The sum of the squares of two consecutive integers is 685. What could the integers be? List all possibilities.

11. A right triangle has a height 8 cm more than twice the length of the base. If the area of the triangle is 96 cm^2, what are the dimensions of the triangle?

12. **T** Jackie mows a strip of uniform width around her 25 m by 15 m rectangular lawn and leaves a patch of lawn that is 60% of the original area. What is the width of the strip?

13. A small flare is launched off the deck of a ship. The height of the flare above the water is given by the function $h(t) = -4.9t^2 + 92t + 9$, where $h(t)$ is measured in metres and t is time in seconds.

a) When will the flare's height be 150 m?

b) How long will the flare's height be above 150 m?

14. A bus company has 4000 passengers daily, each paying a fare of \$2. For each \$0.15 increase, the company estimates that it will lose 40 passengers per day. If the company needs to take in \$10 450 per day to stay in business, what fare should be charged?

15. **C** Describe three possible ways that you could determine the zeros of the quadratic function $f(x) = -2x^2 + 14x - 24$.

Extending

16. The perimeter of a right triangle is 60 cm. The length of the hypotenuse is 6 cm more than twice the length of one of the other sides. Find the lengths of all three sides.

17. Find the zeros of the function $f(x) = 3x - 1 + \frac{1}{x + 1}$.

3.6 The Zeros of a Quadratic Function

GOAL

Use a variety of strategies to determine the number of zeros of a quadratic function.

LEARN ABOUT the Math

Samantha has been asked to predict the number of zeros for each of three quadratic functions without using a graphing calculator. Samantha knows that quadratics have 0, 1, or 2 zeros. The three functions are:

$$f(x) = -2x^2 + 12x - 18$$
$$g(x) = 2x^2 + 6x - 8$$
$$h(x) = x^2 - 4x + 7$$

? How can Samantha predict the number of zeros each quadratic has without graphing?

EXAMPLE 1 Connecting functions to their graphs

Determine the properties of each function that will help you determine the number of x-intercepts each has.

Tara's Solution: Using Properties of the Quadratic Function

$$f(x) = -2x^2 + 12x - 18$$
$$f(x) = -2(x^2 - 6x + 9)$$
$$= -2(x - 3)^2$$

Vertex is (3, 0) and the parabola opens down. This function has one zero.

I decided to find the vertex of the first function. I factored -2 out as a common factor. The trinomial that was left was a perfect square, so I factored it. This put the function in vertex form. Because the vertex is on the x-axis, there is only one zero.

$$g(x) = 2x^2 + 6x - 8$$
$$g(x) = 2(x^2 + 3x - 4)$$
$$= 2(x + 4)(x - 1)$$

This function has two zeros, at $x = -4$ and $x = 1$.

I factored 2 out as a common factor in the second function, then factored the trinomial inside the brackets. I used the factors to find the zeros, so this function has two.

$$\begin{aligned} h(x) &= x^2 - 4x + 7 \\ &= (x^2 - 4x + 4 - 4) + 7 \\ &= (x^2 - 4x + 4) - 4 + 7 \\ &= (x - 2)^2 + 3 \end{aligned}$$

The vertex is (2, 3) and the parabola opens up. This function has no zeros.

This function would not factor, so I found the vertex by completing the square. The vertex is above the x-axis, and the parabola opens up because a is positive. Therefore, the function has no zeros.

Asad's Solution: Using the Quadratic Formula

$$\begin{aligned} f(x) &= -2x^2 + 12x - 18 \\ 0 &= -2x^2 + 12x - 18 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-12 \pm \sqrt{(12)^2 - 4(-2)(-18)}}{2(-2)} \\ &= \frac{-12 \pm \sqrt{144 - 144}}{-4} \\ &= \frac{-12 \pm \sqrt{0}}{-4} \\ &= \frac{-12}{-4} \\ &= 3 \end{aligned}$$

The zeros or x-intercepts occur when each function equals 0. I set $f(x) = 0$, then solved the resulting equation using the quadratic formula with $a = -2$, $b = 12$, and $c = -18$.

The first function has only one value for the x-intercept, so there is only one zero. This can be seen from the value of the **discriminant** (the quantity under the radical sign), which is zero.

This function has one zero.

$$\begin{aligned} g(x) &= 2x^2 + 6x - 8 \\ 0 &= 2x^2 + 6x - 8 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{(6)^2 - 4(2)(-8)}}{2(2)} \\ &= \frac{-6 \pm \sqrt{36 + 64}}{4} \end{aligned}$$

I used the quadratic formula again with $a = 2$, $b = 6$, and $c = -8$. There were two solutions, since the discriminant was positive. So the function has two zeros.

$$= \frac{-6 \pm \sqrt{100}}{4}$$

$$x = \frac{-6 - 10}{4} \quad \text{or} \quad x = \frac{-6 + 10}{4}$$

$$x = -4 \quad \text{or} \quad x = 1$$

This function has two zeros.

$$h(x) = x^2 - 4x + 7$$

I used the quadratic formula with $a = 1$, $b = -4$, and $c = 7$. The discriminant was negative, so there were no real-number solutions. The function has no zeros.

$$0 = x^2 - 4x + 7$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{(-4)^2 - 4(1)(7)}}{2}$$

$$= \frac{4 \pm \sqrt{16 - 28}}{2}$$

$$= \frac{4 \pm \sqrt{-12}}{2}$$

This function has no zeros.

Reflecting

A. Describe the possibilities for the number of zeros of a quadratic function.

B. How can finding the vertex help determine the number of zeros?

C. Why is the factored form useful in determining the number of zeros of a quadratic function?

D. Explain how the quadratic formula can be used to predict the number of zeros of a quadratic function.

APPLY the Math

EXAMPLE 2 Using the discriminant to determine the number of zeros

Find the value of the discriminant to determine the number of zeros of each quadratic function.

a) $f(x) = 2x^2 - 3x - 5$

b) $g(x) = 4x^2 + 4x + 1$

c) $h(x) = -5x^2 + x - 2$

Larry's Solution

a) $b^2 - 4ac = (-3)^2 - 4(2)(-5)$

$= 9 + 40$

$= 49$

Since $49 > 0$, there are two distinct zeros.

The discriminant $b^2 - 4ac$ is the value under the square root sign in the quadratic formula.

If $b^2 - 4ac$ is a positive number, the function has two zeros.

b) $b^2 - 4ac = (4)^2 - 4(4)(1)$

$= 16 - 16$

$= 0$

Therefore, there is one zero.

This time, the discriminant is equal to zero, so the function has only one zero.

c) $b^2 - 4ac = (1)^2 - 4(-5)(-2)$

$= 1 - 40$

$= -39$

Since $-39 < 0$, there are no zeros.

In this function, $b^2 - 4ac$ is a negative number, so the function has no zeros.

EXAMPLE 3 Solving a problem involving a quadratic function with one zero

Determine the value of k so that the quadratic function $f(x) = x^2 - kx + 3$ has only one zero.

Ruth's Solution

$$b^2 - 4ac = 0$$
$$(-k)^2 - 4(1)(3) = 0$$
$$k^2 - 12 = 0$$
$$k^2 = 12$$
$$k = \pm\sqrt{12}$$
$$k = \pm 2\sqrt{3}$$

If there is only one zero, then the discriminant is zero. I put the values for a, b, and c into the equation $b^2 - 4ac = 0$ and solved for k.

Since I had to take the square root of both sides to solve for k, there were two possible values. I expressed the values of k using a mixed radical in simplest form.

EXAMPLE 4 Solving a problem by using the discriminant

A market researcher predicted that the profit function for the first year of a new business would be $P(x) = -0.3x^2 + 3x - 15$, where x is based on the number of items produced. Will it be possible for the business to break even in its first year?

Raj's Solution

$$P(x) = 0$$
$$-0.3x^2 + 3x - 15 = 0$$

At a break-even point, the profit is zero.

$$b^2 - 4ac = (3)^2 - 4(-0.3)(-15)$$
$$= 9 - 18$$
$$= -9$$

I just wanted to know if there was a break-even point and not what it was, so I only needed to know if the profit function had any zeros. I used the value of the discriminant to decide.

Since $b^2 - 4ac < 0$, there are no zeros for this function. Therefore, it is not possible for the business to break even in its first year.

In Summary

Key Idea

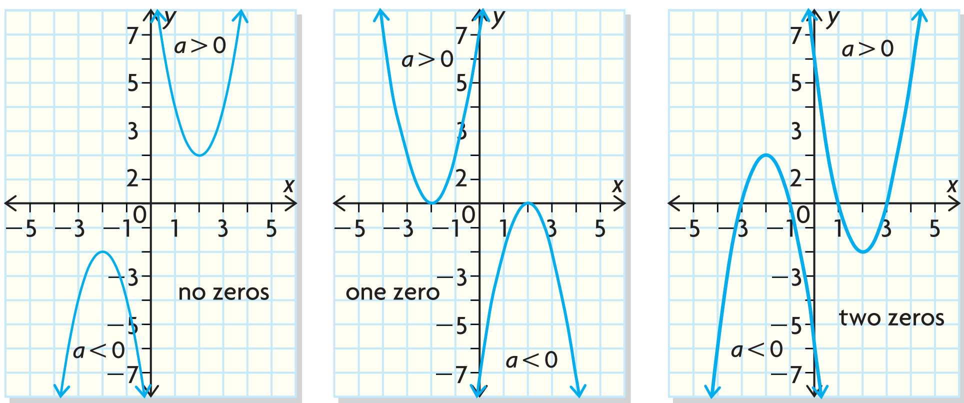
- A quadratic function can have 0, 1, or 2 zeros. You can determine the number of zeros either by graphing or by analyzing the function.

Need to Know

- The number of zeros of a quadratic function can be determined by looking at the graph of the function and finding the number of x-intercepts.
- For a quadratic equation $ax^2 + bx + c = 0$ and its corresponding function $f(x) = ax^2 + bx + c$, see the table below:

Value of the Discriminant	Number of Zeros/Solutions
$b^2 - 4ac > 0$	2
$b^2 - 4ac = 0$	1
$b^2 - 4ac < 0$	0

- The number of zeros can be determined by the location of the vertex relative to the x-axis, and the direction of opening:
  - If $a > 0$, and the vertex is above the x-axis, there are no zeros.
  - If $a > 0$, and the vertex is below the x-axis, there are two zeros.
  - If $a < 0$, and the vertex is above the x-axis, there are two zeros.
  - If $a < 0$, and the vertex is below the x-axis, there are no zeros.
  - If the vertex is on the x-axis, there is one zero.



CHECK Your Understanding

1. Determine the vertex and the direction of opening for each quadratic function. Then state the number of zeros.
   a) $f(x) = 3x^2 - 5$
   b) $f(x) = -4x^2 + 7$
   c) $f(x) = 5x^2 + 3$
   d) $f(x) = 3(x + 2)^2$
   e) $f(x) = -4(x + 3)^2 - 5$
   f) $f(x) = 0.5(x - 4)^2 - 2$
2. Factor each quadratic function to determine the number of zeros.
   a) $f(x) = x^2 - 6x - 16$
   b) $f(x) = 2x^2 - 6x$
   c) $f(x) = 4x^2 - 1$
   d) $f(x) = 9x^2 + 6x + 1$
3. Calculate the value of $b^2 - 4ac$ to determine the number of zeros.
   a) $f(x) = 2x^2 - 6x - 7$
   b) $f(x) = 3x^2 + 2x + 7$
   c) $f(x) = x^2 + 8x + 16$
   d) $f(x) = 9x^2 - 14.4x + 5.76$

PRACTISING

4. **K** Determine the number of zeros. Do not use the same method for all four parts.
   a) $f(x) = -3(x - 2)^2 + 4$
   b) $f(x) = 5(x - 3)(x + 4)$
   c) $f(x) = 4x^2 - 2x$
   d) $f(x) = 3x^2 - x + 5$
5. **A** For each profit function, determine whether the company can break even. If the company can break even, determine in how many ways it can do so.
   a) $P(x) = -2.1x^2 + 9.06x - 5.4$
   b) $P(x) = -0.3x^2 + 2x - 7.8$
   c) $P(x) = -2x^2 + 6.4x - 5.12$
   d) $P(x) = -2.4x^2 + x - 1.2$
6. For what value(s) of k will the function $f(x) = 3x^2 - 4x + k$ have one x-intercept?
7. For what value(s) of k will the function $f(x) = kx^2 - 4x + k$ have no zeros?
8. For what values of k will the function $f(x) = 3x^2 + 4x + k = 0$ have no zeros? one zero? two zeros?
9. The graph of the function $f(x) = x^2 - kx + k + 8$ touches the x-axis at one point. What are the possible values of k?
10. Is it possible for $n^2 + 25$ to equal $-8n$? Explain.
11. Write the equation of a quadratic function that meets each of the given conditions.
    a) The parabola opens down and has two zeros.
    b) The parabola opens up and has no zeros.
    c) The parabola opens down and the vertex is also the zero of the function.

12. The demand function for a new product is $p(x) = -4x + 42.5$, where x is the quantity sold in thousands and p is the price in dollars. The company that manufactures the product is planning to buy a new machine for the plant. There are three different types of machine. The cost function for each machine is shown.

Machine A: $C(x) = 4.1x + 92.16$

Machine B: $C(x) = 17.9x + 19.36$

Machine C: $C(x) = 8.8x + 55.4$

Investigate the break-even quantities for each machine. Which machine would you recommend to the company?

13. Describe how each transformation or sequence of transformations of the function $f(x) = 3x^2$ will affect the number of zeros the function has.
   a) a vertical stretch of factor 2
   b) a horizontal translation 3 units to the left
   c) a horizontal compression of factor 2 and then a reflection in the x-axis
   d) a vertical translation 3 units down
   e) a horizontal translation 4 units to the right and then a vertical translation 3 units up
   f) a reflection in the x-axis, then a horizontal translation 1 unit to the left, and then a vertical translation 5 units up

14. **T** If $f(x) = x^2 - 6x + 14$ and $g(x) = -x^2 - 20x - k$, determine the value of k so that there is exactly one point of intersection between the two parabolas.

15. Determine the number of zeros of the function $f(x) = 4 - (x - 3)(3x + 1)$ without solving the related quadratic equation or graphing. Explain your thinking.

16. **C** Describe how you can determine the number of zeros of a quadratic function if the equation of the function is in
   a) vertex form b) factored form c) standard form

Extending

17. Show that $(x^2 - 1)k = (x - 1)^2$ has one solution for only one value of k.

18. Investigate the number of zeros of the function $f(x) = (k + 1)x^2 + 2kx + k - 1$ for different values of k. For what values of k does the function have no zeros? one zero? two zeros?

3.7 Families of Quadratic Functions

GOAL

Determine the properties of families of quadratic functions.

INVESTIGATE the Math

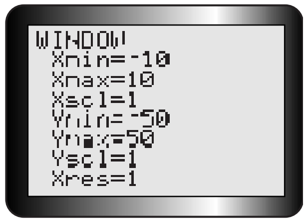
Equations that define quadratic functions can look quite different, yet their graphs can have similar characteristics.

Group 1	Group 2	Group 3
$f(x) = x^2 - 3x - 10$	$m(x) = -2x^2 + 4x + 1$	$r(x) = -3x^2 + 5x - 2$
$g(x) = -2x^2 + 6x + 20$	$n(x) = 0.5x^2 - 1x + 3.5$	$s(x) = 2x^2 + x - 2$
$h(x) = 4x^2 - 12x - 40$	$p(x) = -6x^2 + 12x - 3$	$t(x) = 7x^2 - 2x - 2$
$k(x) = -0.5x^2 + 1.5x + 5$	$q(x) = 10x^2 - 20x + 13$	$u(x) = -4x^2 - 4x - 2$

? What characteristics do the graphs in each of these groups have in common?

A. Graph each of the functions in Group 1 on a graphing calculator. Use the window settings shown. How are the graphs the same? How are they different?

A.



B. Write each of the functions in Group 1 in factored form. What do you notice?

C. Clear all functions, and then graph each of the functions for Group 2 on a graphing calculator. Use the window settings shown.

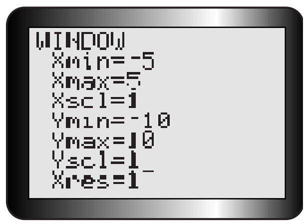
C.

D. How are the graphs the same? How are they different?

E. Write each of the functions in Group 2 in vertex form. What do you notice?

F. Clear all functions, and then graph each of the functions in Group 3 on a graphing calculator. Use the window settings shown. What do these functions have in common?

F.



G. Summarize your findings for each group.

family of parabolas

a group of parabolas that all share a common characteristic

Reflecting

H. Each of the three groups of functions forms a **family of parabolas.** Describe the common characteristics of each of the groups.

I. In each of Groups 1 and 2, what single value was varied to create the family? What transformation is this parameter associated with?

J. What common characteristic appears in all quadratic functions in the same family if the equation is in factored form? vertex form? standard form?

APPLY the Math

EXAMPLE 1 Looking for quadratics that share a vertex

Given the function $f(x) = -3(x + 2)^2 - 1$, determine another quadratic function with the same vertex.

Ian's Solution

$f(x) = -3(x + 2)^2 - 1$

I identified the vertex.

Vertex is $(-2, -1)$.

Family of parabolas is of the form

$f(x) = a(x + 2)^2 - 1$

To get another quadratic function with the same vertex, I needed to change the value of *a* because parabolas with the same vertex are vertically stretched or compressed, but not horizontally or vertically translated.

So another quadratic in the family is $g(x) = 2(x + 2)^2 - 1$.

EXAMPLE 2 Determining a specific equation of a member of the family

Determine the equation of the quadratic function that passes through $(-3, 20)$ if its zeros are 2 and -1.

Preet's Solution

$$\begin{aligned} f(x) &= a(x - 2)[x - (-1)] \\ &= a(x - 2)(x + 1) \\ &= a[x^2 - 2x + x - 2] \\ &= a(x^2 - x - 2) \end{aligned}$$

I wrote the general function of all parabolas that have zeros at 2 and -1, then simplified by expanding.

$f(x) = a(x^2 - x - 2)$

$20 = a[(-3)^2 - (-3) - 2)]$

$20 = 10a$

$a = 2$

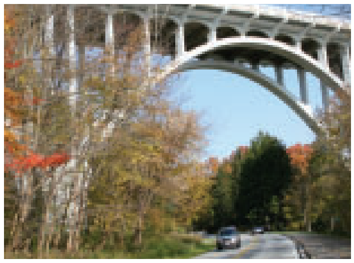
Therefore, $f(x) = 2(x^2 - x - 2)$.

To determine the equation passing through $(-3, 20)$, I had to find the correct value of a.

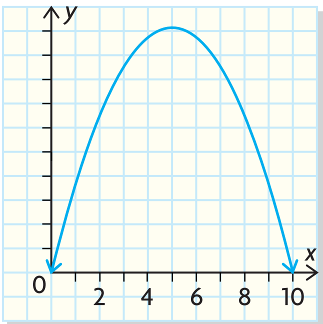
I substituted the point into the equation and solved for a.

EXAMPLE 3 Solving a problem by applying knowledge of the vertex form of a quadratic

A highway overpass has a shape that can be modelled by the equation of a parabola. If the edge of the highway is the origin and the highway is 10 m wide, what is the equation of the parabola if the height of the overpass 2 m from the edge of the highway is 13 m?



Elizabeth's Solution



I drew a sketch. If the edge of the highway is at the origin, then one of the zeros is 0. If the highway is 10 m wide, then the other zero is at (10, 0).

$h = ax(x - 10)$

Since I had the zeros, I wrote the equation in factored form. The equation that would model the overpass would be in the same family, so I needed to find the value of a.

$13 = a(2)(2 - 10)$

$13 = -16a$

$a = -\frac{13}{16}$

(2, 13) is a point on the curve, so I substituted those values into the equation and solved for a. Once I had the value of a, I wrote the equation in factored form.

Therefore, the equation that models the overpass is

$h = -\frac{13}{16}x(x - 10)$

EXAMPLE 4 Selecting a strategy to determine the quadratic function from data

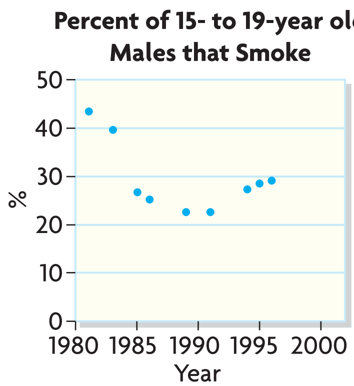
The percent of 15- to 19-year-old males who smoke has been tracked by Health Canada. The data from 1981 to 1996 are given in the table.

Year	1981	1983	1985	1986	1989	1991	1994	1995	1996
Smokers (%)	43.4	39.6	26.7	25.2	22.6	22.6	27.3	28.5	29.1

a) Draw a scatter plot of the data.
b) Draw a curve of good fit.
c) Estimate the location of the vertex.
d) Determine a quadratic function that will model the data.

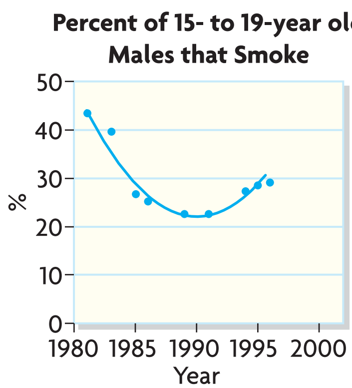
Bryce's Solution

a)



I plotted the points on a graph. They could be represented by a quadratic function.

b)



I drew a curve of good fit by hand so that I could estimate the vertex. Since the values were 22.6 in both 1989 and 1991, I used (1990, 22) as my estimated vertex.

c) The graph models a parabola with the vertex above the x-axis. Estimated vertex: (1990, 22)

I used vertex form with the vertex I knew. I needed to find the value of a to approximate the data. I chose the point (1995, 28.5) as the point on the curve. I substituted the point into the equation and solved for a.

d)
$$\begin{aligned}
f(x) &= a(x - 1990)^2 + 22\\
28.5 &= a(1995 - 1990)^2 + 22\\
28.5 &= a(-5)^2 + 22\\
28.5 &= 25a + 22\\
6.5 &= 25a\\
\frac{65}{25} &= a\\
a &\doteq 0.26
\end{aligned}$$

Therefore, a model for the data is $f(x) = 0.26(x - 1990)^2 + 22$.

In Summary

Key Ideas

- If the value of a is varied in a quadratic function expressed in vertex form, $f(x) = a(x - h)^2 + k$, a family of parabolas with the same vertex and axis of symmetry is created.
- If the value of a is varied in a quadratic function in factored form, $f(x) = a(x - r)(x - s)$, a family of parabolas with the same x-intercepts and axis of symmetry is created.
- If the values of a and b are varied in a quadratic function expressed in standard form, $f(x) = ax^2 + bx + c$, a family of parabolas with the same y-intercept is created.

Need to Know

- The algebraic model of a quadratic function can be determined algebraically.
  - If the zeros are known, write in factored form with a unknown, substitute another known point, and solve for a.
  - If the vertex is known, write in vertex form with a unknown, substitute a known point, and solve for a.

CHECK Your Understanding

1. What characteristics will two parabolas in the family $f(x) = a(x - 3)(x + 4)$ share?

2. How are the parabolas $f(x) = -3(x - 2)^2 - 4$ and $g(x) = 6(x - 2)^2 - 4$ the same? How are they different?

3. What point do the parabolas $f(x) = -2x^2 + 3x - 7$ and $g(x) = 5x^2 + 3x - 7$ have in common?

PRACTISING

4. Determine the equation of the parabola with x-intercepts
a) -4 and 3, and that passes through (2, 7)
b) 0 and 8, and that passes through $(-3, -6)$
c) $\sqrt{7}$ and $-\sqrt{7}$, and that passes through $(-5, 3)$
d) $1 - \sqrt{2}$ and $1 + \sqrt{2}$, and that passes through (2, 4)

5. Determine the equation of the parabola with vertex
a) $(-2, 5)$ and that passes through $(4, -8)$
b) (1, 6) and that passes through $(0, -7)$
c) $(4, -5)$ and that passes through $(-1, -3)$
d) (4, 0) and that passes through (11, 8)

6. Determine the equation of the quadratic function $f(x) = ax^2 - 6x - 7$ if $f(2) = 3$.

7. a) Sketch the graph of $f(x) = (x - 2)(x + 6)$.
b) Use your graph to sketch the graph of $g(x) = -2(x - 2)(x + 6)$.
c) Sketch the graph of $h(x) = 3(x - 2)(x + 6)$.

8. **K** Determine the equation of the parabola with x-intercepts ± 4 and passing through (3, 6).

9. Determine the equation of the quadratic function that passes through $(-4, 5)$ if its zeros are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

10. **T** A tunnel with a parabolic arch is 12 m wide. If the height of the arch 4 m from the left edge is 6 m, can a truck that is 5 m tall and 3.5 m wide pass through the tunnel? Justify your decision.

11. **A** A projectile is launched off the top of a platform. The table gives the height of the projectile at different times during its flight.

Time (s)	0	1	2	3	4	5	6
Height (m)	11	36	51	56	51	36	11

a) Draw a scatter plot of the data.
b) Draw a curve of good fit.
c) Determine the equation that will model this set of data.

12. Jason tossed a ball over a motion detector and it recorded these data.

Time (s)	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Height above Ground (m)	0	2.1875	3.75	4.6875	5	4.6875	3.75	2.1875	0

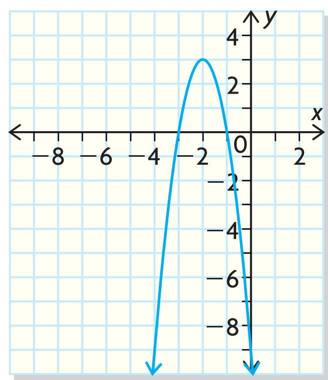
a) Draw a scatter plot of the data.
b) Draw a curve of good fit.
c) Determine an algebraic expression that models the data. Express the function in standard form.

13. Students at an agricultural school collected data showing the effect of different annual amounts of water (rainfall plus irrigation), x, in hectare-metres $(\text{ha} \cdot \text{m})$, on the yield of broccoli, y, in hundreds of kilograms per hectare (100 kg/ha).

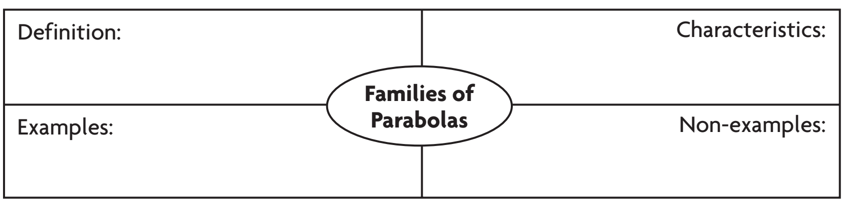
Amount of Water, x (ha • m)	0.30	0.45	0.60	0.75	0.90	1.05	1.20	1.35	1.50
Yield, y (100 kg/ha)	35	104	198	287	348	401	427	442	418

a) Draw a scatter plot and a curve of good fit.
b) Estimate the location of the vertex.
c) Determine an algebraic model for the data.

14. What is the equation of the parabola at the right if the point $(-4, -9)$ is on the graph?



15. Complete the chart shown. Include what you know about families of parabolas in standard, vertex, and factored form.
C



Extending

16. A parabolic bridge is 40 m wide. Determine the height of the bridge 12 m in from the outside edge, if the height 5 m in from the outside edge is 8 m.

17. A family of cubic equations with zeros -3, 1, and 5 can be represented by the function $f(x) = a(x + 3)(x - 1)(x - 5)$. Which equation describes the cubic in the family that passes through the point (3, 6)?

3.8 Linear–Quadratic Systems

GOAL

Solve problems involving the intersection of a linear and a quadratic function.

LEARN ABOUT the Math

Adam has decided to celebrate his birthday by going skydiving. He loves to freefall, so he will wait for some time before opening his parachute.

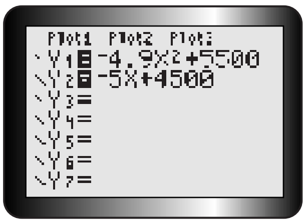
- His height after jumping from the airplane before he opens his parachute can be modelled by the quadratic function $h_1(t) = -4.9t^2 + 5500$, where t is time in seconds and $h_1(t)$ is the height above the ground, in metres, t seconds after jumping out.
- After he releases his parachute, he begins falling at a constant rate. His height above the ground can be modelled by the linear function $h_2(t) = -5t + 4500$.

? According to these models, how long after jumping out of the airplane should Adam release his parachute? At what height will this occur?

EXAMPLE 1 Selecting a strategy to solve a linear–quadratic system of equations

Determine when the two functions have the same height values.

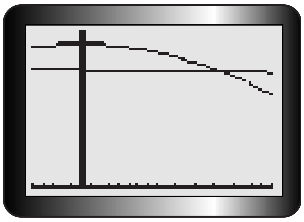
Kobi's Solution: Using a Graphical Approach



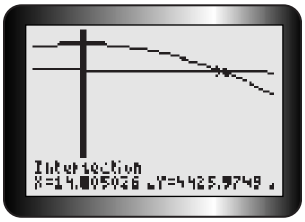
I graphed both functions on my graphing calculator. I was looking for the time at which both function values were the same. I looked for the point of intersection of the two graphs.

Tech Support

For help using the graphing calculator to find points of intersection, see Technical Appendix, B-12.



I adjusted the window settings until both graphs and the point of intersection on the right was visible. I was only interested in this point since time must be positive.



I used the intersection operation to locate the point of intersection: (14.8, 4426). That means that Adam should release his parachute 14.8 s after jumping out of the airplane. He will be 4426 m above the ground at that time.

Christina's Solution: Using Algebra

Just before parachute opens:
$h(t) = -4.9t^2 + 5500$

Just after parachute opens:
$h(t) = -5t + 4500$

At the moment the parachute is opened, the heights represented by each equation will be the same, so I set the right-hand sides of the equations equal to each other. This resulted in a quadratic equation that I needed to solve.

$$-4.9t^2 + 5500 = -5t + 4500$$
$$-4.9t^2 + 5t + 1000 = 0$$

I put the equation into standard form and then used the quadratic formula with $a = -4.9$, $b = 5$, and $c = 1000$ to solve for t.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-5 \pm \sqrt{5^2 - 4(-4.9)(1000)}}{2(-4.9)}$$
$$= \frac{-5 \pm \sqrt{25 + 19600}}{-9.8}$$
$$= \frac{-5 \pm \sqrt{19625}}{-9.8}$$

$t \doteq 14.8$ or $t \doteq -13.78 \leftarrow$ inadmissible

I got two possible values for t, but the time after jumping cannot be negative, so the only answer was $t = 14.8$ s.

Adam should open his parachute after 14.8 s.

$$h_2 = -5(14.8) + 4500$$
$$= 4426$$

Adam will be about 4426 m above the ground when he opens his parachute.

To find the height above the ground, I substituted my value for t into one of the equations. I chose the second one because it was an easier calculation.

Reflecting

A. Explain how you would determine the point of intersection of a linear function and a quadratic function graphically and algebraically.

B. What are an advantage and a disadvantage of each method?

C. Do you think that a linear function always intersects a quadratic function in two places? Why or why not?

APPLY the Math

EXAMPLE 2 Selecting a strategy to predict the number of points of intersection

Determine the number of points of intersection of the quadratic and linear functions $f(x) = 3x^2 + 12x + 14$ and $g(x) = 2x - 8$.

Julianne's Solution

$$3x^2 + 12x + 14 = 2x - 8$$
$$3x^2 + 10x + 22 = 0$$
$$b^2 - 4ac = (10)^2 - 4(3)(22)$$
$$= 100 - 264$$
$$= -164$$

If there is a point of intersection, the values of $f(x)$ and $g(x)$ will be equal at that point, so I set the expressions for $f(x)$ and $g(x)$ equal to each other. I put the resulting quadratic function into standard form. I then calculated the value of the discriminant.

The line and the parabola do not intersect.

Since $-164 < 0$, there are no real roots.

EXAMPLE 3 Solving a problem involving the point of intersection

Justin is skeet shooting. The height of the skeet is modelled by the function $h(t) = -5t^2 + 32t + 2$, where $h(t)$ is the height in metres t seconds after the skeet is released. The path of Justin's bullet is modelled by the function $g(t) = 31.5t + 1$, with the same units. How long will it take for the bullet to hit the skeet? How high off the ground will the skeet be when it is hit?

Stephanie's Solution

$$h(t) = -5t^2 + 32t + 2$$
$$g(t) = 31.5t + 1$$
$$-5t^2 + 32t + 2 = 31.5t + 1$$
$$-5t^2 + 0.5t + 1 = 0$$
$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-0.5 \pm \sqrt{(0.5)^2 - 4(-5)(1)}}{-10}$$
$$= \frac{-0.5 \pm \sqrt{20.25}}{-10}$$

I needed to find the point of intersection of the quadratic and the linear functions. I set them equal to each other. Then I put the resulting quadratic equation into standard form. I used the quadratic formula to solve for t.

$$t = \frac{-0.5 + 4.5}{-10} \quad \text{or} \quad t = \frac{-0.5 - 4.5}{-10}$$
$$t = -0.4 \quad \text{or} \quad t = 0.5$$

I got two possible values for t, but time cannot be negative, so I couldn't use the solution $t = -0.4$.

The bullet will hit the skeet after 0.5 s.

$$g(0.5) = 31.5(0.5) + 1$$
$$= 16.75$$

I substituted the value of t into $g(t)$ to solve for the height.

The skeet will be 16.75 m off the ground when it is hit.

In Summary

Key Ideas

- A linear function and a quadratic function can intersect at a maximum of two points.
- The point(s) of intersection of a line and a parabola can be found
  - graphically
  - algebraically

Need to Know

- To determine the points of intersection algebraically, use substitution to replace $f(x)$ in the quadratic function with the expression for $g(x)$ from the linear function. This results in a quadratic equation whose solutions correspond to the x-coordinates of the points of intersection.
- In many situations, one of the two solutions will be inadmissible.

CHECK Your Understanding

1. Find the point(s) of intersection by graphing.

a) $f(x) = x^2, g(x) = x + 6$

b) $f(x) = -2x^2 + 3, g(x) = 0.5x + 3$

c) $f(x) = (x - 3)^2 + 1, g(x) = -2x - 2$

2. Determine the point(s) of intersection algebraically.

a) $f(x) = -x^2 + 6x - 5, g(x) = -4x + 19$

b) $f(x) = 2x^2 - 1, g(x) = 3x + 1$

c) $f(x) = 3x^2 - 2x - 1, g(x) = -x - 6$

3. Determine the number of points of intersection of $f(x) = 4x^2 + x - 3$ and $g(x) = 5x - 4$ without solving.

PRACTISING

4. Determine the point(s) of intersection of each pair of functions.

K **a)** $f(x) = -2x^2 - 5x + 20, g(x) = 6x - 1$

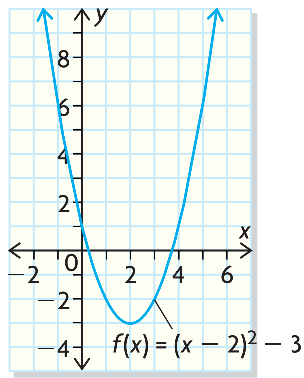
b) $f(x) = 3x^2 - 2, g(x) = x + 7$

c) $f(x) = 5x^2 + x - 2, g(x) = -3x - 6$

d) $f(x) = -4x^2 - 2x + 3, g(x) = 5x + 4$

5. An integer is two more than another integer. Twice the larger integer is one more than the square of the smaller integer. Find the two integers.

6. The revenue function for a production by a theatre group is $R(t) = -50t^2 + 300t$, where t is the ticket price in dollars. The cost function for the production is $C(t) = 600 - 50t$. Determine the ticket price that will allow the production to break even.



7. a) Copy the graph of $f(x) = (x - 2)^2 - 3$. Then draw lines with slope -4 that intersect the parabola at **(i)** one point, **(ii)** two points, and **(iii)** no points.
   b) Write the equations of the lines from part (a).
   c) How are all of the lines with slope -4 that do not intersect the parabola related?

8. Determine the value of k such that $g(x) = 3x + k$ intersects the quadratic function $f(x) = 2x^2 - 5x + 3$ at exactly one point.

9. Determine the value(s) of k such that the linear function $g(x) = 4x + k$ does not intersect the parabola $f(x) = -3x^2 - x + 4$.

10. **A** A daredevil jumps off the CN Tower and falls freely for several seconds before releasing his parachute. His height, $h(t)$, in metres, t seconds after jumping can be modelled by

    $h_1(t) = -4.9t^2 + t + 360$ before he released his parachute; and
    $h_2(t) = -4t + 142$ after he released his parachute.

    How long after jumping did the daredevil release his parachute?

11. **T** A quadratic function is defined by $f(x) = 3x^2 + 4x - 2$. A linear function is defined by $g(x) = mx - 5$. What value(s) of the slope of the line would make it a tangent to the parabola?

12. A punter kicks a football. Its height, $h(t)$, in metres, t seconds after the kick is given by the equation $h(t) = -4.9t^2 + 18.24t + 0.8$. The height of an approaching blocker's hands is modelled by the equation $g(t) = -1.43t + 4.26$, using the same t. Can the blocker knock down the punt? If so, at what point will it happen?

13. **C** Given a quadratic function $f(x)$ and a linear function $g(x)$, describe two ways you could determine the number of points of intersection of the two functions without solving for them.

Extending

14. Determine the coordinates of any points of intersection of the functions $x^2 - 2x + 3y + 6 = 0$ and $2x + 3y + 6 = 0$.

15. In how many ways could the graphs of two parabolas intersect? Draw a sketch to illustrate each possibility.

16. Determine the equation of the line that passes through the points of intersection of the graphs of the quadratic functions $f(x) = x^2 - 4$ and $g(x) = -3x^2 + 2x + 8$.

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 3.5, Examples 1, 2, and 3.
- Try Chapter Review Questions 12, 13, and 14.

Q: How can you determine the solutions to a quadratic equation?

A: Arrange the equation into standard form $ax^2 + bx + c = 0$, then:

- try to factor; the numbers that make each factor zero are the solutions to the original equation
- graph the corresponding function $f(x) = ax^2 + bx + c = 0$ and determine its x-intercepts; these points are the solutions to the original equation
- use the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Study Aid

- See Lesson 3.6, Examples 2, 3, and 4.
- Try Chapter Review Questions 15 and 16.

Q: How can you use the discriminant to determine the number of solutions of a quadratic equation?

A: The value of the discriminant, $b^2 - 4ac$, determines the number of roots of a quadratic equation. If $b^2 - 4ac > 0$, there are two distinct roots. If $b^2 - 4ac = 0$, there is one root. If $b^2 - 4ac < 0$, there are no roots.

Study Aid

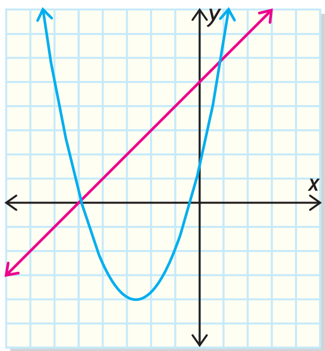
- See Lesson 3.7, Example 2.
- Try Chapter Review Questions 18 and 19.

Q: What characteristics do the members of the family of parabolas $f(x) = a(x - 2)(x + 6)$ have in common? the family of parabolas $g(x) = a(x - 2)^2 - 5$? the family of parabolas $h(x) = ax^2 + bx - 7$?

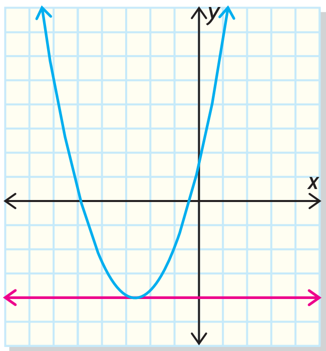
A: The members of the first family of parabolas will all have the same x-intercepts, 2 and -6, and the same axis of symmetry, $x = -2$. The members of the second family will have the same vertex, $(2, -5)$. The members of the third family will have the same y-intercept, $y = -7$.

Q: In how many ways can a line intersect a parabola?

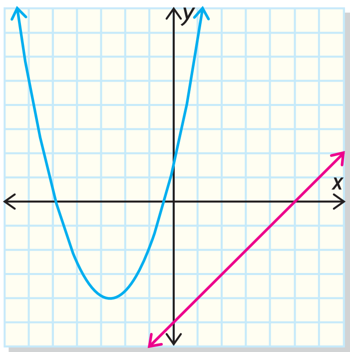
A: A line can intersect a parabola in at most two places.



It may intersect a parabola at only one point.



There may be no points of intersection.



Q: **How can you determine the points of intersection between a linear and a quadratic function?**

A1: You can graph both functions on the same set of axes and determine the point(s) of intersection from the graphs.

A2: You could equate the two functions. This results in a quadratic equation whose solutions are the x-coordinates of the points of intersection.

Study Aid

- See Lesson 3.8, Examples 1 and 3.
- Try Chapter Review Questions 21 and 23.

PRACTICE Questions

Lesson 3.1

1. Consider the quadratic function $f(x) = -3(x - 2)^2 + 5$.

a) State the direction of opening, the vertex, and the axis of symmetry.

b) State the domain and range.

c) Graph the function.

2. Consider the quadratic function $f(x) = 4(x - 2)(x + 6)$.

a) State the direction of opening and the zeros of the function.

b) Determine the coordinates of the vertex.

c) State the domain and range.

d) Graph the function.

3. Determine the equation of the axis of symmetry of the parabola with points $(-5, 3)$ and $(3, 3)$ equally distant from the vertex on either side of it.

Lesson 3.2

4. For each quadratic function, state the maximum or minimum value and where it will occur.

a) $f(x) = -3(x - 4)^2 + 7$

b) $f(x) = 4x(x + 6)$

5. The height, $h(t)$, in metres, of the trajectory of a football is given by $h(t) = 2 + 28t - 4.9t^2$, where t is the time in flight, in seconds. Determine the maximum height of the football and the time when that height is reached.

Lesson 3.3

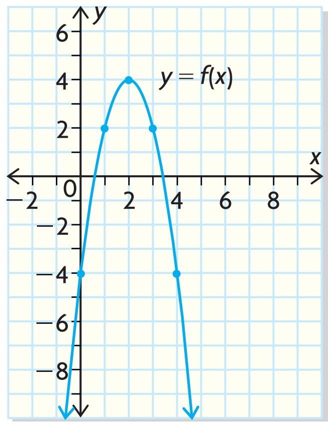
6. Describe the relationship between $f(x) = x^2$, $g(x) = \sqrt{x}$, and $h(x) = -\sqrt{x}$.

7. Is the inverse of a quadratic function also a function? Give a reason for your answer.

8. a) Given the graph of $f(x)$, sketch the graph of the inverse relation.

b) State the domain and range of the inverse relation.

c) Is the inverse relation a function? Why or why not?



Lesson 3.4

9. Express each number as a mixed radical in simplest form.

a) $\sqrt{98}$

b) $-5\sqrt{32}$

c) $4\sqrt{12} - 3\sqrt{48}$

d) $(3 - 2\sqrt{7})^2$

10. The area of a triangle can be calculated from Heron's formula,

$$A = \sqrt{s(s - a)(s - b)(s - c)},$$

where a, b, and c are the side lengths and $s = \frac{a + b + c}{2}$. Calculate the area of a triangle with side lengths 5, 7, and 10. Leave your answer in simplest radical form.

11. What is the perimeter of a right triangle with legs 6 cm and 3 cm? Leave your answer in simplest radical form.

Lesson 3.5

12. Determine the x-intercepts of the quadratic function $f(x) = 2x^2 + x - 15$.

13. The population of a Canadian city is modelled by $P(t) = 12t^2 + 800t + 40\ 000$, where t is the time in years. When $t = 0$, the year is 2007.

a) According to the model, what will the population be in 2020?

b) In what year is the population predicted to be 300 000?

14. A rectangular field with an area of 8000 m^2 is enclosed by 400 m of fencing. Determine the dimensions of the field to the nearest tenth of a metre.

Lesson 3.6

15. The height, $h(t)$, of a projectile, in metres, can be modelled by the equation $h(t) = 14t - 5t^2$, where t is the time in seconds after the projectile is released. Can the projectile ever reach a height of 9 m? Explain.

16. Determine the values of k for which the function $f(x) = 4x^2 - 3x + 2kx + 1$ has two zeros. Check these values in the original equation.

17. Determine the break-even points of the profit function $P(x) = -2x^2 + 7x + 8$, where x is the number of dirt bikes produced, in thousands.

Lesson 3.7

18. Determine the equation of the parabola with roots $2 + \sqrt{3}$ and $2 - \sqrt{3}$, and passing through the point (2, 5).

19. Describe the characteristics that the members of the family of parabolas $f(x) = a(x + 3)^2 - 4$ have in common. Which member passes through the point $(-2, 6)$?

20. An engineer is designing a parabolic arch. The arch must be 15 m high, and 6 m wide at a height of 8 m.

a) Determine a quadratic function that satisfies these conditions.

b) What is the width of the arch at its base?

Lesson 3.8

21. Calculate the point(s) of intersection of $f(x) = 2x^2 + 4x - 11$ and $g(x) = -3x + 4$.

22. The height, $h(t)$, of a baseball, in metres, at time t seconds after it is tossed out of a window is modelled by the function $h(t) = -5t^2 + 20t + 15$. A boy shoots at the baseball with a paintball gun. The trajectory of the paintball is given by the function $g(t) = 3t + 3$. Will the paintball hit the baseball? If so, when? At what height will the baseball be?

23. a) Will the parabola defined by $f(x) = x^2 - 6x + 9$ intersect the line $g(x) = -3x - 5$? Justify your answer.

b) Change the slope of the line so that it will intersect the parabola in two locations.

3 Chapter Self-Test

1. You are given $f(x) = -5x^2 + 10x - 5$.

a) Express the function in factored form and determine the vertex.

b) Identify the zeros, the axis of symmetry, and the direction of opening.

c) State the domain and range.

d) Graph the function.

2. For each function, state whether it will have a maximum or a minimum value. Describe the method you would choose to calculate the maximum or minimum value.

a) $f(x) = -2x^2 - 8x + 3$ **b)** $f(x) = 3(x - 1)(x + 5)$

3. You can choose whether you are provided the equation of a quadratic function in standard form, factored form, or vertex form. If you needed to know the information listed, which form would you choose and why?

a) the vertex

b) the y-intercept

c) the zeros

d) the axis of symmetry

e) the domain and range

4. Determine the maximum area of a rectangular field that can be enclosed by 2400 m of fencing.

5. Determine the equation of the inverse of $f(x) = 2(x - 1)^2 - 3$.

6. a) Simplify $(2 - \sqrt{8})(3 + \sqrt{2})$.

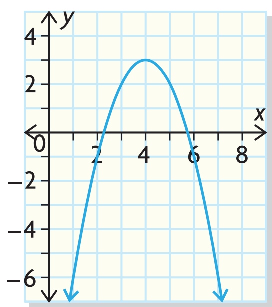
b) Simplify $(3 + \sqrt{5})(5 - \sqrt{10})$.

c) Explain why the answer to part (a) has fewer terms than the answer to part (b).

7. Calculate the value of k such that $kx^2 - 4x + k = 0$ has one root.

8. Does the linear function $g(x) = 6x - 5$ intersect the quadratic function $f(x) = 2x^2 - 3x + 2$? How can you tell? If it does intersect, determine the point(s) of intersection.

9. Determine the equation in standard form of the parabola shown below.



3 Chapter Task

Baseball Bonanza

Vernon Wells hits a baseball that travels for 142 m before it lands. The flight of the ball can be modelled by a quadratic function in which x is the horizontal distance the ball has travelled away from Vernon, and $h(x)$ is the height of the ball at that distance.

There are many quadratic equations you could use to model the distance and height, but you want to find one that is close to reality.

? What is the function that will model the height of Vernon's ball accurately over time?

A. Assume that the ball was between 0.6 m and 1.5 m above the ground when it was hit.

- **a)** What would $h(142)$ be?
- **b)** What happens when $x = 0$?
- **c)** What are the possible values for $h(x)$ when $x = 0$?
- **d)** What would be a good range of values for the height of the ball? Are some values for the height unreasonable?

B. Explain why each function is not a good model of the situation, and support your claim with reasons and a well-labelled sketch.

- **a)** $h(x) = -0.5x(x - 142)$
- **b)** $h(x) = -0.5x^2 + 71x + 1$
- **c)** $h(x) = -0.0015x^2 + 0.213x + 1.2$

C. Determine an equation that models the path of the ball, given this additional information:

- **a)** The ball was 1.2 m off the ground when it was hit.
- **b)** The ball reached a maximum height of 17 m when it was approximately 70 m away from Vernon.

Explain the method you are using to get the equation, and show all of your steps. Why did you approach the problem this way?

D. Use the model you created to graph the flight of Vernon's ball.

Task *Checklist*

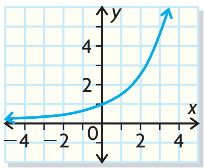
- ✔ Did you state your reasons that the given models were not reasonable?
- ✔ Did you draw a well-labelled graph, including some values?
- ✔ Did you show your work in your choice of method for part C?
- ✔ Did you support your choice of method in part C?

Chapters 1–3 Cumulative Review

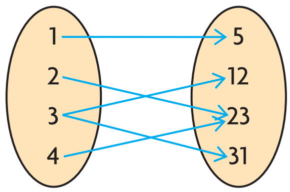
Multiple Choice

1. Identify the relation that is not a function.

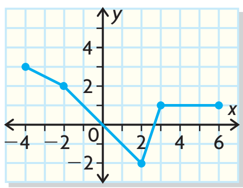
a)



c)



b)



d) $\{(8, 9), (3, 2), (5, 7), (1, 0), (4, 6)\}$

2. For the graph of $f(x) = \sqrt{x}$, identify the transformation that would *not* be applied to $f(x)$ to obtain the graph of $y = 2f(-2x) + 3$.

a) vertical stretch by factor of 2
b) reflection in x-axis
c) vertical translation up 3 units
d) horizontal compression by factor of $\frac{1}{2}$

3. An American visitor to Canada uses this function to convert from temperature in degrees Celsius into degrees Fahrenheit: $f(x) = 2x + 30$.
Identify $f^{-1}(x)$.

a) $f^{-1}(x) = \dfrac{x + 30}{2}$ **c)** $f^{-1}(x) = \dfrac{x - 2}{30}$

b) $f^{-1}(x) = \dfrac{x - 30}{2}$ **d)** $f^{-1}(x) = \dfrac{x + 2}{30}$

4. The range of $f(x) = -|x - 2| + 3$ is

a) $\{y \in \mathbf{R} \mid y \leq 3\}$ **c)** $\{y \in \mathbf{R} \mid 2 \leq y \leq 3\}$
b) $\{y \in \mathbf{R} \mid y \geq 3\}$ **d)** $\{y \in \mathbf{R} \mid 0 \leq y \leq 2\}$

5. Which pairs of functions are equivalent?

a) i **b)** i and ii **c)** i and iii **d)** iii and iv

i) $h(x) = (x + 6)(x + 3)(x - 6)$ and $b(x) = (x + 3)(x^2 - 36)$
ii) $b(t) = (3t + 2)^3$ and $c(t) = 27t^3 + 54t^2 + 36t + 8$
iii) $h(t) = (4 - x)^3$ and $h(t) = (x - 4)^3$
iv) $f(x) = (x^2 - 4x) - (2x^2 + 2x - 4) - (x^2 + 1)$ and $b(x) = (2x - 5)(2x - 1)$

6. Which expression has the restrictions $y \neq -1, 0, \frac{1}{2}$ on its variable?

a) $\dfrac{3y}{y - 2} \times \dfrac{4(y - 2)}{6y}$

b) $\dfrac{5y(y + 3)}{4y} \times \dfrac{(y - 5)}{(y + 3)}$

c) $\dfrac{(3y + 1)}{(2y - 1)} \div \dfrac{3y(y + 1)}{2y - 1}$

d) $\dfrac{10y}{y + 2} \div \dfrac{5}{2(y + 2)}$

7. Identify the correct product of $\dfrac{x^2 - 5x + 6}{x^2 - 1} \times \dfrac{x^2 - 4x - 5}{x^2 - 4}$.

a) $\dfrac{(x + 3)(x - 5)}{(x - 1)(x + 2)}$ **c)** $\dfrac{(x - 3)(x - 5)}{(x - 1)(x - 2)}$

b) $\dfrac{(x - 3)(x + 5)}{(x + 1)(x + 2)}$ **d)** $\dfrac{(x - 3)(x - 5)}{(x - 1)(x + 2)}$

8. Identify the correct sum of $\dfrac{5x - 6}{x + 1} + \dfrac{3x}{x - 4}$.

a) $\dfrac{2x^2 + 23x + 24}{(x + 1)(x - 4)}$ **c)** $\dfrac{15x^2 - 18x}{(x + 1)(x - 4)}$

b) $\dfrac{8x^2 - 23x + 24}{(x + 1)(x - 4)}$ **d)** $\dfrac{8x^2 - 29x + 24}{(x + 1)(x - 4)}$

9. Given the quadratic function $f(x) = 3x^2 - 6x + 15$, identify the coordinates of the vertex.

a) $(1, 12)$ **c)** $(12, 1)$
b) $(-1, -12)$ **d)** $(12, -1)$

10. When the equation of a quadratic function is in factored form, which feature is most easily determined?

a) y-intercepts **c)** vertex
b) x-intercepts **d)** maximum value

11. The height, h, in metres, of a baseball after Bill hits it with a bat is described by the function $h(t) = 0.8 + 29.4t - 4.9t^2$, where t is the time in seconds after the ball is struck. What is the maximum height of the ball?

a) 4.9 m **b)** 29.4 m **c)** 44.9 m **d)** 25 m

12. It costs a bus company \$225 to run a minibus on a ski trip, plus \$30 per passenger. The bus has seating for 22 passengers, and the company charges \$60 per fare if the bus is full. For each empty seat, the company has to increase the ticket price by \$5. How many empty seats should the bus run with to maximize profit from this trip?

a) 8 **b)** 6 **c)** 10 **d)** 2

13. Without drawing the graph, identify the function that has two zeros.

a) $n(x) = -x^2 - 6x - 9$
b) $m(x) = 4(x + 1)^2 + 0.5$
c) $h(x) = -5(x + 1.3)^2$
d) $g(x) = -2(x + 3.6)^2 + 4.1$

14. The graph of function $f(x) = x^2 - kx + k + 8$ touches the x-axis at one point. What are the possible values of k?

a) $k = 1$ or $k = 8$ **c)** $k = 0$ or $k = 1$
b) $k = -4$ or $k = 8$ **d)** $k = -8$ or $k = 4$

15. For $f(x) = 2(x - 3)^2 + 5, x \geq 3$, determine the equation for f^{-1}.

a) $y = 3 + \sqrt{\frac{x - 5}{2}}, x \geq 5$

b) $y = 3 - \sqrt{\frac{x - 5}{2}}, x \geq 5$

c) $y = 3 \pm \sqrt{\frac{x - 5}{2}}$

d) $y = 3 + \sqrt{\frac{x + 5}{2}}, x \leq 5$

16. The relation that is also a function is

a) $x^2 + y^2 = 25$ **c)** $x^2 = y$
b) $y^2 = x$ **d)** $x^2 - y^2 = 25$

17. Given $f(x) = x^2 - 5x + 3$, then

a) $f(-1) = -3$ **c)** $f(-1) = -1$
b) $f(-1) = 7$ **d)** $f(-1) = 9$

18. Which of the following statements is not true?

a) The horizontal line test can be used to show that a relation is a function.
b) The set of all possible input values of a function is called the domain.
c) The equation $y = 3x + 5$ describes a function.
d) This set of ordered pairs describes a function: $\{(0, 1), (1, 2), (3, -3), (4, -1)\}$.

19. The range that best corresponds to $f(x) = \frac{3}{x}$ is

a) $\{y \in \mathbf{R}\}$ **c)** $\{y \in \mathbf{R} \mid y < 0\}$
b) $\{y \in \mathbf{R} \mid y > 0\}$ **d)** $\{y \in \mathbf{R} \mid y \neq 0\}$

20. If $f(x) = 5x - 7$, then

a) $f^{-1}(x) = 7x - 5$ **c)** $f^{-1}(x) = \frac{x - 5}{7}$
b) $f^{-1}(x) = x - 7$ **d)** $f^{-1}(x) = \frac{x + 7}{5}$

21. The inverse of $g(x) = x^2 - 5x - 6$ is

a) $g^{-1}(x) = \left(x - \frac{1}{2}\right)^2 - \frac{49}{4}$

b) $g^{-1}(x) = \frac{1}{2} \pm \sqrt{x - \frac{49}{4}}$

c) $g^{-1}(x) = \frac{1}{2} \pm \sqrt{x + \frac{49}{4}}$

d) $g^{-1}(x) = \left(x + \frac{1}{2}\right)^2 + \frac{49}{4}$

22. Which of the following statements is false?

a) The domain of f is the range of f^{-1}.
b) The graph of the inverse can be found by reflecting $y = f(x)$ in the line $y = x$.
c) The domain of f^{-1} is the range of f.
d) To determine the equation of the inverse, interchange x and y and solve for x.

23. If $f(x) = 3(x + 2)^2 - 5$, the domain must be restricted to which interval if the inverse is to be a function?

a) $x \geq -5$ **c)** $x \geq 2$
b) $x \geq -2$ **d)** $x \geq 5$

24. The inverse of $f(x) = \sqrt{x - 1}$ is

a) $f^{-1}(x) = x^2 + 1, x \leq 1$
b) $f^{-1}(x) = x^2 - 1, x \geq 1$
c) $f^{-1}(x) = x^2 + 1, x \geq 1$
d) $f^{-1}(x) = x^2 - 1, x \leq 1$

25. What transformations are applied to $y = f(x)$ to obtain the graph of $y = af(x - p) + q$, if $a < 0$, $p < 0$, and $q < 0$?

a) Vertical stretch by a factor of $|a|$, followed by a translation $|p|$ units to the left and $|q|$ units down
b) Reflection in the x-axis, vertical stretch by a factor of $|a|$, followed by a translation $|p|$ units to the right and $|q|$ units down
c) Reflection in the x-axis, vertical stretch by a factor of $|a|$, followed by a translation $|p|$ units to the left and $|q|$ units down
d) Reflection in the x-axis, vertical stretch by a factor of $|a|$, followed by a translation $|p|$ units to the right and $|q|$ units up

26. The vertex form of the equation $y = -2x^2 - 12x - 19$ is

a) $y = -2x(x + 6) - 19$
b) $y = -2(x - 3)(x + 6)$
c) $y = -2(x + 3)^2 - 1$
d) $y = -2(x - 3)^2 + 1$

27. The coordinates of the vertex for the graph of $y = (x + 2)(x - 3)$ are

a) $(-2, 3)$ **c)** $(2, 3)$
b) $\left(-\frac{1}{2}, -\frac{21}{4}\right)$ **d)** $\left(\frac{1}{2}, -\frac{25}{4}\right)$

28. The profit function for a new product is given by $P(x) = -4x^2 + 28x - 40$, where x is the number sold in thousands. How many items must be sold for the company to break even?

a) 2000 or 5000 **c)** 5000 or 7000
b) 2000 or 3500 **d)** 3500 or 7000

29. Which of the following statements is not true for the equation of a quadratic function?

a) In standard form, the y-intercept is clearly visible.
b) In vertex form, the break-even points are clearly visible.
c) In factored form, the x-intercepts are clearly visible.
d) In vertex form, the coordinates of the vertex are clearly visible.

30. State the value of the discriminant, D, and the number of roots for $7x^2 + 12x + 6 = 0$.

a) $D = 312, n = 2$ **c)** $D = 312, n = 1$
b) $D = 24, n = 2$ **d)** $D = -24, n = 0$

31. The simplified form of $\frac{7}{ab} - \frac{2}{b} + \frac{1}{3a^2}$ is

a) $\frac{6}{ab - b + 3a^2}, a, b \neq 0$
b) $\frac{21a - 6a^2 + b}{3a^2b}, a, b \neq 0$
c) $\frac{7a - 2a^2 + b}{3a^2b}, a, b \neq 0$
d) $\frac{7a - 2b + ab}{3a^3b^2}, a, b \neq 0$

32. The simplified form of $\frac{x^2 - 4}{x + 3} \div \frac{2x + 4}{x^2 - 9}$ is

a) $\frac{2(x - 2)(x + 2)^2}{(x + 3)^2(x - 3)}$
b) $\frac{(x^2 - 4)(x - 3)}{2x + 4}$
c) $\frac{(x - 2)(x - 3)}{2}$
d) $\frac{2(x - 3)}{x - 2}$

Investigations

33. Studying Functions

Analyze two of the following functions in depth.

a) $f(x) = 3x^2 - 24x + 50$

b) $g(x) = 5 - 2\sqrt{3x + 6}$

c) $h(x) = \dfrac{1}{\frac{1}{3}(x - 6)} - 2$

Include:

i) the domain and range

ii) the relationship to the parent function, including all applied transformations

iii) a sketch of the function

34. Charity Walk

Sacha and Jill set off at the same time on a 30 km walk for charity. Sacha, who has trained all year for this event, walks 1.4 km/h faster than Jill, but sees a friend on the route and stops to talk for 20 min. Even with this delay, Sacha finishes the walk 2 h ahead of Jill.

How fast was each person walking, and how long did it take for each person to finish the walk?

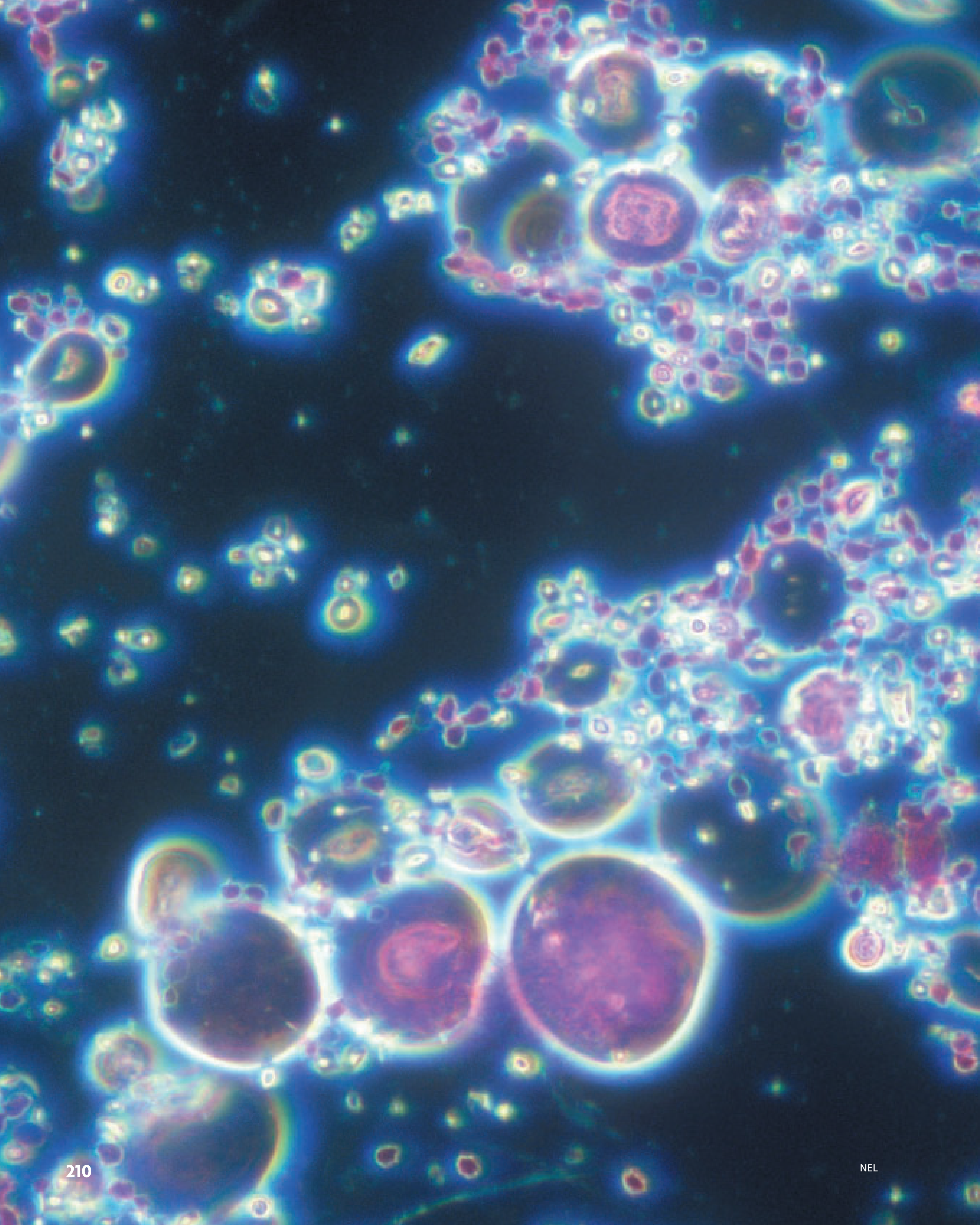
35. Ski Trip

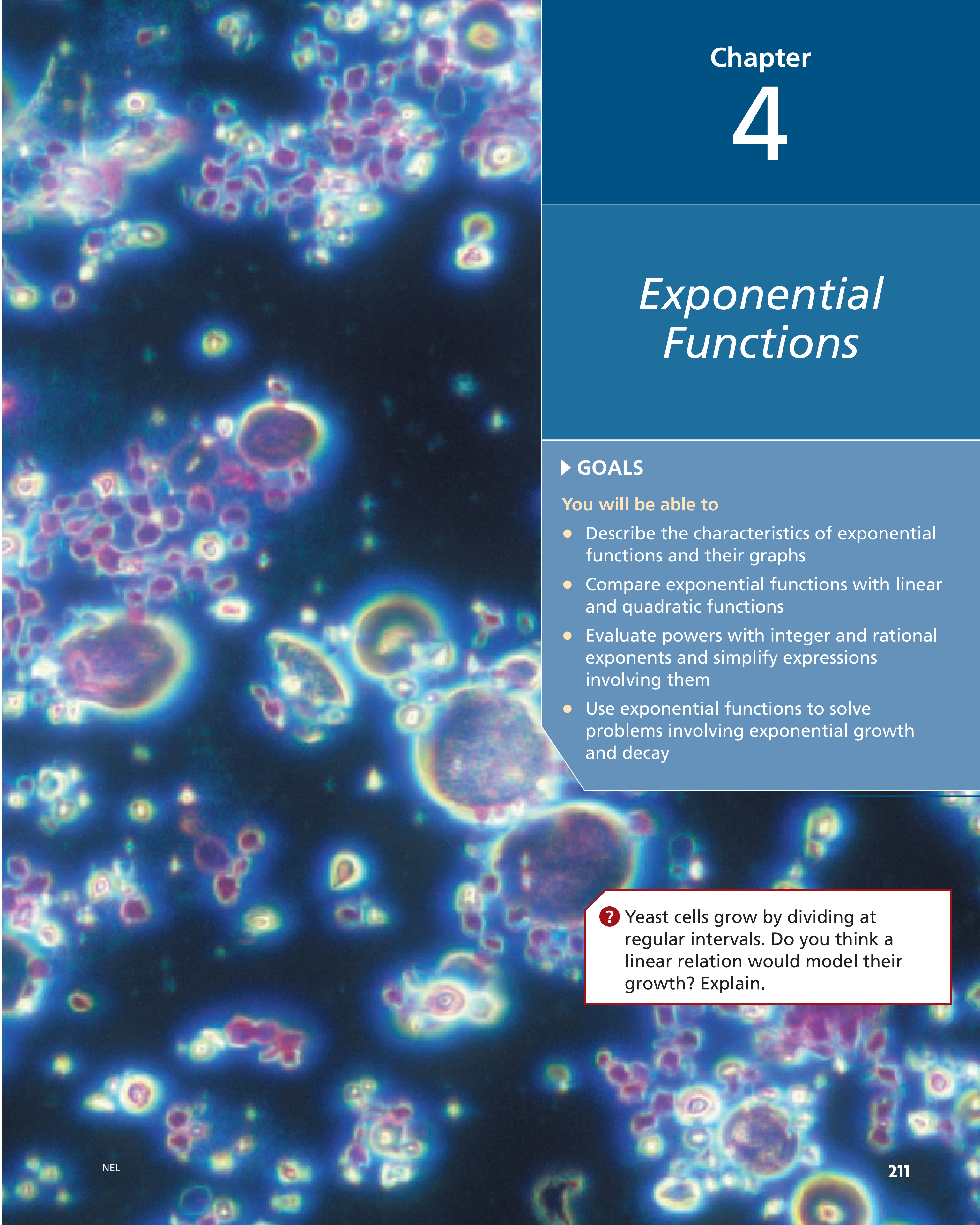
Josh is running a ski trip over March Break. Last year he had 25 students go and each paid \$550. This year he will increase the price and knows that for each \$50 price increase, 2 fewer students will go on the trip. The bus costs a flat fee of \$5500, and hotel and lift tickets cost \$240 per person.

Determine

a) the number of students who must go for Josh to break even

b) the cost of the trip that will maximize his profit





Chapter 4

Exponential Functions

GOALS

You will be able to

- Describe the characteristics of exponential functions and their graphs
- Compare exponential functions with linear and quadratic functions
- Evaluate powers with integer and rational exponents and simplify expressions involving them
- Use exponential functions to solve problems involving exponential growth and decay

? Yeast cells grow by dividing at regular intervals. Do you think a linear relation would model their growth? Explain.

4 Getting Started

SKILLS AND CONCEPTS You Need

Study | ***Aid***

- For help, see Essential Skills Appendix.

Question	Appendix
1, 2, 4, and 7	A-3
6	A-2
9	A-7

1. Evaluate.

a) 7^2 c) 5^{-1} e) 100^2

b) 2^5 d) 10^0 f) 2^{-3}

2. Evaluate.

a) $(-3)^2$ c) -4^2 e) $(-5)^3$

b) $(-3)^3$ d) $(-4)^2$ f) -5^3

3. Predict whether the power $(-5)^{120}$ will result in a positive or negative answer. Explain how you know.

4. Evaluate.

a) $(3^2)^2$ c) $[(-4)^2]^3$ e) $[(2^2)^2]^2$

b) $(7^2)^4$ d) $[-(10^2)]^3$ f) $-[(2^2)^2]^0$

5. Evaluate.

a) $(\sqrt{49})^2$ b) $3\sqrt{64}$ c) $\sqrt{4}\sqrt{16}$ d) $\dfrac{\sqrt{9}}{\sqrt{81}}$

6. Evaluate.

a) $\dfrac{5}{8}+\dfrac{5}{3}$ c) $\dfrac{7}{8}\div\dfrac{2}{3}$ e) $-\dfrac{4}{3}+\left(\dfrac{9}{10}\div\dfrac{5}{12}\right)$

b) $\dfrac{5}{8}-\dfrac{5}{3}$ d) $\dfrac{1}{5}-\dfrac{3}{8}\left(\dfrac{4}{3}\right)$ f) $-\dfrac{9}{10}\left(\dfrac{3}{8}+\dfrac{7}{3}\right)$

7. Simplify.

a) $a^2(a^5)$ b) $b^{12}\div b^8$ c) $(c^3)^4$ d) $d(d^6)d^3$

8. Determine the exponent that makes each equation true.

a) $9^x = 81$ c) $(-5)^a = -125$

b) $8^m = 256$ d) $-10^r = -100\ 000\ 000$

9. Evaluate the following formulas for $r = 2.5$ cm and $h = 4.8$ cm.

a) the volume of a cylinder: $V = \pi r^2 h$

b) the volume of a sphere: $V = \dfrac{4}{3}\pi r^3$

10. For each set of data, calculate the differences. Identify whether or not the data represent a linear or quadratic relationship. Explain.

a)

x	*y*	First Differences	Second Differences
−4	12		
−2	7		
0	2		
2	−3		
4	−8		
6	−13		

b)

x	*y*	First Differences	Second Differences
−3	9		
−2	10		
−1	12		
0	15		
1	19		
2	24		

APPLYING What You Know

Binary Code

The numbers we work with every day are written in base 10.

For example, 235 means $2 \times 10^2 + 3 \times 10^1 + 5 \times 10^0$.

Binary numbers are numbers that are written in base 2.

For example, 1011 means $1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$.
Each number consists of only the digits 0 and 1.

Computers use binary code to do their calculations. Each letter, number, or symbol needs a separate binary code.

? Is there a relationship between the number of digits and the possible number of codes?

A. Copy and complete the table shown.

Number of Digits	Possible Codes	Number of Possible Codes	Possible Codes as a Power of Two
1	0, 1	2	2^1
2	00, 01, 10, 11	4	2^2
3	000, 001, 010, 011, 100, 101, 110, 111	8	
4			
5			
6			
7			

B. Select any two numbers (other than the last two) from the third column, and calculate their product. Write the two numbers and their product as powers of two. Repeat this several times with different pairs of numbers.

C. What is the relationship between the exponents of the powers that you multiplied and the exponent of the resulting product?

D. What rule for multiplying powers with the same base does this suggest?

E. Select any two numbers (other than the first two) from the third column, and divide the greater by the lesser. Write the two numbers and their quotient as powers of two. Repeat this several times with different pairs of numbers.

F. What is the relationship between the exponents of the powers that you divided and the exponent of the resulting quotient?

G. What rule for dividing powers with the same base does this suggest?

H. How can you predict the possible number of codes if you know the number of digits?

4.1 Exploring Growth and Decay

YOU WILL NEED

- two different types of balls that bounce (e.g., basketball, racquetball, soccer ball, golf ball)
- graphing calculator with a motion detector (CBR)

GOAL

Collect data and study the characteristics of rapidly decaying functions.

EXPLORE the Math

When you drop a ball it will bounce several times.

? Is the height of each bounce related to the height of the previous bounce?

A. Set the CBR to "Ball Bounce" mode. Work with a partner. One of you holds the ball, while the other holds the CBR 0.5 m over the ball. When the CBR is triggered, drop the ball.

B. Let the ball bounce at least 5 times while you collect the data. Use the trace key to determine the height of each bounce.

C. Copy the table. In the first row, record the original height of the ball, then record the bounce number and bounce height for next bounces.

Bounce Number	Bounce Height	First Differences
0		
1		
2		

D. Calculate and complete the first-differences column.

E. Repeat parts A to D for two additional starting heights.

F. After recording data for three different starting heights, plot the bounce height versus bounce number on the same graph. Use a different colour for each set of data. Draw a dashed curve through each set of points.

G. Describe the shape of each graph. Does each set of data represent a function? How do you know?

H. State the domain and range of each graph.

I. Repeat the exploration with a different type of ball, and explain how the height of each bounce is related to its previous bounce.

Reflecting

J. Why was a dashed curve used in part F instead of a solid one?

K. Look at the first-differences column. Describe how the bounce height changed from one bounce to the next. Was this pattern the same for each *type* of ball? Explain.

L. Did the type of ball you used influence the graph? Explain. Did the initial height of the drop influence the graph? Explain.

M. What happens to the bounce height as the bounce number increases? If you continue the pattern indefinitely, will the bounce height ever reach zero? Explain.

In Summary

Key Ideas

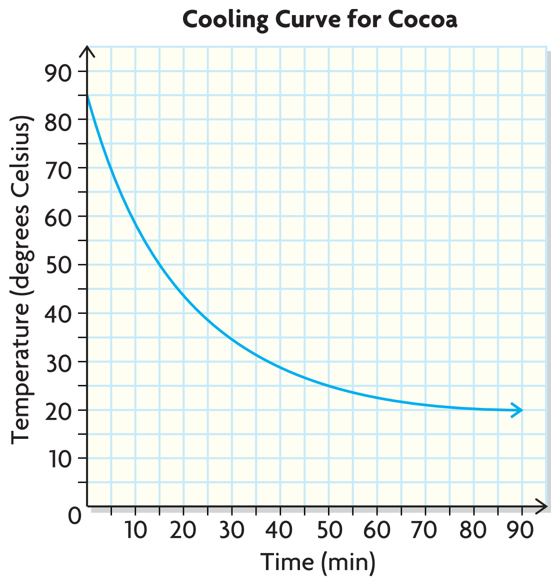
- Some real-world situations can be modelled by functions whose first differences follow a multiplicative pattern.
- The scatter plots for these situations show increasing or decreasing nonlinear patterns.

Need to Know

- The domain and range of a function should be considered in terms of the situation it is modelling.

FURTHER Your Understanding

1. A cup of hot cocoa left on a desk in a classroom had its temperature measured once every minute. The graph shows the relationship between the temperature of the cocoa, in degrees Celsius, and time, in minutes.

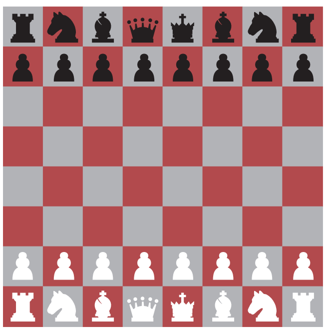


a) What characteristics of this graph are the same as the graph(s) you drew in the ball-bounce experiment?

b) What was the temperature of the cocoa at the start of the experiment?

c) What is the temperature of the classroom?

2. A folktale tells of a man who helps a king solve a problem. In return, the king offers the man anything he desires. The man asks for one grain of rice on a square of a chessboard and then double the number of grains of rice for each subsequent square.



a) Complete the table of values for the first 10 squares.

Number of Squares on the Chessboard	Number of Grains on that Square	First Differences
1	1	
2	2	
3	4	

b) Create a scatter plot of the data in the first two columns.

c) Compare this graph and the first differences of the data with your graphs and data for the ball-bounce experiment. How are they the same and how are they different?

4.2 Working with Integer Exponents

GOAL

Investigate powers that have integer or zero exponents.

LEARN ABOUT the Math

The metric system of measurement is used in most of the world. A key feature of the system is its ease of use. Since all units differ by multiples of 10, it is easy to convert from one unit to another. Consider the chart listing the prefix names and their factors for the unit of measure for length, the metre.

Name	Symbol	Multiple of the Metre	Multiple as a Power of 10
terametre	Tm	1 000 000 000 000	10^{12}
gigametre	Gm	1 000 000 000	10^{9}
megametre	Mm	1 000 000	10^{6}
kilometre	km	1 000	10^{3}
hectometre	hm	100	10^{2}
decametre	dam	10	10^{1}
metre	m	1	
decimetre	dm	0.1	
centimetre	cm	0.01	
millimetre	mm	0.001	
micrometre	μm	0.000 1	
nanometre	nm	0.000 01	
picometre	pm	0.000 001	
femtometre	fm	0.000 000 001	
attometre	am	0.000 000 000 001	

? How can powers be used to represent metric units for lengths less than 1 metre?

EXAMPLE 1 — Using reasoning to define zero and negative integer exponents

Use the table to determine how multiples of the unit metre that are less than or equal to 1 can be expressed as powers of 10.

Jemila's Solution

Multiples	Powers
1000	10^3
$1000 \div 10 = 100$	$10^3 \div 10 = 10^2$
$100 \div 10 = 10$	$10^2 \div 10 = 10^1$
$10 \div 10 = 1$	$10^1 \div 10 = 10^0$
$1 \div 10 = 0.1$ $= \frac{1}{10}$	$10^0 \div 10 = 10^{-1}$
$0.1 \div 10 = 0.01$ $= \frac{1}{100}$ $= \frac{1}{10^2}$	$10^{-1} \div 10 = 10^{-2}$
$0.01 \div 10 = 0.001$ $= \frac{1}{1000}$ $= \frac{1}{10^3}$	$10^{-2} \div 10 = 10^{-3}$

I think that $x^{-n} = \frac{1}{x^n}$ is the rule for negative exponents.

As I moved down the table, the powers of 10 decreased by 1, while the multiples were divided by 10. To come up with the next row in the table, I divided the multiples and the powers by 10.

If I continue this pattern, I'll get $10^0 = 1$, $10^{-1} = 0.1$, $10^{-2} = 0.01$, etc.

I rewrote each decimal as a fraction and each denominator as a power of 10.

I noticed that $10^0 = 1$ and $10^{-n} = \frac{1}{10^n}$.

I don't think it mattered that the base was 10. The relationship would be true for any base.

EXAMPLE 2 — Connecting the concept of an exponent of 0 to the exponent quotient rule

Use the quotient rule to show that $10^0 = 1$.

David's Solution

$$\frac{10^6}{10^6} = 1$$

I can divide any number except 0 by itself to get 1. I used a power of 10.

$$\frac{10^6}{10^6} = 10^{6-6} = 10^0$$

When you divide powers with the same base, you subtract the exponents.

Therefore, $10^0 = 1$.

I applied the rule to show that a power with zero as the exponent must be equal to 1.

Reflecting

A. What type of number results when x^{-n} is evaluated if x is a positive integer and $n > 1$?

B. How is 10^2 related to 10^{-2}? Why do you think this relationship holds for other opposite exponents?

C. Do you think the rules for multiplying and dividing powers change if the powers have negative exponents? Explain.

APPLY the Math

EXAMPLE 3 Representing powers with integer bases in rational form

Evaluate.

a) 5^{-3} **b)** $(-4)^{-2}$ **c)** -3^{-4}

Stergios's Solution

a) $5^{-3} = \frac{1}{5^3}$

$= \frac{1}{125}$

5^{-3} is what you get if you divide 1 by 5^3. I evaluated the power.

b) $(-4)^{-2} = \frac{1}{(-4)^2}$

$= \frac{1}{16}$

$(-4)^{-2}$ is what you get if you divide 1 by $(-4)^2$. Since the negative sign is in the parentheses, the square of the number is positive.

c) $-3^{-4} = -\frac{1}{3^4}$

$= -\frac{1}{81}$

In this case, the negative sign is not inside the parentheses, so the entire power is negative. I knew that $3^{-4} = \frac{1}{3^4}$.

If the base of a power involving a negative exponent is a fraction, it can be evaluated in a similar manner.

Communication *Tip*

Rational numbers can be written in a variety of forms. The term *rational form* means "Write the number as an integer, or as a fraction."

EXAMPLE 4 Representing powers with rational bases as rational numbers

Evaluate $\left(\frac{2}{3}\right)^{-3}$.

Sadira's Solution

$$\left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^{3}}$$

$\left(\frac{2}{3}\right)^{-3}$ is what you get if you divide 1 by $\left(\frac{2}{3}\right)^{3}$.

$$= \frac{1}{\left(\frac{8}{27}\right)}$$

$$= 1 \times \frac{27}{8}$$

Dividing by a fraction is the same as multiplying by its reciprocal, so I used this to evaluate the power.

$$= \frac{27}{8}$$

EXAMPLE 5 Selecting a strategy for expressions involving negative exponents

Evaluate $\frac{3^5 \times 3^{-2}}{(3^{-3})^2}$.

Kayleigh's Solution: Using Exponent Rules

$$\frac{3^5 \times 3^{-2}}{(3^{-3})^2} = \frac{3^{5+(-2)}}{3^{-3\times 2}}$$

I simplified the numerator and denominator separately. Then I divided the numerator by the denominator. I added exponents for the numerator, multiplied exponents for the denominator, and subtracted exponents for the final calculation.

$$= \frac{3^3}{3^{-6}}$$

$$= 3^{3-(-6)}$$

$$= 3^9$$

$$= 19\ 683$$

Tech Support

For help with evaluating powers on a graphing calculator, see Technical Appendix, B-15.

Derek's Solution: Using a Calculator

I entered the expression into my calculator. I made sure I used parentheses around the entire numerator and denominator so that the calculator would compute those values before dividing.

In Summary

Key Ideas

- An integer base raised to a negative exponent is equivalent to the reciprocal of the same base raised to the opposite exponent.

$$b^{-n} = \frac{1}{b^n}, \quad \text{where} \quad b \neq 0$$

- A fractional base raised to a negative exponent is equivalent to the reciprocal of the same base raised to the opposite exponent.

$$\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \left(\frac{b}{a}\right)^n, \quad \text{where} \quad a \neq 0, b \neq 0$$

- A number (or expression), other than 0, raised to the power of zero is equal to 1.

$$b^0 = 1, \text{ where } b \neq 0$$

Need to Know

- When multiplying powers with the same base, add exponents.

$$b^m \times b^n = b^{m+n}$$

- When dividing powers with the same base, subtract exponents.

$$b^m \div b^n = b^{m-n} \quad \text{if} \quad b \neq 0$$

- To raise a power to a power, multiply exponents.

$$(b^m)^n = b^{mn}$$

- In simplifying numerical expressions involving powers, it is customary to present the answer as an integer, a fraction, or a decimal.
- In simplifying algebraic expressions involving powers, it is customary to present the answer with positive exponents.

CHECK Your Understanding

1. Rewrite each expression as an equivalent expression with a positive exponent.

a) 5^{-4} **c)** $\frac{1}{2^{-4}}$ **e)** $\left(\frac{3}{11}\right)^{-1}$

b) $\left(-\frac{1}{10}\right)^{-3}$ **d)** $-\left(\frac{6}{5}\right)^{-3}$ **f)** $\frac{7^{-2}}{8^{-1}}$

2. Write each expression as a single power with a positive exponent.

a) $(-10)^8(-10)^{-8}$ **c)** $\frac{2^8}{2^{-5}}$ **e)** $(-9^4)^{-1}$

b) $6^{-7} \times 6^5$ **d)** $\frac{11^{-3}}{11^5}$ **f)** $[(7^{-3})^{-2}]^{-2}$

3. Which is the greater power, 2^{-5} or $\left(\frac{1}{2}\right)^{-5}$? Explain.

PRACTISING

4. Simplify, then evaluate each expression. Express answers in rational form.

a) $2^{-3}(2^{7})$ **c)** $\dfrac{5^4}{5^6}$ **e)** $(4^{-3})^{-1}$

b) $(-8)^3(-8)^{-3}$ **d)** $\dfrac{3^{-8}}{3^{-6}}$ **f)** $(7^{-1})^2$

5. Simplify, then evaluate each expression. Express answers in rational form.

a) $3^3(3^2)^{-1}$ **c)** $\dfrac{(12^{-1})^3}{12^{-3}}$ **e)** $(3^{-2}(3^3))^{-2}$

b) $(9 \times 9^{-1})^{-2}$ **d)** $\dfrac{(5^3)^{-2}}{5^{-6}}$ **f)** $9^7(9^3)^{-2}$

6. Simplify, then evaluate each expression. Express answers in rational form.

a) $10(10^4(10^{-2}))$ **c)** $\dfrac{6^{-5}}{(6^2)^{-2}}$ **e)** $2^8 \times \left(\dfrac{2^{-5}}{2^6}\right)$

b) $8(8^2)(8^{-4})$ **d)** $\dfrac{4^{-10}}{(4^{-4})^3}$ **f)** $13^{-5} \times \left(\dfrac{13^2}{13^8}\right)^{-1}$

7. Evaluate. Express answers in rational form.

a) $16^{-1} - 2^{-2}$ **d)** $\left(\dfrac{1}{5}\right)^{-1} + \left(-\dfrac{1}{2}\right)^{-2}$

b) $(-3)^{-1} + 4^0 - 6^{-1}$ **e)** $5^{-3} + 10^{-3} - 8(1000^{-1})$

c) $\left(-\dfrac{2}{3}\right)^{-1} + \left(\dfrac{2}{5}\right)^{-1}$ **f)** $3^{-2} - 6^{-2} + \dfrac{3}{2}(-9)^{-1}$

8. Evaluate. Express answers in rational form.

a) $5^2(-10)^{-4}$ **c)** $\dfrac{12^{-1}}{(-4)^{-1}}$ **e)** $(8^{-1})\left(\dfrac{2^{-3}}{4^{-1}}\right)$

b) $16^{-1}(2^5)$ **d)** $\dfrac{(-9)^{-2}}{(3^{-1})^2}$ **f)** $\dfrac{(-5)^3(-25)^{-1}}{(-5)^{-2}}$

9. Evaluate. Express answers in rational form.
K
a) $(-4)^{-3}$ **c)** $-(5)^{-3}$ **e)** $(-6)^{-3}$
b) $(-4)^{-2}$ **d)** $-(5)^{-2}$ **f)** $-(6)^{-2}$

10. Without using your calculator, write the given numbers in order from least to greatest. Explain your thinking.
T

$$(0.1)^{-1}, 4^{-1}, 5^{-2}, 10^{-1}, 3^{-2}, 2^{-3}$$

11. Evaluate each expression for $x = -2$, $y = 3$, and $n = -1$. Express answers in rational form.
A

a) $(x^n + y^n)^{-2n}$ **c)** $\left(\dfrac{x^n}{y^n}\right)^n$

b) $(x^2)^n(y^{-2n})x^{-n}$ **d)** $\left(\dfrac{xy^n}{(xy)^{2n}}\right)^{2n}$

12. Kendra, Erik, and Vinh are studying. They wish to evaluate $3^{-2} \times 3$. Kendra notices errors in each of her friends' solutions, shown here.

Erik's solution	Vinh's solution
$3^{-2} \times 3$	$3^{-2} \times 3$
$= 3^{-1}$	$= 3^{-2}$
$= -\dfrac{1}{3^{1}}$	$= \dfrac{1}{3^{2}}$
$= -\dfrac{1}{3}$	$= \dfrac{1}{9}$

a) Explain where each student went wrong.
b) Create a solution that demonstrates the correct steps.

13. Evaluate using the laws of exponents.

a) $2^3 \times 4^{-2} \div 2^2$ **d)** $4^{-1}(4^2 + 4^0)$ **g)** $\dfrac{3^{-2} \times 2^{-3}}{3^{-1} \times 2^{-2}}$

b) $(2 \times 3)^{-1}$ **e)** $\dfrac{2^5}{3^{-2}} \times \dfrac{3^{-1}}{2^4}$ **h)** $\dfrac{4^{-2} + 3^{-1}}{3^{-2} + 2^{-3}}$

c) $\left(\dfrac{3^{-1}}{2^{-1}}\right)^{-2}$ **f)** $(5^0 + 5^2)^{-1}$ **i)** $\dfrac{5^{-1} - 2^{-2}}{5^{-1} + 2^{-2}}$

14. Find the value of each expression for $a = 1$, $b = 3$, and $c = 2$.
a) ac^c **c)** $(ab)^{-c}$ **e)** $(-a \div b)^{-c}$ **g)** $(a^b b^a)^c$
b) $a^c b^c$ **d)** $(b \div c)^{-a}$ **f)** $(a^{-1}b^{-2})^c$ **h)** $[(b)^{-a}]^{-c}$

15. **a)** Explain the difference between evaluating $(-10)^3$ and evaluating 10^{-3}.
C **b)** Explain the difference between evaluating $(-10)^4$ and evaluating -10^4.

Extending

16. Determine the exponent that makes each equation true.

a) $16^x = \dfrac{1}{16}$ **c)** $2^x = 1$ **e)** $25^n = \dfrac{1}{625}$

b) $10^x = 0.01$ **d)** $2^n = 0.25$ **f)** $12^n = \dfrac{1}{144}$

17. If $10^{2y} = 25$, determine the value of 10^{-y}, where $y > 0$.

18. Simplify.

a) $(x^2)^{5-r}$ **d)** $x^{3(7-r)}x^r$

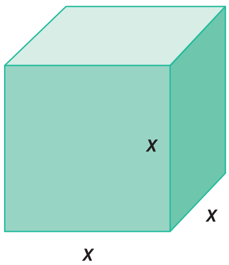
b) $(b^{2m+3m}) \div (b^{m-n})$ **e)** $(a^{10-p})\left(\dfrac{1}{a}\right)^p$

c) $(b^{2m+3n}) \div (b^{m-n})$ **f)** $[(3x^4)^{6-m}]\left(\dfrac{1}{x}\right)^m$

4.3 Working with Rational Exponents

GOAL

Investigate powers involving rational exponents and evaluate expressions containing them.



The volume of this cube is $V(x) = x^3$ and the area of its base is $A(x) = x^2$.
In this cube, x is the side length and can be called

- the square root of A, since if squared, the result is $A(x)$
- the cube root of V, since if cubed, the result is $V(x)$

LEARN ABOUT the Math

? What exponents can be used to represent the side length x as the square root of area and the cube root of volume?

EXAMPLE 1 Representing a side length by rearranging the area formula

Express the side length x as a power of A and V.

Ira's Solution

$A = x^2$

$x = A^n$

$A = (x)(x)$

I used the area formula for the base. Since I didn't know what power to use, I used the variable n to write x as a power of A.

$A = A^n \times A^n$

$A = A^{n+n}$

$A^1 = A^{2n}$

I rewrote the area formula, substituting A^n for x.

Since I was multiplying powers with the same base, I added the exponents.

Therefore,

$1 = 2n$

I set the two exponents equal to each other. I solved this equation.

$\frac{1}{2} = n$

The exponent that represents a square root is $\frac{1}{2}$.

Therefore, $x = A^{\frac{1}{2}} = \sqrt{A}$.

EXAMPLE 2 Representing a side length by rearranging the volume formula

Sienna's Solution

$V = x^3$

$x = V^n$

$V = (x)(x)(x)$

I used the volume formula for a cube. I represented the edge length x as a power of the volume V. I used the variable n.

$V = V^n \times V^n \times V^n$

I rewrote the volume formula, substituting V^n for x.

$V = V^{n+n+n}$

I added the exponents.

$V^1 = V^{3n}$

Therefore,

$1 = 3n$

I set the two exponents equal to each other. I solved this equation.

$\frac{1}{3} = n$

The exponent that represents a cube root is $\frac{1}{3}$.

Therefore, $x = V^{\frac{1}{3}} = \sqrt[3]{V}$.

Reflecting

A. Why could x be expressed as both a square root and a cube root?

B. Make a conjecture about the meaning of $x^{\frac{1}{n}}$. Explain your reasoning.

C. Do the rules for multiplying powers with the same base still apply if the exponents are rational numbers? Create examples to illustrate your answer.

APPLY the Math

EXAMPLE 3 Connecting radical notation and exponents

Express the following in radical notation. Then evaluate.

a) $49^{-\frac{1}{2}}$ **b)** $(-8)^{\frac{1}{3}}$ **c)** $10\,000^{\frac{1}{4}}$

Donato's Solution

a) $49^{-\frac{1}{2}} = \frac{1}{49^{\frac{1}{2}}}$

I wrote the power using the reciprocal of its base and its opposite exponent. An exponent of $\frac{1}{2}$ means square root.

$= \frac{1}{\sqrt{49}}$

$= \frac{1}{7}$

I evaluated the power.

index (plural indices)

the number at the left of the radical sign. It tells which root is indicated: 3 for cube root, 4 for fourth root, etc. If there is no number, the square root is intended.

b) $(-8)^{\frac{1}{3}} = \sqrt[3]{-8}$
$= -2$

An exponent of $\frac{1}{3}$ means cube root. I wrote the root as a radical, using an index of 3. That means the number is multiplied by itself three times to get -8. The number is -2.

c) $10\,000^{\frac{1}{4}} = \sqrt[4]{10\,000}$
$= 10$

An exponent of $\frac{1}{4}$ means the fourth root, since
$10\,000^{\frac{1}{4}} \times 10\,000^{\frac{1}{4}} \times 10\,000^{\frac{1}{4}} \times 10\,000^{\frac{1}{4}} = 10\,000^{1}$.

That number must be 10.

EXAMPLE 4 Selecting an approach to evaluate a power

Evaluate $27^{\frac{2}{3}}$.

Cory's Solutions

$27^{\frac{2}{3}}$

I know that the exponent $\frac{1}{3}$ indicates a cube root. So I used the power-of-a-power rule to separate the exponents:

$\frac{2}{3} = 2 \times \frac{1}{3}$ and $\frac{2}{3} = \frac{1}{3} \times 2$

$= 27^{\frac{1}{3}\times 2}$	$= 27^{2\times\frac{1}{3}}$
$= (27^{\frac{1}{3}})^2$	$= (27^2)^{\frac{1}{3}}$
$= (\sqrt[3]{27})^2$	$= \sqrt[3]{27^2}$
$= (3)^2$	$= \sqrt[3]{729}$
$= 9$	$= 9$

To see if the order in which I applied the exponents mattered, I calculated the solution in two ways.

In the first way, I evaluated the cube root before squaring the result.

In the other way, I squared the base and then took the cube root of the result.

Both ways resulted in 9.

EXAMPLE 5 Evaluating a power with a rational exponent

Evaluate.

a) $(-27)^{\frac{4}{3}}$ **b)** $(16)^{-0.75}$

Casey's Solutions

a) $(-27)^{\frac{4}{3}} = ((-27)^{\frac{1}{3}})^4$

$= (\sqrt[3]{-27})^4$

$= (-3)^4$

$= 81$

I rewrote the exponent as $4 \times \frac{1}{3}$.
I represented $(-27)^{\frac{1}{3}}$ as $\sqrt[3]{-27}$.
I calculated the cube root of -27.
I evaluated the power.

b) $16^{-0.75} = 16^{-\frac{3}{4}}$

$= \dfrac{1}{16^{\frac{3}{4}}}$

$= \dfrac{1}{(\sqrt[4]{64})^3}$

$= \dfrac{1}{2^3}$

$= \dfrac{1}{8}$

I rewrote the power, changing the exponent from -0.75 to its equivalent fraction.

I expressed $16^{-\frac{3}{4}}$ as a rational number, using 1 as the numerator and $16^{\frac{3}{4}}$ as the denominator.

I determined the fourth root of 64 and cubed the result.

The rules of exponents also apply to powers involving rational exponents.

EXAMPLE 6 Representing an expression involving the same base as a single power

Simplify, and then evaluate $\dfrac{8^{\frac{5}{6}}\sqrt{8}}{8^{\frac{5}{3}}}$.

Lucia's Solution

$\dfrac{8^{\frac{5}{6}}\sqrt{8}}{8^{\frac{5}{3}}} = \dfrac{8^{\frac{5}{6}}8^{\frac{1}{2}}}{8^{\frac{5}{3}}}$

$= \dfrac{8^{\frac{5}{6}+\frac{1}{2}}}{8^{\frac{5}{3}}}$

To simplify, I converted the radical into exponent form.

Since the bases were the same, I wrote the numerator as a single power by adding exponents, then I subtracted exponents to simplify the whole expression.

$$= \frac{8^{\frac{4}{3}}}{8^{\frac{5}{3}}}$$

$$= 8^{\frac{4}{3} - \frac{5}{3}}$$

$$= 8^{-\frac{1}{3}}$$

Once I had simplified to a single power of 8, the number was easier to evaluate.

$$= \frac{1}{8^{\frac{1}{3}}}$$

$$= \frac{1}{2}$$

I checked my work on my calculator.

In Summary

Key Ideas

- A number raised to a rational exponent is equivalent to a radical. The rational exponent $\frac{1}{n}$ indicates the nth root of the base. If $n > 1$ and $n \in N$, then $b^{\frac{1}{n}} = \sqrt[n]{b}$, where $b \neq 0$.
- If the numerator of a rational exponent is not 1, and if m and n are positive integers, then $b^{\frac{m}{n}} = (\sqrt[n]{b})^m = \sqrt[n]{b^m}$, where $b \neq 0$.

Need to Know

- The exponent laws that apply to powers with integer exponents also apply to powers with rational exponents. Included are the product-of-powers rule $a^n \times b^n = (ab)^n$ and the quotient of powers rule $a^n \div b^n = \left(\frac{a}{b}\right)^n$.
- The power button on a scientific calculator can be used to evaluate rational exponents.
- Some roots of negative numbers do not have real solutions. For example, -16 does not have a real-number square root, since whether you square a positive or negative number, the result is positive.
- Odd roots can have negative bases, but even ones cannot.

CHECK Your Understanding

1. Write in radical form. Then evaluate without using a calculator.
   a) $49^{\frac{1}{2}}$
   b) $100^{\frac{1}{2}}$
   c) $(-125)^{\frac{1}{3}}$
   d) $16^{0.25}$
   e) $81^{\frac{1}{4}}$
   f) $-(144)^{0.5}$

2. Write in exponent form, then evaluate. Express answers in rational form.
   a) $\sqrt[9]{512}$
   b) $\sqrt[3]{-27}$
   c) $\sqrt[3]{27^2}$
   d) $(\sqrt[3]{-216})^5$
   e) $\sqrt[5]{\frac{-32}{243}}$
   f) $\sqrt[4]{\left(\frac{16}{81}\right)^{-1}}$

3. Write as a single power.
   a) $8^{\frac{2}{3}}(8^{\frac{1}{3}})$
   b) $8^{\frac{2}{3}} \div 8^{\frac{1}{3}}$
   c) $(-11)^2(-11)^{\frac{3}{4}}$
   d) $(7^{\frac{5}{6}})^{-\frac{6}{5}}$
   e) $\dfrac{9^{\frac{-1}{5}}}{9^{\frac{2}{3}}}$
   f) $10^{-\frac{4}{5}}(10^{\frac{1}{15}}) \div 10^{\frac{2}{3}}$

PRACTISING

4. Write as a single power, then evaluate. Express answers in rational form.
   a) $\sqrt{5}\sqrt{5}$
   b) $\dfrac{\sqrt[3]{-16}}{\sqrt[3]{2}}$
   c) $\dfrac{\sqrt{28}\sqrt{4}}{\sqrt{7}}$
   d) $\dfrac{\sqrt[4]{18}\,(\sqrt[4]{9})}{\sqrt[4]{2}}$

5. Evaluate.
   a) $49^{\frac{1}{2}} + 16^{\frac{1}{2}}$
   b) $27^{\frac{2}{3}} - 81^{\frac{3}{4}}$
   c) $16^{\frac{3}{4}} + 16^{\frac{3}{4}} - 81^{-\frac{1}{4}}$
   d) $128^{-\frac{5}{7}} - 16^{0.75}$
   e) $16^{\frac{3}{2}} + 16^{-0.5} + 8 - 27^{\frac{2}{3}}$
   f) $81^{\frac{1}{2}} + \sqrt[3]{8} - 32^{\frac{4}{5}} + 16^{\frac{3}{4}}$

6. Write as a single power, then evaluate. Express answers in rational form.
   a) $4^{\frac{1}{5}}(4^{0.3})$
   b) $100^{0.2}(100^{\frac{-7}{10}})$
   c) $\dfrac{64^{\frac{4}{3}}}{64}$
   d) $\dfrac{27^{-1}}{27^{\frac{-2}{3}}}$
   e) $\dfrac{(16^{-2.5})^{-0.2}}{16^{\frac{3}{4}}}$
   f) $\dfrac{(8^{-2})(8^{2.5})}{(8^6)^{-0.25}}$

7. Predict the order of these six expressions in terms of value from lowest to highest. Check your answers with your calculator. Express answers to three decimal places.
   a) $\sqrt[4]{623}$
   b) $125^{\frac{2}{5}}$
   c) $\sqrt[10]{10.24}$
   d) $80.9^{\frac{1}{4}}$
   e) $17.5^{\frac{5}{8}}$
   f) $21.4^{\frac{3}{2}}$

8. The volume of a cube is 0.015 625 m^3. Determine the length of each side.

A

9. Use your calculator to determine the values of $27^{\frac{4}{3}}$ and $27^{1.3333}$. Compare the two answers. What do you notice?

10. Explain why $(-100)^{0.2}$ is possible to evaluate while $(-100)^{0.5}$ is not.

C

11. Write $125^{\frac{-2}{3}}$ in radical form, then evaluate. Explain each of your steps.

K

12. Evaluate.

a) $-256^{0.375}$

b) $15.625^{\frac{4}{3}}$

c) $\sqrt[3]{-0.027^4}$

d) $(-3.375)^{\frac{2}{3}}$

e) $\sqrt[4]{(0.0016)^3}$

f) $(-7776)^{1.6}$

13. The power 4^3 means that 4 is multiplied by itself three times. Explain the meaning of $4^{2.5}$.

14. State whether each expression is true or false.

a) $9^{\frac{1}{2}} + 4^{\frac{1}{2}} = (9 + 4)^{\frac{1}{2}}$

b) $9^{\frac{1}{2}} + 4^{\frac{1}{2}} = (9 \times 4)^{\frac{1}{2}}$

c) $\left(\frac{1}{a} + \frac{1}{b}\right)^{-1} = a + b$

d) $\left(\frac{1}{a} \times \frac{1}{b}\right)^{-1} = ab$

e) $\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^6 = x^2 + y^2$

f) $\left[\left(x^{\frac{1}{3}}\right)\left(y^{\frac{1}{3}}\right)\right]^6 = x^2y^2$

15. a) What are some values of m and n that would make $(-2)^{\frac{m}{n}}$ undefined?

T

b) What are some values of m and n that would make $(6)^{\frac{m}{n}}$ undefined?

Extending

16. Given that $x^y = y^x$, what could x and y be? Is there a way to find the answer graphically?

17. Mary must solve the equation $1.225 = (1 + i)^{12}$ to determine the value of each dollar she invested for a year at the interest rate i per year. Her friend Bindu suggests that she begin by taking the 12th root of each side of the equation. Will this work? Try it and solve for the variable i. Explain why it does or does not work.

18. Solve.

a) $\left(\frac{1}{16}\right)^{\frac{1}{4}} - \sqrt[3]{\frac{8}{27}} = \sqrt{x^2}$

b) $\sqrt[3]{\frac{1}{8}} - \sqrt[4]{x^4} + 15 = \sqrt[4]{16}$

4.4 Simplifying Algebraic Expressions Involving Exponents

GOAL

Simplify algebraic expressions involving powers and radicals.

LEARN ABOUT the Math

The ratio of the surface area to the volume of microorganisms affects their ability to survive. An organism with a higher surface area-to-volume ratio is more buoyant and uses less of its own energy to remain near the surface of a liquid, where food is more plentiful.

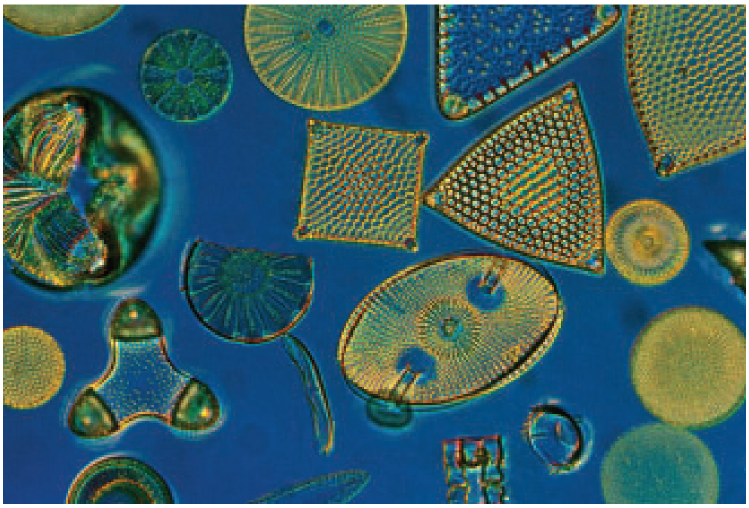
Mike is calculating the surface area-to-volume ratio for different-sized cells. He assumes that the cells are spherical.

For a sphere,

$$SA(r) = 4\pi r^2 \quad \text{and} \quad V(r) = \tfrac{4}{3}\pi r^3.$$

He substitutes the value of the radius into each formula and then divides the two expressions to calculate the ratio.

Radius (μm)	Surface Area/ Volume
1	$\frac{4\pi}{\left(\frac{4}{3}\pi\right)}$
1.5	$\frac{9\pi}{4.5\pi}$
2	$\frac{16\pi}{\left(\frac{32}{3}\pi\right)}$
2.5	$\frac{25\pi}{\left(\frac{125}{6}\pi\right)}$
3	$\frac{36\pi}{36\pi}$
3.5	$\frac{49\pi}{\left(\frac{343}{6\pi}\right)}$



? How can Mike simplify the calculation he uses?

EXAMPLE 1 Representing the surface area-to-volume ratio

Simplify $\frac{SA(r)}{V(r)}$, given that $SA(r) = 4\pi r^2$ and $V(r) = \frac{4}{3}\pi r^3$.

Bram's Solution

$$\frac{SA(r)}{V(r)}$$

I used the formulas for *SA* and *V* and wrote the ratio.

$$= \frac{4\cancel{\pi}^{1} r^2}{\frac{4}{3}\cancel{\pi}_{1} r^3}$$

The numerator and denominator have a factor of π, so I divided both by π.

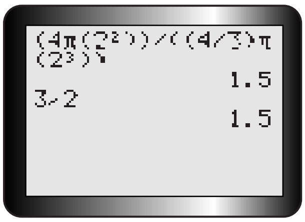
I started to simplify the expression by dividing the coefficients.

$$\left(4 \div \frac{4}{3} = 4 \times \frac{3}{4} = 3\right)$$

$$= 3r^{-1}$$

$$= \frac{3}{r}$$

The bases of the powers were the same, so I subtracted exponents to simplify the part of the expression involving *r*.



I used a calculator and substituted $r = 2$ in the unsimplified ratio first and my simplified expression next.

Each version gave me the same answer, so I think that they are equivalent, but the second one took far fewer keystrokes!

Reflecting

A. How can you use the simplified ratio to explain why the values in Mike's table kept decreasing?

B. Is it necessary to simplify an algebraic expression before you substitute numbers and perform calculations? Explain.

C. What are the advantages and disadvantages to simplifying an algebraic expression prior to performing calculations?

D. Do the exponent rules used on algebraic expressions work the same way as they do on numerical expressions? Explain by referring to Bram's work.

APPLY the Math

EXAMPLE 2 Connecting the exponent rules to the simplification of algebraic expressions

Simplify $\dfrac{(2x^{-3}y^2)^3}{(x^3y^{-4})^2}$.

Adnan's Solution

$$\frac{(2x^{-3}y^2)^3}{(x^3y^{-4})^2} = \frac{(2)^3(x^{-3})^3(y^2)^3}{(x^3)^2(y^{-4})^2}$$

I used the product-of-powers rule to raise each factor in the numerator to the third power and to square each factor in the denominator. Then I multiplied exponents.

$$= \frac{8x^{-9}y^6}{x^6y^{-8}}$$

$$= 8x^{-9-6}y^{6-(-8)}$$

$$= 8x^{-15}y^{14}$$

$$= \frac{8y^{14}}{x^{15}}$$

I simplified the whole expression by subtracting exponents of terms with the same base.

One of the powers had a negative exponent. To write it with positive exponents, I used its reciprocal.

EXAMPLE 3 Selecting a computational strategy to evaluate an expression

Evaluate the expression $\dfrac{(x^{2n+1})(x^{3n-1})}{x^{2n-5}}$ for $x = -3$ and $n = 2$.

Bonnie's Solution: Substituting, then Simplifying

$$\frac{(x^{2n+1})(x^{3n-1})}{x^{2n-5}} = \frac{(-3)^{2(2)+1}(-3)^{3(2)-1}}{(-3)^{2(2)-5}}$$

I substituted the values for x and n into the expression.

$$= \frac{(-3)^5(-3)^5}{(-3)^{-1}}$$

$$= \frac{(-243)(-243)}{\frac{1}{-3}}$$

$$= -177\ 147$$

Then I evaluated the numerator and denominator separately, before dividing one by the other.

Alana's Solution: Simplifying, then Substituting

$$\frac{(x^{2n+1})(x^{3n-1})}{x^{2n-5}}$$

Each power had the same base, so I simplified by using exponent rules before I substituted.

$$= \frac{x^{(2n+1)+(3n-1)}}{x^{2n-5}}$$

I added the exponents in the numerator to express it as a single power.

$$= \frac{x^{5n}}{x^{2n-5}}$$

$$= x^{(5n)-(2n-5)}$$

Then I subtracted the exponents in the denominator to divide the powers.

$$= x^{3n+5}$$

Once I had a single power, I substituted -3 for x and 2 for n and evaluated.

$$= (-3)^{3(2)+5}$$

$$= (-3)^{11}$$

$$= -177\ 147$$

EXAMPLE 4 Simplifying an expression involving powers with rational exponents

Simplify $\dfrac{(27a^{-3}b^{12})^{\frac{1}{3}}}{(16a^{-8}b^{12})^{\frac{1}{2}}}$.

Jane's Solution

$$\frac{(27a^{-3}b^{12})^{\frac{1}{3}}}{(16a^{-8}b^{12})^{\frac{1}{2}}} = \frac{27^{\frac{1}{3}}a^{\frac{-3}{3}}b^{\frac{12}{3}}}{16^{\frac{1}{2}}a^{\frac{-8}{2}}b^{\frac{12}{2}}}$$

In the numerator, I applied the exponent $\frac{1}{3}$ to each number or variable inside the parentheses, using the power-of-a-power rule. I did the same in the denominator, applying the exponent $\frac{1}{2}$ to the numbers and variables.

$$= \frac{3a^{-1}b^{4}}{4a^{-4}b^{6}}$$

$$= \frac{3}{4}a^{-1+4}b^{4-6}$$

I simplified by subtracting the exponents.

$$= \frac{3}{4}a^{3}b^{-2}$$

$$= \frac{3a^{3}}{4b^{2}}$$

I expressed the answer with positive exponents.

Sometimes it is necessary to express an expression involving radicals using exponents in order to simplify it.

EXAMPLE 5 Representing an expression involving radicals as a single power

Simplify $\left(\frac{\sqrt[5]{x^8}}{\sqrt{x^3}}\right)^3$.

Albino's Solution

$$\left(\frac{\sqrt[5]{x^8}}{\sqrt{x^3}}\right)^3 = \left(\frac{x^{\frac{8}{5}}}{x^{\frac{3}{2}}}\right)^3$$

Since this is a fifth root divided by a square root, I couldn't write it as a single radical.

I changed the radical expressions to exponential form and used exponent rules to simplify.

$$= \left(x^{\frac{8}{5}-\frac{3}{2}}\right)^3$$

$$= \left(x^{\frac{16}{10}-\frac{15}{10}}\right)^3$$

$$= \left(x^{\frac{1}{10}}\right)^3$$

$$= x^{\frac{3}{10}}$$

$$= \sqrt[10]{x^3}$$

When I got a single power, I converted it to radical form.

In Summary

Key Idea

- Algebraic expressions involving powers containing integer and rational exponents can be simplified with the use of the exponent rules in the same way numerical expressions can be simplified.

Need to Know

- When evaluating an algebraic expression by substitution, simplify prior to substituting. The answer will be the same if substitution is done prior to simplifying, but the number of calculations will be reduced.
- Algebraic expressions involving radicals can often be simplified by changing the expression into exponential form and applying the rules for exponents.

CHECK Your Understanding

1. Simplify. Express each answer with positive exponents.

a) $x^4(x^3)$ **c)** $\frac{m^5}{m^{-3}}$ **e)** $(y^3)^2$

b) $(p^{-3})(p)^5$ **d)** $\frac{a^{-4}}{a^{-2}}$ **f)** $(k^6)^{-2}$

2. Simplify. Express each answer with positive exponents.

a) $y^{10}(y^4)^{-3}$

b) $(x^{-3})^{-3}(x^{-1})^5$

c) $\dfrac{(n^{-4})^3}{(n^{-3})^{-4}}$

d) $\dfrac{w^4(w^{-3})}{(w^{-2})^{-1}}$

e) $\dfrac{(x^{-1})^4x}{x^{-3}}$

f) $\dfrac{(b^{-7})^2}{b(b^{-5})b^9}$

3. Consider the expression $\dfrac{x^7(y^2)^3}{x^5y^4}$.

a) Substitute $x = -2$ and $y = 3$ into the expression, and evaluate it.

b) Simplify the expression. Then substitute the values for x and y to evaluate it.

c) Which method seems more efficient?

PRACTISING

4. Simplify. Express answers with positive exponents.

a) $(pq^2)^{-1}(p^3q^3)$

b) $\left(\dfrac{x^3}{y}\right)^{-2}$

c) $\dfrac{(ab)^{-2}}{b^5}$

d) $\dfrac{m^2n^2}{(m^3n^{-2})^2}$

e) $\dfrac{(w^2x)^2}{(x^{-1})^2w^3}$

f) $\left(\dfrac{(ab)^{-1}}{a^2b^{-3}}\right)^{-2}$

5. Simplify. Express answers with positive exponents.

a) $(3xy^4)^2(2x^2y)^3$

b) $\dfrac{(2a^3)^2}{4ab^2}$

c) $\dfrac{(10x)^{-1}y^3}{15x^3y^{-3}}$

d) $\dfrac{(3m^4n^2)^2}{12m^{-2}n^6}$

e) $\dfrac{p^{-5}(r^3)^2}{(p^2r)^2(p^{-1})^2}$

f) $\left(\dfrac{(x^3y)^{-1}(x^4y^3)}{(x^2y^{-3})^{-2}}\right)^{-1}$

6. Simplify. Express answers with positive exponents.

K

a) $(x^4)^{\frac{1}{2}}(x^6)^{-\frac{1}{3}}$

b) $\dfrac{9(c^8)^{0.5}}{(16c^{12})^{0.25}}$

c) $\dfrac{\sqrt{25m^{-12}}}{\sqrt{36m^{10}}}$

d) $\sqrt[3]{\dfrac{(10x^3)^2}{(10x^6)^{-1}}}$

e) $\left(\dfrac{(32x^5)^{-2}}{(x^{-1})^{10}}\right)^{0.2}$

f) $\dfrac{\sqrt[10]{1024x^{20}}}{\sqrt[9]{512x^{27}}}$

7. Evaluate each expression. Express answers in rational form with positive exponents.

a) $(16x^6y^4)^{\frac{1}{2}}$ for $x = 2, y = 1$

b) $\dfrac{(9p^{-2})^{\frac{1}{2}}}{6p^2}$ for $p = 3$

c) $\dfrac{(81x^4y^6)^{\frac{1}{2}}}{8(x^9y^3)^{\frac{1}{3}}}$ for $x = 10, y = 5$

d) $\left(\dfrac{(25a^4)^{-1}}{(7a^{-2}b)^2}\right)^{\frac{1}{2}}$ for $a = 11, b = 10$

8. Evaluate. Express answers in rational form with positive exponents.

a) $(\sqrt{10\,000x})^{\frac{3}{2}}$ for $x = 16$

b) $\left(\dfrac{(4x^3)^4}{(x^3)^6}\right)^{-0.5}$ for $x = 5$

c) $(-2a^2b)^{-3}\sqrt{25a^4b^6}$ for $a = 1$, $b = 2$

d) $\sqrt{\dfrac{(18m^{-5}n^2)(32m^2n)}{4mn^{-3}}}$ for $m = 10$, $n = 1$

9. Simplify. Express answers in rational form with positive exponents.

a) $(36m^4n^6)^{0.5}(81m^{12}n^8)^{0.25}$

b) $\left(\dfrac{(6x^3)^2(6y^3)}{(9xy)^6}\right)^{-\frac{1}{3}}$

c) $\left(\dfrac{\sqrt{64a^{12}}}{(a^{1.5})^{-6}}\right)^{\frac{2}{3}}$

d) $\left(\dfrac{(x^{18})^{\frac{-1}{6}}}{\sqrt[5]{243x^{10}}}\right)^{0.5}$

10. If $M = \dfrac{(16x^8y^{-4})^{\frac{1}{4}}}{32x^{-2}y^8}$, determine values for x and y so that
T

a) $M = 1$ **b)** $M > 1$ **c)** $0 < M < 1$ **d)** $M < 0$

11. The volume and surface area of a cylinder are given, respectively, by the
A formulas

$$V = \pi r^2 h \quad \text{and} \quad SA = 2\pi rh + 2\pi r^2.$$

a) Determine an expression, in simplified form, that represents the surface area-to-volume ratio for a cylinder.

b) Calculate the ratio for a radius of 0.8 cm and a height of 12 cm.

12. If $x = -2$ and $y = 3$, write the three expressions in order from least to greatest.

$$\frac{y^{-4}(x^2)^{-3}y^{-3}}{x^{-5}(y^{-4})^2},\ \frac{x^{-3}(y^{-1})^{-2}}{(x^{-5})(y^4)},\ (y^{-5})(x^5)^{-2}(y^2)(x^{-3})^{-4}$$

13. How is simplifying algebraic expressions like simplifying numerical ones?
C How is it different?

Extending

14. **a)** The formula for the volume of a sphere of radius r is $V(r) = \frac{4}{3}\pi r^3$. Solve this equation for r. Write two versions, one in radical form and one in exponential form.

b) Determine the radius of a sphere with a volume of $\frac{256\pi}{3}$ m^3.

15. Simplify $\dfrac{\sqrt{x(x^{2n+1})}}{\sqrt[3]{x^{3n}}}$, $x > 0$.

4 Mid-Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 4.4, Examples 2, 3, and 4.
- Try Mid-Chapter Review Questions 9, 10, and 11.

Q: How do you evaluate an expression involving a negative exponent?

A: To evaluate a number raised to a negative exponent, you can take the reciprocal of the number, change the sign of the exponent, and then evaluate the equivalent expression.

$$b^{-x} = \frac{1}{b^x}, \quad b \neq 0$$

EXAMPLE

$$5^{-3} = \frac{1}{5^3} = \frac{1}{125} \quad \text{and} \quad \left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$$

Study Aid

- See Lesson 4.2, Examples 1, 3, and 4.
- Try Mid-Chapter Review Questions 1, 2, and 3.

Q: How do you simplify algebraic expressions involving rational exponents?

A: You can use the same exponent rules you use to simplify and evaluate numerical expressions.

Study Aid

- See Lesson 4.3, Examples 3, 4, and 5.
- Try Mid-Chapter Review Questions 5 to 8.

Q: How do you evaluate an expression involving a rational exponent?

A: The denominator of a rational exponent indicates the index of the root of the base. The numerator has the same meaning as an integer exponent. These can be evaluated in two different ways:

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \quad \text{or} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

EXAMPLE

$$\begin{aligned} 27^{\frac{2}{3}} &= (27^{\frac{1}{3}})^2 \\ &= (\sqrt[3]{27})^2 \\ &= 3^2 \\ &= 9 \end{aligned} \quad \text{or} \quad \begin{aligned} 27^{\frac{2}{3}} &= (27^2)^{\frac{1}{3}} \\ &= \sqrt[3]{27^2} \\ &= \sqrt[3]{243} \\ &= 9 \end{aligned}$$

PRACTICE Questions

Lesson 4.2

1. Write as a single power. Express your answers with positive exponents.
   a) $5(5^4)$
   b) $\dfrac{(-8)^4}{(-8)^5}$
   c) $(9^3)^6$
   d) $\dfrac{3(3)^6}{3^5}$
   e) $\left(\dfrac{1}{10}\right)^6\left(\dfrac{1}{10}\right)^{-4}$
   f) $\left(\dfrac{(7)^2}{(7)^4}\right)^{-1}$

2. Evaluate. Express answers in rational form.
   a) $4^{-2} - 8^{-1}$
   b) $(4 + 8)^0 - 5^{-2}$
   c) $25^{-1} + 3(5^{-1})^2$
   d) $\left(-\dfrac{1}{2}\right)^3 + 4^{-3}$

3. Evaluate. Express answers in rational form.
   a) $\left(\dfrac{4}{7}\right)^2$
   b) $\left(-\dfrac{2}{5}\right)^3$
   c) $\left(\dfrac{-2}{3}\right)^{-3}$
   d) $\dfrac{(-3)^{-2}}{(-3)^{-5}}$

Lesson 4.3

4. What restrictions are there on the value of x in $x^{-\frac{1}{2}}$? Are these restrictions different for $x^{\frac{1}{2}}$? Explain.

5. Evaluate. Express answers in rational form.
   a) $\left(\dfrac{49}{81}\right)^{\frac{1}{2}}$
   b) $\sqrt{\dfrac{100}{121}}$
   c) $\left(\dfrac{16}{9}\right)^{-0.5}$
   d) $((-125)^{\frac{1}{3}})^{-3}$
   e) $\sqrt[4]{(-9)^{-2}}$
   f) $\dfrac{-\sqrt[3]{512}}{\sqrt[5]{-1024}}$

6. Copy and complete the table. Express values in the last column in rational form.

	Exponential Form	Radical Form	Evaluation of Expression
a)	$100^{\frac{1}{2}}$		
b)	$16^{0.25}$		
c)		$\sqrt{121}$	
d)	$(-27)^{\frac{5}{3}}$		
e)	$49^{2.5}$		
f)		$\sqrt[10]{1024}$	

7. Evaluate. Express answers to three decimals.
   a) $-456^{\frac{4}{7}}$
   b) $98^{0.75}$
   c) $\left(\dfrac{5}{8}\right)^{\frac{5}{8}}$
   d) $(\sqrt[5]{-1000})^3$

8. Evaluate $-8^{\frac{4}{3}}$ and $(-8)^{\frac{4}{3}}$. Explain the difference between the two.

Lesson 4.4

9. Simplify. Express answers with positive exponents.
   a) $\dfrac{(x^{-3})x^5}{x^7}$
   b) $\dfrac{(n^{-4})n^{-6}}{(n^{-2})^7}$
   c) $\left(\dfrac{(y^2)^6}{y^9}\right)^{-2}$
   d) $\dfrac{(-2x^5)^3}{8x^{10}}$
   e) $(3a^2)^{-3}(9a^{-1})^2$
   f) $\dfrac{(4r^{-6})(-2r^2)^5}{(-2r)^4}$

10. Simplify. Express answers with positive exponents.
   a) $\dfrac{x^{0.5}y^{1.8}}{x^{0.3}y^{2.5}}$
   b) $\dfrac{(mn^3)^{-\frac{1}{2}}}{m^{\frac{1}{2}}n^{-\frac{5}{2}}}$
   c) $\dfrac{\sqrt{x^2y^4}}{(x^{-2}y^3)^{-1}}$
   d) $\left(\dfrac{2abc^3}{(2a^2b^3c)^2}\right)^{-2}$
   e) $\dfrac{\sqrt[4]{81p^8}}{\sqrt{9p^4}}$
   f) $\dfrac{\sqrt[6]{(8x^6)^2}}{\sqrt[4]{625x^8}}$

11. Evaluate each expression for $a = 2$ and $b = 3$. Express values in rational form.
   a) $\left(\dfrac{b^3}{a^{\frac{5}{2}}}\right)^2\left(\dfrac{2a^4}{b^5}\right)$
   b) $\sqrt{\dfrac{9b^3(ab)^2}{(a^2b^3)^3}}$

12. Simplify.
   a) $(a^{10+2p})(a^{-p-8})$
   b) $(2x^2)^{3-2m}\left(\dfrac{1}{x}\right)^{2m}$
   c) $[(c)^{2n-3m}](c^3)^m \div (c^2)^n$
   d) $(x^{4n-m})\left(\dfrac{1}{x^3}\right)^{m+n}$

4.5 Exploring the Properties of Exponential Functions

YOU WILL NEED
- graphing calculator
- graph paper

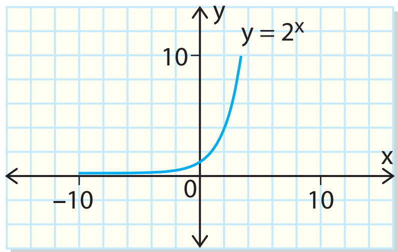
GOAL

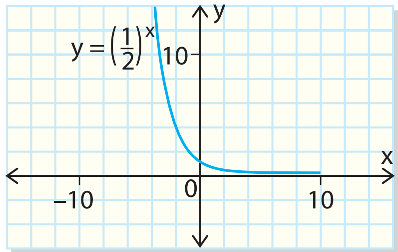
Determine the characteristics of the graphs and equations of exponential functions.

EXPLORE the Math

Functions such as $f(x) = 2^x$ and $g(x) = (\frac{1}{2})^x$ are examples of **exponential functions**. These types of functions can model many different phenomena, including population growth and the cooling of a liquid.

exponential function
a function of the form $y = a(b^x)$





? What are the characteristics of the graph of the exponential function $f(x) = b^x$, and how does it compare with the graphs of quadratic and linear functions?

A. Create a tables of values for each of the following functions. $g(x) = x$, $h(x) = x^2$, and $k(x) = 2^x$, where $-3 \le x \le 5$

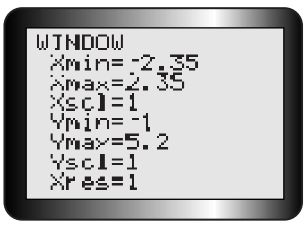
B. In each of your tables, calculate the first and second differences. Describe the difference patterns for each type of function.

C. Graph each function on graph paper and draw a smooth curve through each set of points. Label each curve with the appropriate equation.

D. State the domain and range of each function.

E. For each function, describe how values of the dependent variable, y, change as the values of the independent variable, x, increase and decrease.

F. Use a graphing calculator to graph the functions $y = 2^x$, $y = 5^x$, and $y = 10^x$. Graph all three functions on the same graph. Use the WINDOW settings shown.



G. For each function, state
- the domain and range
- the intercepts
- the equations of any **asymptotes**, if present

H. Examine the y-values as x increases and decreases. Which curve increases faster as you trace to the right? Which one decreases faster as you trace to the left?

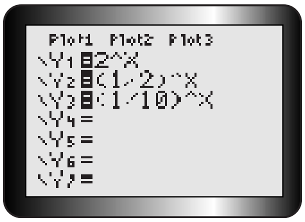
Tech Support

For help tracing functions on the graphing calculator, see Technical Appendix, B-2.

I. Delete the second and third functions $(y = 5^x$ and $y = 10^x)$, and replace them with $y = \left(\frac{1}{2}\right)^x$ and $y = \left(\frac{1}{10}\right)^x$ (or $y = 0.5^x$ and $y = 0.1^x)$.

J. For each new function, state
- the domain and range
- the intercepts
- the equations of any asymptotes

K. Describe how each of the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = \left(\frac{1}{10}\right)^x$ differs from $y = 2^x$ as the x-values increase and as they decrease.



L. Investigate what happens when the base of an exponential function is negative. Try $y = (-2)^x$. Discuss your findings.

M. Compare the features of the graphs of $f(x) = b^x$ for each group. Think about the domain, range, intercepts, and asymptotes.

i) different values of b when $b > 1$
ii) different values of b when $0 < b < 1$
iii) values of b when $0 < b < 1$, compared with values of $b > 1$
iii) values of $b < 0$ compared with values of $b > 0$

Reflecting

N. How do the differences for exponential functions differ from those for linear and quadratic functions? How can you tell that a function is exponential from its differences?

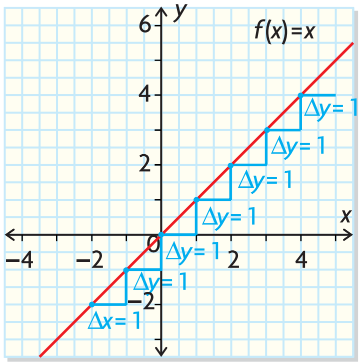
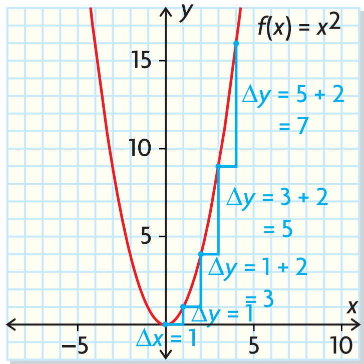
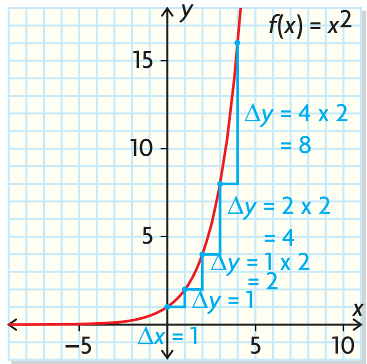
O. The base of an exponential function of the form $f(x) = b^x$ cannot be 1. Explain why this restriction is necessary.

P. Explain how you can distinguish an exponential function from a quadratic function and a linear function by using
- the graphs of each function
- a table of values for each function
- the equation of each function

In Summary

Key Ideas

- Linear, quadratic, and exponential functions have unique first-difference patterns that allow them to be recognized.

<table>
<tr><th>Linear</th><th>Quadratic</th><th>Exponential</th></tr>
<tr><td>Linear functions have constant first differences.</td><td>Quadratic functions have first differences that are related by addition. Their second differences are constant.</td><td>Exponential functions have first differences that are related by multiplication. Their second finite differences are not constant.</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
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- The exponential function $f(x) = b^x$ is
  - an increasing function representing growth when $b > 1$
  - a decreasing function representing decay when $0 < b < 1$

Need to Know

- The exponential function $f(x) = b^x$ has the following characteristics:
  - If $b > 0$, then the function is defined, its domain is $\{x \in \mathbf{R}\}$, and its range is $\{y \in \mathbf{R} \mid y \geq 0\}$.
  - If $b > 1$, then the greater the value of b, the faster the growth.
  - If $0 < b < 1$, then the lesser the value of b, the faster the decay.
  - The function has the x-axis, $y = 0$, as horizontal *asymptote*.
  - The function has a y-intercept of 1.
- Linear, quadratic, and exponential functions can be recognized from their graphs. Linear functions are represented by straight lines, quadratic functions by parabolas, and exponential functions by quickly increasing or decreasing curves with a horizontal asymptote.
- A function in which the variables have exponent 1 (e.g., $f(x) = 2x$) is linear. A function with a single squared term (e.g., $f(x) = 3x^2 - 1$) is quadratic. A function with a positive base (0 and 1 excluded) and variable exponent (e.g., $f(x) = 5^x$) is exponential.

FURTHER Your Understanding

1. Use differences to identify the type of function represented by the table of values.

a)

x	y
−4	5
−3	8
−2	13
−1	20
0	29
1	40

c)

x	y
−2	−2.75
0	−2
2	1
4	13
6	61
8	253

b)

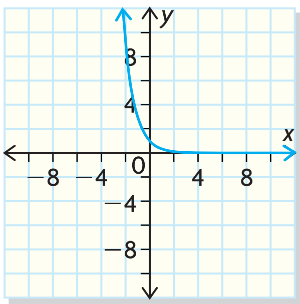
x	y
−5	32
−4	16
−3	8
−2	4
−1	2
0	1

d)

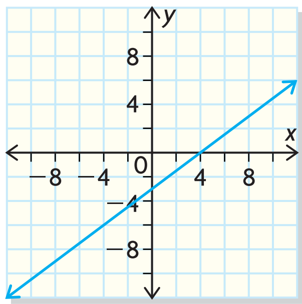
x	y
0.5	0.9
0.75	1.1
1	1.3
1.25	1.5
1.5	1.7
1.75	1.9

2. What type of function is represented in each graph? Explain how you know.

a)



c)



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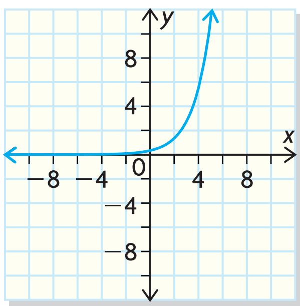
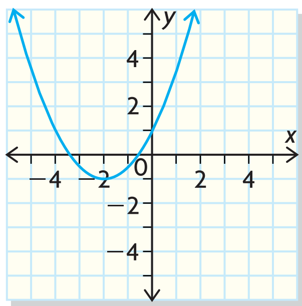


d)



4.6 Transformations of Exponential Functions

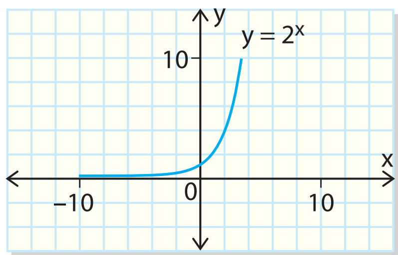
YOU WILL NEED
- graphing calculator

GOAL

Investigate the effects of transformations on the graphs and equations of exponential functions.

INVESTIGATE the Math

Recall the graph of the function $f(x) = 2^x$.
- It is an increasing function.
- It has a y-intercept of 1.
- Its asymptote is the line $y = 0$.



? If $f(x) = 2^x$, how do the parameters a, k, d, and c in the function $g(x) = af(k(x - d)) + c$ affect the size and shape of the graph of $f(x)$?

Tech Support

You can adjust to these settings by pressing ZOOM and 4 . ZDecimal .

A. Use your graphing calculator to graph the function $f(x) = 2^x$. Use the window settings shown.

B. Predict what will happen to the function $f(x) = 2^x$ if it is changed to
- $g(x) = 2^x + 1$ or $h(x) = 2^x - 1$
- $p(x) = 2^{x+1}$ or $q(x) = 2^{x-1}$

C. Copy and complete the table by graphing the given functions, one at a time, as Y2. Keep the graph of $f(x) = 2^x$ as Y1 for comparison. For each function, sketch the graph on the same grid and describe how its points and features have changed.

<table>
<tr><th>Function</th><th>Sketch</th><th>Table of Values</th><th>Description of Changes of New Graph</th></tr>
<tr><td>$g(x) = 2^x + 1$</td><td></td><td><table><tr><th>x</th><th>$y = 2^x$</th><th>$y = 2^x + 1$</th></tr><tr><td>-1</td><td></td><td></td></tr><tr><td>0</td><td></td><td></td></tr><tr><td>1</td><td></td><td></td></tr></table></td><td></td></tr>
<tr><td>$h(x) = 2^x - 1$</td><td></td><td><table><tr><th>x</th><th>$y = 2^x$</th><th>$y = 2^x - 1$</th></tr><tr><td>-1</td><td></td><td></td></tr><tr><td>0</td><td></td><td></td></tr><tr><td>1</td><td></td><td></td></tr></table></td><td></td></tr>
<tr><td>$p(x) = 2^{x+1}$</td><td></td><td><table><tr><th>x</th><th>$y = 2^x$</th><th>$y = 2^{x+1}$</th></tr><tr><td>-1</td><td></td><td></td></tr><tr><td>0</td><td></td><td></td></tr><tr><td>1</td><td></td><td></td></tr></table></td><td></td></tr>
<tr><td>$q(x) = 2^{x-1}$</td><td></td><td><table><tr><th>x</th><th>$y = 2^x$</th><th>$y = 2^{x-1}$</th></tr><tr><td>-1</td><td></td><td></td></tr><tr><td>0</td><td></td><td></td></tr><tr><td>1</td><td></td><td></td></tr></table></td><td></td></tr>
</table>

D. Describe the types of transformations you observed in part C. Comment on how the features and points of the original graph were changed by the transformations.

E. Predict what will happen to the function $f(x) = 2^x$ if it is changed to

- $g(x) = 3(2^x)$
- $h(x) = 0.5(2^x)$
- $j(x) = -(2^x)$

F. Create a table like the one in part C using the given functions in part E. Graph each function one at a time, as Y2. Keep the graph of $f(x) = 2^x$ as Y1 for comparison. In your table, sketch the graph on the same grid, complete the table of values, and describe how its points and features have changed.

G. Describe the types of transformations you observed in part F. Comment on how the features and points of the original graph were changed by the transformations.

H. Predict what will happen to the function $f(x) = 2^x$ if it is changed to
- $g(x) = 2^{2x}$
- $h(x) = 2^{0.5x}$
- $j(x) = 2^{-x}$

I. Create a table like the one in part C using the given functions in part H. Graph each function one at a time, as Y2. Keep the graph of $f(x) = 2^x$ as Y1 for comparison. In your table, sketch the graph, complete the table of values, and describe how its points and features have changed.

J. Describe the types of transformations you observed in part I. Comment on how the features and points of the original graph were changed by such transformations.

K. Choose several different bases for the original function. Experiment with different kinds of transformations. Are the changes in the function affected by the value of the base?

L. Summarize your findings by describing the roles that the parameters a, k, d, and c play in the function defined by $f(x) = ab^{k(x-d)} + c$.

Reflecting

M. Which transformations change the shape of the curve? Explain how the equation is changed by these transformations.

N. Which transformations change the location of the asymptote? Explain how the equation is changed by these transformations.

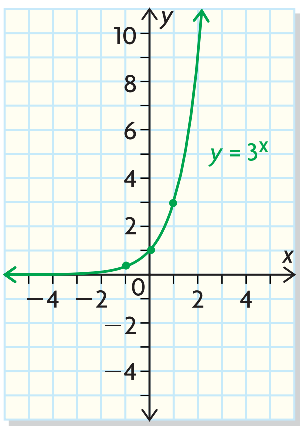
O. Do the transformations affect $f(x) = b^x$ in the same way they affect $f(x) = x$, $f(x) = x^2$, $f(x) = \frac{1}{x}$, $f(x) = \sqrt{x}$, and $f(x) = |x|$? Explain.

APPLY the Math

EXAMPLE 1 Using reasoning to predict the shape of the graph of an exponential function

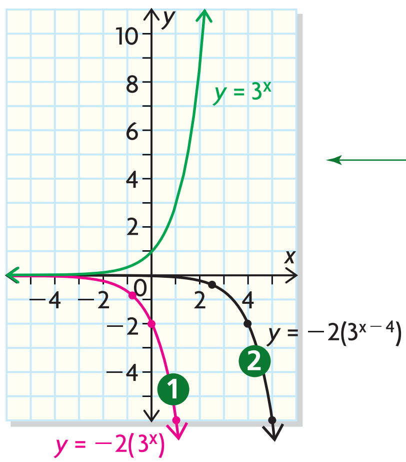
Use transformations to sketch the function $y = -2(3^{x-4})$. State the domain and range.

J.P.'s Solution



I began by sketching the graph of $y = 3^x$.

Three of its key points are (0, 1), (1, 3), and $(-1, \frac{1}{3})$. The asymptote is the x-axis, $y = 0$.



The function I really want to graph is $y = -2(3^{x-4})$. The base function, $y = 3^x$, was changed by multiplying all y-values by -2, resulting in a vertical stretch of factor 2 and a reflection in the x-axis.

Subtracting 4 from x results in a translation of 4 units to the right.

I could perform these two transformations in either order, since one affected only the x-coordinate and the other affected only the y-coordinate. I did the stretch first.

1 With vertical stretches and reflection in the x-axis (multiplying by -2, graphed in red), my key points had their y-values doubled:

$(0, 1) \rightarrow (0, -2)$, $(1, 3) \rightarrow (1, -6)$, and $(-1, \frac{1}{3}) \rightarrow (-1, -\frac{2}{3})$

The asymptote $y = 0$ was not affected.

2 With translations (subtracting 4, graphed in black), the key points changed by adding 4 to the x-values:

$(0, -2) \rightarrow (4, -2)$, $(1, -6) \rightarrow (5, -6)$, and $(-1, -\frac{2}{3}) \rightarrow (3, -\frac{2}{3})$

This shifted the curve 4 units to the right. The asymptote $y = 0$ was not affected.

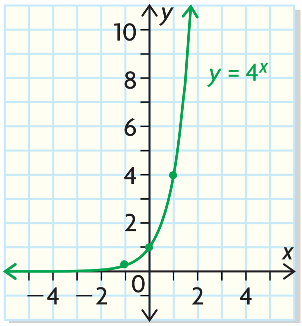
The domain of the original function, $\{x \in \mathbf{R}\}$, was not changed by the transformations.

The range, determined by the equation of the asymptote, was $y > 0$ for the original function. There was no vertical translation, so the asymptote remained the same, but, due to the reflection in the x-axis, the range changed to $\{y \in \mathbf{R} \mid y < 0\}$.

EXAMPLE 2 Connecting the graphs of different exponential functions

Use transformations to sketch the graph of $y = 4^{-2x-4} + 3$.

Ilia's Solution



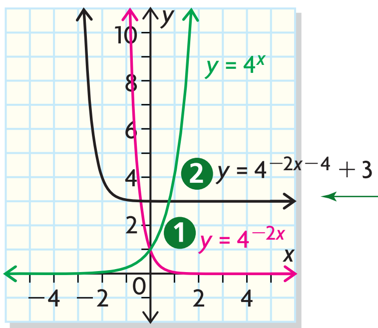
I began by sketching the graph of the base curve, $y = 4^x$. It has the line $y = 0$ as its asymptote, and three of its key points are (0, 1), (1, 4), and $\left(-1, \frac{1}{4}\right)$.

I factored the exponent to see the different transformations clearly:

$y = 4^{-2(x+2)} + 3$

The x-values were multiplied by -2, resulting in a horizontal compression of factor $\frac{1}{2}$, as well as a reflection in the y-axis.

There were two translations: 2 units to the left and 3 units up.



I applied the transformations in the proper order.

The table shows how the key points and the equation of the asymptote change:

Point or Asymptote	Horizontal Stretch and Reflection	Horizontal Translation	Vertical Translation
(0, 1)	(0, 1)	$(-2, 1)$	$(-2, 4)$
(1, 4)	$\left(-\frac{1}{2}, 4\right)$	$\left(-2\frac{1}{2}, 4\right)$	$\left(-2\frac{1}{2}, 7\right)$
$\left(-1, \frac{1}{4}\right)$	$\left(\frac{1}{2}, \frac{1}{4}\right)$	$\left(-1\frac{1}{2}, \frac{1}{4}\right)$	$\left(-1\frac{1}{2}, 3\frac{1}{4}\right)$
$y = 0$	$y = 0$	$y = 0$	$y = 3$

1 There was one stretch and one reflection, each of which applied only to the x-coordinate: a horizontal compression of factor $\frac{1}{2}$ and a reflection in the y-axis (shown in red).

2 There were two translations: 2 units to the left and 3 units up (shown in black).

EXAMPLE 3 Communicating the relationship among different exponential functions

Compare and contrast the functions defined by $f(x) = 9^x$ and $g(x) = 3^{2x}$.

Pinder's Solution: Using Exponent Rules

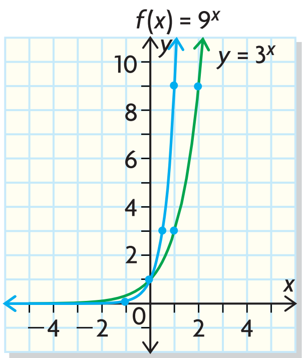
$$\begin{aligned} f(x) &= 9^x \\ &= (3^2)^x \\ &= 3^{2x} \\ &= g(x) \end{aligned}$$

Both functions are the same.

9 is a power of 3, so, to make it easier to compare 9^x with 3^{2x}, I substituted 3^2 for 9 in the first equation.

By the power-of-a-power rule, $f(x)$ has the same equation as $g(x)$.

Kareem's Solution



Both functions are the same.

$f(x) = 9^x$ is an exponential function with a y-intercept of 1 and the line $y = 0$ as its asymptote. Also, $f(x) = 9^x$ passes through the points (1, 9) and $\left(-1, \frac{1}{9}\right)$.

$g(x) = 3^{2x}$ is the base function $y = 3^x$ after a horizontal compression of factor $\frac{1}{2}$. This means that the key points change by multiplying their x-values by $\frac{1}{2}$. The point (1, 3) becomes (0.5, 3) and (2, 9) becomes (1, 9). When I plotted these points, I got points on the curve of $f(x)$.

EXAMPLE 4 Connecting the verbal and algebraic descriptions of transformations of an exponential curve

An exponential function with a base of 2 has been stretched vertically by a factor of 1.5 and reflected in the y-axis. Its asymptote is the line $y = 2$. Its y-intercept is (0, 3.5). Write an equation of the function and discuss its domain and range.

Louise's Solution

$$y = a2^{k(x-d)} + c$$

I began by writing the general form of the exponential equation with a base of 2.

$$y = 1.5(2^{-x}) + c$$

Since the function had been stretched vertically by a factor of 1.5, $a = 1.5$. The function has also been reflected in the y-axis, so $k = -1$. There was no horizontal translation, so $d = 0$.

$$y = 1.5(2^{-x}) + 2$$

Since the horizontal asymptote is $y = 2$ the function has been translated vertically by 2 units, so $c = 2$.

$$\begin{aligned} y &= 1.5(2^{-(0)}) + 2 \\ &= 1.5(1) + 2 \\ &= 3.5 \end{aligned}$$

I substituted $x = 0$ into the equation and calculated the y-intercept. It matched the stated y-intercept, so my equation seemed to represent this function.

The original domain is $\{x \in \mathbf{R}\}$. The transformations didn't change this.

The range changed, since there was a vertical translation. The asymptote moved up 2 units along with the function, so the range is $\{y \in \mathbf{R} \mid y > 2\}$.

In Summary

Key Ideas

- In functions of the form $g(x) = af(k(x - d)) + c$, the constants a, k, d, and c change the location or shape of the graph of $f(x)$. The shape is dependent on the value of the base function $f(x) = b^x$, as well as on the values of a and k.
- Functions of the form $g(x) = af(k(x - d)) + c$ can be graphed by applying the appropriate transformations to the key points of the base function $f(x) = b^x$, one at a time, following the order of operations. The horizontal asymptote changes when vertical translations are applied.

Need to Know

- In exponential functions of the form $g(x) = a\,b^{k(x-d)} + c$:
  - If $|a| > 1$, a vertical stretch by a factor of $|a|$ occurs. If $0 < |a| < 1$, a vertical compression by a factor of $|a|$ occurs. If a is also negative, then the function is reflected in the x-axis.
  - If $|k| > 1$, a horizontal compression by a factor of $\left|\frac{1}{k}\right|$ occurs. If $0 < |k| < 1$, a horizontal stretch by a factor of $\left|\frac{1}{k}\right|$ occurs. If k is also negative, then the function is reflected in the y-axis.
  - If $d > 0$, a horizontal translation of d units to the right occurs. If $d < 0$, a horizontal translation to the left occurs.
  - If $c > 0$, a vertical translation of c units up occurs. If $c < 0$, a vertical translation of c units down occurs.
  - You might have to factor the exponent to see what the transformations are. For example, if the exponent is $2x + 2$, it is easier to see that there was a horizontal stretch of 2 and a horizontal translation of 1 to the left if you factor to $2(x + 1)$.
  - When transforming functions, consider the order. You might perform stretches and reflections followed by translations, but if the stretch involves a different coordinate than the translation, the order doesn't matter.
  - The domain is always $\{x \in \mathbf{R}\}$. Transformations do not change the domain.
  - The range depends on the location of the horizontal asymptote and whether the function is above or below the asymptote. If it is above the asymptote, its range is $y > c$. If it is below, its range is $y < c$.

CHECK Your Understanding

1. Each of the following are transformations of $f(x) = 3^x$. Describe each transformation.

a) $g(x) = 3^x + 3$

b) $g(x) = 3^{x+3}$

c) $g(x) = \frac{1}{3}(3^x)$

d) $g(x) = 3^{\frac{x}{3}}$

2. For each transformation, state the base function and then describe the transformations in the order they could be applied.

a) $f(x) = -3(4^{x+1})$

b) $g(x) = 2\left(\frac{1}{2}\right)^{2x} + 3$

c) $h(x) = 7(0.5^{x-4}) - 1$

d) $k(x) = 5^{3x-6}$

3. State the y-intercept, the equation of the asymptote, and the domain and range for each of the functions in questions 1 and 2.

PRACTISING

4. Each of the following are transformations of $h(x) = (\frac{1}{2})^x$. Use words to describe the sequence of transformations in each case.

   a) $g(x) = -\left(\frac{1}{2}\right)^{2x}$

   b) $g(x) = 5\left(\frac{1}{2}\right)^{-(x-3)}$

   c) $g(x) = -4\left(\frac{1}{2}\right)^{3x+9} - 6$

5. Let $f(x) = 4^x$. For each function that follows,
   K
   - state the transformations that must be applied to $f(x)$
   - state the y-intercept and the equation of the asymptote
   - sketch the new function
   - state the domain and range

   a) $g(x) = 0.5f(-x) + 2$
   b) $h(x) = -f(0.25x + 1) - 1$
   c) $g(x) = -2f(2x - 6)$
   d) $h(x) = f(-0.5x + 1)$

6. Compare the functions $f(x) = 6^x$ and $g(x) = 3^{2x}$.

7. A cup of hot liquid was left to cool in a room whose temperature was 20 °C.
   C The temperature changes with time according to the function $T(t) = 80\left(\frac{1}{2}\right)^{\frac{t}{30}} + 20$. Use your knowledge of transformations to sketch this function. Explain the meaning of the y-intercept and the asymptote in the context of this problem.

8. The doubling time for a certain type of yeast cell is 3 h. The number of cells after t hours is described by $N(t) = N_0 2^{\frac{t}{3}}$, where N_0 is the initial population.

   a) How would the graph and the equation change if the doubling time were 9 h?

   b) What are the domain and range of this function in the context of this problem?

9. Match the equation of the functions from the list to the appropriate graph at the top of the next page.

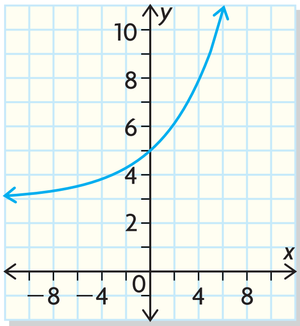
   a) $f(x) = -\left(\frac{1}{4}\right)^{-x} + 3$

   b) $y = \left(\frac{1}{4}\right)^{x} + 3$

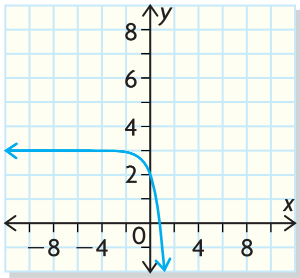
   c) $g(x) = -\left(\frac{5}{4}\right)^{-x} + 3$

   d) $h(x) = 2\left(\frac{5}{4}\right)^{x} + 3$

i)



iii)



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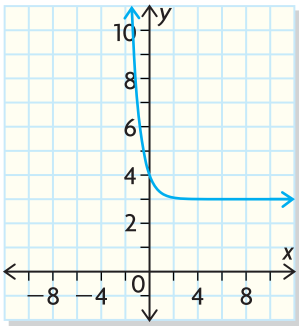

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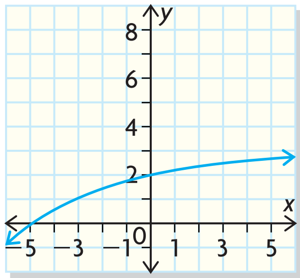
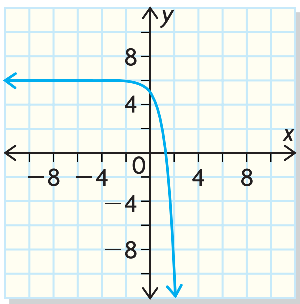
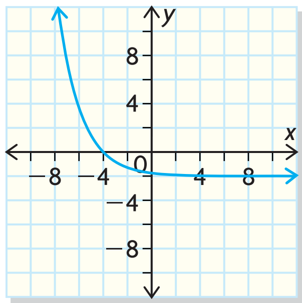


10. Each graph represents a transformation of the function $f(x) = 2^x$. Write an equation for each one.

a)



b)



11. State the transformations necessary (and in the proper order) to transform
T $f(x) = 2^{x+1} + 5$ to $g(x) = \frac{1}{4}(2^x)$.

Extending

12. Use your knowledge of transformations to sketch the function
$f(x) = \frac{-3}{2^{x+2}} - 1$.

13. Use your knowledge of transformations to sketch the function
$g(x) = 4 - 2\left(\frac{1}{3}\right)^{-0.5x+1}$.

14. State the transformations necessary (and in the proper order) to transform
$m(x) = -\left(\frac{3}{2}\right)^{2x-2}$ to $n(x) = -\left(\frac{9}{4}\right)^{-x+1} + 2$.

4.7 Applications Involving Exponential Functions

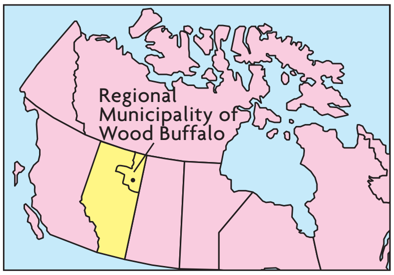
YOU WILL NEED

- graphing calculator

GOAL

Use exponential functions to solve problems involving exponential growth and decay.

The regional municipality of Wood Buffalo, Alberta, has experienced a large population increase in recent years due to the discovery of one of the world's largest oil deposits. Its population, 35 000 in 1996, has grown at an annual rate of approximately 8%.



? How long will it take for the population to double at this growth rate?

LEARN ABOUT the Math

EXAMPLE 1 Selecting a strategy to determine the doubling rate

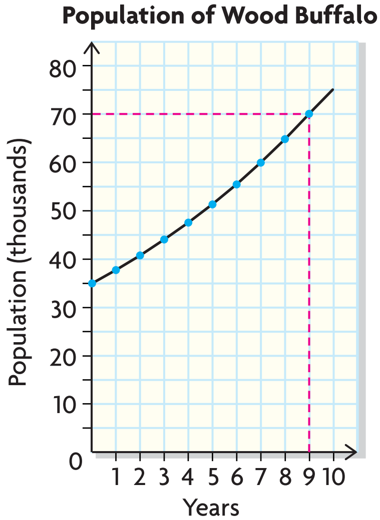
Carter's Solution: Using a Table of Values and a Graph

$0.08(35) + 35 = 35(0.08 + 1) = 35(1.08)$

When you add 8% of a number to the number, the new value is 108% of the old one. This is the same as multiplying by 1.08, so I created the table of values by repeatedly multiplying by 1.08.

I did this 10 times, once for each year, and saw that the population doubled to 70 000 after 9 years of growth.

Time (year from 1996)	0	1	2	3	4	5	6	7	8	9	10
Population (thousands)	35.0	37.8	40.8	44.1	47.6	51.4	55.5	60.0	64.8	70.0	75.6



I plotted the points and drew a smooth curve through the data.

I drew a horizontal line across the graph at 70 000 and saw that it touched the curve at 9 years.

Sonja's Solution: Creating an Algebraic Model

$P(1) = 35(1.08) = 37.8$

$P(2) = 37.8(1.08) = 40.8$

Substituting P(1) into P(2):

$$P(2) = 35(1.08)(1.08)$$
$$= 35(1.08)^2$$

So, $P(3) = 35(1.08)^2(1.08)$
$$= 35(1.08)^3$$

To calculate the population after 1 year, I needed to multiply 35 by 1.08. For each additional year, I repeatedly multiply by 1.08. Repeated multiplication can be represented with exponents. The value of the exponent will correspond to the number for the year.

This led to an algebraic model.

Therefore, $P(n) = 35(1.08)^n$

$$P(6) = 35(1.08)^6$$
$$= 35(1.586\ 874\ 323)$$
$$= 55.540\ 601\ 3$$

$$P(9) = 35(1.08)^9$$
$$= 35(1.999\ 004\ 627)$$
$$= 69.965\ 161\ 95 \doteq 70$$

The population would double in approximately 9 years at an 8% rate of growth.

Since population is a function of time, I expressed the relationship in function notation. I used $P(n)$, where the exponent, n, would represent the number of years after 1996 and $P(n)$ would represent the population in thousands.

I guessed that it would take 6 years for the population to double. I substituted $n = 6$ into the expression for the function, but it was too low.

I tried values for n until I got an answer that was close to the target of 70; $n = 9$ was pretty close.

Reflecting

A. Which features of the function indicate that it is exponential?

B. Describe what each part of the equation $P(n) = 35(1.08)^n$ represents in the context of the problem *and* the features of the graph.

C. Compare Carter's and Sonja's solutions. Which approach do you think is better? Why?

APPLY the Math

EXAMPLE 2 Solving an exponential decay problem, given the equation

A 200 g sample of radioactive polonium-210 has a *half-life* of 138 days. This means that every 138 days, the amount of polonium left in a sample is half of the original amount. The mass of polonium, in grams, that remains after t days can be modelled by $M(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{138}}$.

a) Determine the mass that remains after 5 years.

b) How long does it take for this 200 g sample to decay to 110 g?

Zubin's Solution: Using the Algebraic Model

a)
$$\begin{aligned} 5 \text{ years} &= 5(365) \text{ days} \\ &= 1825 \text{ days} \end{aligned}$$

Since the half-life is measured in days, I converted the number of years to days before substituting into the function.

$$\begin{aligned} M(1825) &= 200\left(\frac{1}{2}\right)^{\frac{1825}{138}} \\ &\doteq 200(0.000\,104\,5) \\ &\doteq 0.021 \end{aligned}$$

I used my calculator to determine the answer.

There is approximately 0.02 g of polonium-210 left after 5 years.

b)
$$M(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{138}}$$

I began by writing the equation and substituting the amount of the sample remaining.

$$110 = 200\left(\frac{1}{2}\right)^{\frac{t}{138}}$$

I needed to isolate t in the equation, so I divided each side by 200.

$$\frac{110}{200} = \left(\frac{1}{2}\right)^{\frac{t}{138}}$$

I didn't know how to isolate t, so I used guess and check to find the answer.

$$0.55 = \left(\frac{1}{2}\right)^{\frac{t}{138}}$$

$$M(100) = 200\left(\frac{1}{2}\right)^{\frac{100}{138}}$$

$$\doteq 121 \text{ g}$$

I knew that if the exponent was 1 (t = 138 days), the original amount would be halved, but the amount I needed to find was 110 g, so the exponent needed to be less than 1. I guessed 100 days, which I substituted into the original equation. I calculated the answer.

It was too high, which meant that my guess was too low.

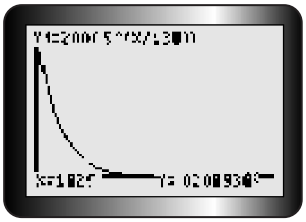
$$M(119) = 200\left(\frac{1}{2}\right)^{\frac{119}{138}}$$

$$\doteq 110 \text{ g}$$

I guessed and checked a few more times until I found the answer of approximately 119 days.

Barry's Solution: Using a Graphical Model

a)



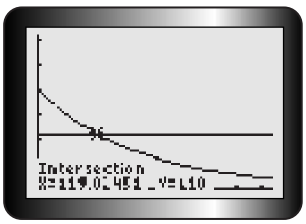
I graphed $M(t)$, then used the value operation. I had to change 5 years into days.

There is about 0.02 g remaining after 5 years.

b)

I graphed $M(t)$, and I graphed a horizontal line to represent 110 g.

I knew that the point where the line met the curve would represent the answer.



I used the "Intersect" operation on the graphing calculator to find the point. The x-value represents the number of days.

It takes approximately 119 days for the sample to decay to 110 g.

Tech Support

For help determining the point(s) of intersection between two functions, see Technical Appendix, B-12.

EXAMPLE 3 Solving a problem by determining an equation for a curve of good fit

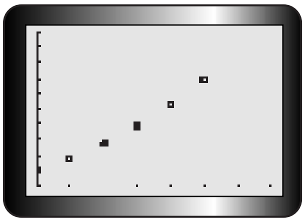
A biologist tracks the population of a new species of frog over several years. From the table of values, determine an equation that models the frog's population growth, and determine the number of years before the population triples.

Year	0	1	2	3	4	5
Population	400	480	576	691	829	995

Tech Support

For help creating scatter plots on a graphing calculator, see Technical Appendix, B-11.

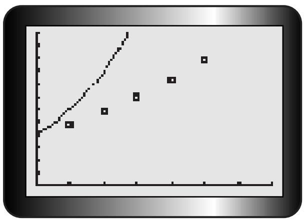
Fred's Solution



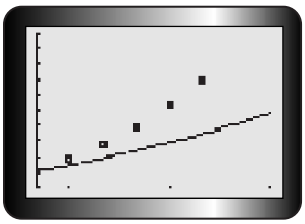
I used my graphing calculator to create a scatter plot.

The equation is of the form $P(t) = ab^t$, where
- $P(t)$ represents the population in year t
- a is the initial population
- b is the base of the exponential function

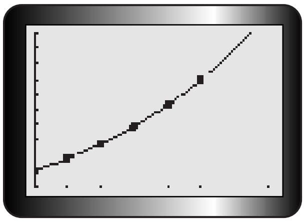
Since the function is increasing, $b > 1$. The initial population occurs when $x = 0$. That means that $a = 400$. If $b = 2$, then the population would have doubled, but it went up by only 80 in the first year, so the value of b must be less than 2.



I tried $b = 1.5$ and entered the equation $P(t) = 400(1.5)^t$ into the equation editor. The graph rose too quickly, so 1.5 is too great for b.



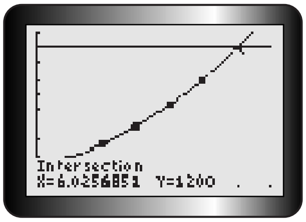
I changed the equation to $P(t) = 400(1.1)^t$. I graphed the equation on the calculator. I checked to see if the curve looked right. It rose too slowly, so b must be between 1.1 and 1.5.



I continued this process until I found a good fit when $b = 1.2$.

The equation that models this population is

$P(t) = 400(1.2)^t$



To determine the year that the population tripled, I graphed the line $P = 1200$ and found the intersection point of the curve and the line.

From this graph, I determined that the frog population tripled in approximately 6 years.

EXAMPLE 4 Representing a real-world problem with an algebraic model

A new car costs $24 000. It loses 18% of its value each year after it is purchased. This is called *depreciation*. Determine the value of the car after 30 months.

Gregg's Solution

$y = ab^x$

The car's value decreases each year. Another way to think about the car *losing* 18% of its value each year is to say that it *keeps* 82% of its value. To determine its value, I multiplied its value in the previous year by 0.82. The repeated multiplication suggested that this relationship is exponential. That makes sense, since this has to be a decreasing function where $0 < b < 1$.

$$V(n) = 24(0.82)^n$$

I used V and n to remind me of what they represented.

The base of the exponential function that models the value of the car is 0.82. The initial value is \$24 000, which is the value of a and the exponent n is measured in years.

$$\begin{aligned} n &= 30 \text{ months} \\ &= 30 \div 12 \text{ years} \\ &= 2.5 \text{ years} \end{aligned}$$

I converted 30 months to years to get my answer.

$$\begin{aligned} V(2.5) &= 24(0.82)^{2.5} \\ &= 24(0.608\,884\,097) \\ &\doteq 14.6 \end{aligned}$$

The car is worth about \$14 600 after 30 months.

In Summary

Key Ideas

- The exponential function $f(x) = ab^x$ and its graph can be used as a model to solve problems involving exponential growth and decay. Note that
  - $f(x)$ is the final amount or number
  - a is the initial amount or number
  - for exponential growth, $b = 1 +$ growth rate; for exponential decay, $b = 1 -$ decay rate
  - x is the number of growth or decay periods

Need to Know

- For situations that can be modeled by an exponential function:
  - If the *growth rate* (as a percent) is given, then the base of the power in the equation can be obtained by *adding* the rate, as a decimal, to 1. For example, a growth rate of 8% involves multiplying repeatedly by 1.08.
  - If the *decay rate* (as a percent) is given, then the base of the power in the equation is obtained by *subtracting* the rate, as a decimal, from 1. For example, a decay rate of 8% involves multiplying repeatedly by 0.92.
  - One way to tell the difference between growth and decay is to consider whether the quantity in question (e.g., light intensity, population, dollar value) has increased or decreased.
  - The units for the growth/decay rate and for the number of growth/decay periods must be the same. For example, if light intensity decreases "per metre," then the number of decay periods in the equation is measured in metres, too.

CHECK Your Understanding

1. Solve each exponential equation. Express answers to the nearest hundredth of a unit.

a) $A = 250(1.05)^{10}$

b) $P = 9000\left(\frac{1}{2}\right)^{8}$

c) $500 = N_0(1.25)^{1.25}$

d) $625 = P(0.71)^{9}$

2. Complete the table.

	Function	Exponential Growth or Decay?	Initial Value	Growth or Decay Rate
a)	$V(t) = 20(1.02)^t$			
b)	$P(n) = (0.8)^n$			
c)	$A(x) = 0.5(3)^x$			
d)	$Q(w) = 600\left(\frac{5}{8}\right)^w$			

3. The growth in population of a small town since 1996 is given by the function $P(n) = 1250(1.03)^n$.

a) What is the initial population? Explain how you know.

b) What is the growth rate? Explain how you know.

c) Determine the population in the year 2007.

d) In which year does the population reach 2000 people?

4. A computer loses its value each month after it is purchased. Its value as a function of time, in months, is modelled by $V(m) = 1500(0.95)^m$.

a) What is the initial value of the computer? Explain how you know.

b) What is the rate of depreciation? Explain how you know.

c) Determine the value of the computer after 2 years.

d) In which month after it is purchased does the computer's worth fall below $900?

PRACTISING

5. In 1990, a sum of $1000 is invested at a rate of 6% per year for 15 years.

a) What is the growth rate?

b) What is the initial amount?

c) How many growth periods are there?

d) Write an equation that models the growth of the investment, and use it to determine the value of the investment after 15 years.

6. A species of bacteria has a population of 500 at noon. It doubles every 10 h. The function that models the growth of the population, P, at any hour, t, is

K

$$P(t) = 500\left(2^{\frac{t}{10}}\right).$$

a) Why is the exponent $\frac{t}{10}$?
b) Why is the base 2?
c) Why is the multiplier 500?
d) Determine the population at midnight.
e) Determine the population at noon the next day.
f) Determine the time at which the population first exceeds 2000.

7. Which of these functions describe exponential decay? Explain.

a) $g(x) = -4(3)^x$
b) $h(x) = 0.8(1.2)^x$
c) $j(x) = 3(0.8)^{2x}$
d) $k(x) = \frac{1}{3}(0.9)^{\frac{x}{2}}$

8. A town with a population of 12 000 has been growing at an average rate of 2.5% for the last 10 years. Suppose this growth rate will be maintained in the future. The function that models the town's growth is

$$P(n) = 12(1.025^n)$$

where $P(n)$ represents the population (in thousands) and n is the number of years from now.

a) Determine the population of the town in 10 years.
b) Determine the number of years until the population doubles.
c) Use this equation (or another method) to determine the number of years ago that the population was 8000. Answer to the nearest year.
d) What are the domain and range of the function?

9. A student records the internal temperature of a hot sandwich that has been left to cool on a kitchen counter. The room temperature is 19 °C. An equation that models this situation is

A

$$T(t) = 63(0.5)^{\frac{t}{10}} + 19$$

where T is the temperature in degrees Celsius and t is the time in minutes.

a) What was the temperature of the sandwich when she began to record its temperature?
b) Determine the temperature, to the nearest degree, of the sandwich after 20 min.
c) How much time did it take for the sandwich to reach an internal temperature of 30 °C?

10. In each case, write an equation that models the situation described. Explain what each part of each equation represents.

a) the percent of colour left if blue jeans lose 1% of their colour every time they are washed

b) the population if a town had 2500 residents in 1990 and grew at a rate of 0.5% each year after that for t years

c) the population of a colony if a single bacterium takes 1 day to divide into two; the population is P after t days

11. A population of yeast cells can double in as little as 1 h. Assume an initial population of 80 cells.

a) What is the growth rate, in percent per hour, of this colony of yeast cells?

b) Write an equation that can be used to determine the population of cells at t hours.

c) Use your equation to determine the population after 6 h.

d) Use your equation to determine the population after 90 min.

e) Approximately how many hours would it take for the population to reach 1 million cells?

f) What are the domain and range for this situation?

12. A collector's hockey card is purchased in 1990 for \$5. The value increases by 6% every year.

a) Write an equation that models the value of the card, given the number of years since 1990.

b) Determine the increase in value of the card in the 4th year after it was purchased (from year 3 to year 4).

c) Determine the increase in value of the card in the 20th year after it was purchased.

13. Light intensity in a lake falls by 9% per metre of depth relative to the surface.

a) Write an equation that models the intensity of light per metre of depth. Assume that the intensity is 100% at the surface.

b) Determine the intensity of light at a depth of 7.5 m.

14. A disinfectant is advertised as being able to kill 99% of all germs with each application.

a) Write an equation that represents the percent of germs left with n applications.

b) Suppose a kitchen countertop has 10 billion (10^{10}) germs. How many applications are required to eliminate all of the germs?

15. T A town has a population of 8400 in 1990. Fifteen years later, its population grew to 12 500. Determine the average annual growth rate of this town's population.

16. C A group of yeast cells grows by 75% every 3 h. At 9 a.m., there are 200 yeast cells.

a) Write an equation that models the number of cells, given the number of hours after 9 a.m.

b) Explain how each part of your equation is related to the given information.

Extending

17. In the year 2002, a single baby girl born in Alberta was given the name Nevaeh.
- Two years later, there were 18 girls (including the first one) with that name.
- By 2005, there were 70 girls with the name (*National Post,* Wed., May 24, 2006, p. A2).

a) Investigate whether or not this is an example of exponential growth.
b) Determine what the growth rate might be, and create a possible equation to model the growth in the popularity of this name.
c) Discuss any limitations of your model.

18. Psychologist H. Ebbinghaus performed experiments in which he had people memorize lists of words and then tested their memory of the list. He found that the percent of words they remembered can be modelled by

$$R(T) = \frac{100}{1 + 1.08T^{0.21}}$$

where $R(T)$ is the percent of words remembered after T hours. This equation is now known as the "forgetting curve," even though it actually models the percent of words remembered!
a) Graph this function with technology. Describe its features and decide whether or not it is an example of exponential decay.
b) Predict the percent of words remembered after 24 h.

Curious | *Math*

Zeno's Paradox

Zeno of Elea (c. 490–425 BCE), a Greek philosopher and mathematician, is famous for his paradoxes that deal with motion. (A paradox is a statement that runs counter to common sense, but may actually be true.) Zeno suggested that it is impossible to get to point *B* from point *A*.

He illustrated his point of view with a story.

Achilles (point *A*) and Tortoise agreed to have a race. Tortoise was given a head start (point *B*). After the race started, Achilles travelled half the distance between himself and Tortoise (point *C*). And again, after a period of time, he travelled half the remaining distance between himself and Tortoise (point *D*). Each time he arrived at the halfway point, there was a new, and smaller, halfway point. So if you look at it this way, there is an infinite number of halfway points and Achilles would never catch up to Tortoise.

1. What is the function that models this problem, if Tortoise was given a head start of 1000 m? Does this function support Zeno's paradox? Explain.

2. Who will win the race between Achilles and Tortoise? Explain.

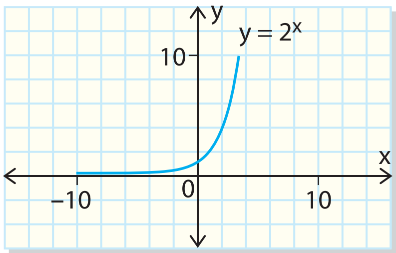
4 Chapter Review

FREQUENTLY ASKED Questions

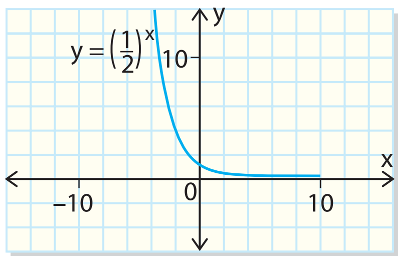
Q: How can you identify an exponential function from
- its equation?
- its graph?
- a table of values?

A: The exponential function has the form $f(x) = b^x$, where the variable is an exponent.

The shape of its graph depends upon the parameter b.



If $b > 1$, then the curve increases as x increases.



If $0 < b < 1$, then the curve decreases as x increases.

In each case, the function has the x-axis (the line $y = 0$) as its horizontal asymptote.

A differences table for an exponential function shows that the differences are never constant, as they are for linear and quadratic functions. They are related by a multiplication pattern.

x	$y = 3^x$	First Differences	Second Differences
0	1		
		2	
1	3		4
		6	
2	9		12
		18	
3	27		36
		54	
4	81		108
		162	
5	243		324
		486	
6	729		

Study *Aid*
- See Lesson 4.5.
- Try Chapter Review Questions 9 and 10.

Study Aid

- See Lesson 4.6, Examples 1, 2, and 3.
- Try Chapter Review Questions 11 and 12.

Q: How can transformations help in drawing the graphs of exponential functions?

A: Functions of the form $g(x) = af(k(x - d)) + c$ can be graphed by applying the appropriate transformations to the key points and asymptotes of the parent function $f(x) = b^x$, following an appropriate order—often, stretches and compressions, then reflections, and finally translations.

In functions of the form $g(x) = ab^{k(x-d)} + c$, the constants a, k, d, and c change the location or shape of the graph of $f(x)$. The shape of the graph of $g(x)$ depends on the value of the base of the function, $f(x) = b^x$.

- a represents the vertical stretch or compression factor. If $a < 0$, then the function has also been reflected in the x-axis.
- k represents the horizontal stretch or compression factor. If $k < 0$, then the function has also been reflected in the y-axis.
- c represents the number of units of vertical translation up or down.
- d represents the number of units of horizontal translation right or left.

Study Aid

- See Lesson 4.7, Examples 1, 2, 3 and 4.
- Try Chapter Review Questions 13 to 17.

Q: How can exponential functions model growth and decay? How can you use them to solve problems?

A: Exponential functions can be used to model phenomena exhibiting repeated multiplication of the same factor.

Each formula is modelled after the exponential function

$$y = ab^x$$

The amount in the future

The present value or initial amount

The growth/decay factor: if growth, then $b > 1$; if decay, then $0 < b < 1$

The number of periods of growth or decay

When solving problems, list these four elements of the equation and fill in the data as you read the problem. This will help you organize the information and create the equation you require to solve the problem.

Here are some examples:

Growth	Decay
Cell division (doubling bacteria, yeast cells, etc.): $P(t) = P_0(2)^{\frac{t}{D}}$	Radioactivity or half-life: $N(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{H}}$
Population growth: $P(n) = P_0(1 + r)^n$	Depreciation of assets: $V(n) = V_0(1 - r)^n$
Growth in money: $A(n) = P(1 + i)^n$	Light intensity in water: $V(n) = 100(1 - r)^n$

PRACTICE Questions

Lesson 4.2

1. **a)** If $x > 1$, which is greater, x^{-2} or x^2? Why?
b) Are there values of x that make the statement $x^{-2} > x^2$ true? Explain.

2. Write each as a single power. Then evaluate. Express answers in rational form.

a) $(-7)^3(-7)^{-4}$

b) $\dfrac{(-2)^8}{(-2)^3}$

c) $\dfrac{(5)^{-3}(5)^6}{5^3}$

d) $\dfrac{4^{-10}(4^{-3})^6}{(4^{-4})^8}$

e) $(11)^9\left(\dfrac{1}{11}\right)^7$

f) $\left(\dfrac{(-3)^7(-3)^4}{(-3^4)^3}\right)^{-3}$

Lesson 4.3

3. Express each radical in exponential form and each power in radical form.

a) $\sqrt[3]{x^7}$

b) $y^{\frac{8}{5}}$

c) $(\sqrt{p})^{11}$

d) $m^{1.25}$

4. Evaluate. Express answers in rational form.

a) $\left(\dfrac{2}{5}\right)^{-3}$

b) $\left(\dfrac{16}{225}\right)^{-0.5}$

c) $\dfrac{(81)^{-0.25}}{\sqrt[3]{-125}}$

d) $(\sqrt[3]{-27})^4$

e) $(\sqrt[5]{-32})(\sqrt[6]{64})^5$

f) $\sqrt[6]{((-2)^3)^2}$

5. Simplify. Write with only positive exponents.

a) $a^{\frac{3}{2}}(a^{-\frac{3}{2}})$

b) $\dfrac{b^{0.8}}{b^{-0.2}}$

c) $\dfrac{c\left(c^{\frac{5}{6}}\right)}{c^2}$

d) $\dfrac{d^{-5}d^{\frac{11}{2}}}{(d^{-3})^2}$

e) $((e^{-2})^{\frac{7}{2}})^{-2}$

f) $((f^{-\frac{1}{6}})^{\frac{6}{5}})^{-1}$

6. Explain why $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$, for $a > 0$ and $b > 0$.

Lesson 4.4

7. Evaluate each expression for the given values. Express answers in rational form.

a) $(5x)^2(2x)^3$; $x = -2$

b) $\dfrac{8m^{-5}}{(2m)^{-3}}$; $m = 4$

c) $\dfrac{2w(3w^{-2})}{(2w)^2}$; $w = -3$

d) $\dfrac{(9y)^2}{(3y^{-1})^3}$; $y = -2$

e) $(6(x^{-4})^3)^{-1}$; $x = -2$

f) $\dfrac{(-2x^{-2})^3(6x)^2}{2(-3x^{-1})^3}$; $x = \dfrac{1}{2}$

8. Simplify. Write each expression using only positive exponents. All variables are positive.

a) $\sqrt[3]{27x^3y^9}$

b) $\sqrt{\dfrac{a^6b^5}{a^8b^3}}$

c) $\dfrac{m^{\frac{3}{2}}n^{-2}}{m^{\frac{7}{2}}n^{-\frac{3}{2}}}$

d) $\dfrac{\sqrt[4]{x^{-16}(x^6)^{-6}}}{(x^4)^{-\frac{11}{2}}}$

e) $((-x^{0.5})^3)^{-1.2}$

f) $\dfrac{\sqrt{x^6(y^3)^{-2}}}{(x^3y)^{-2}}$

Lesson 4.5

9. Identify the type of function (linear, quadratic, or exponential) for each table of values.

a)

x	y
−5	−38
0	−3
5	42
10	97
15	162
20	237

b)

x	y
0	−45
2	−15
4	15
6	45
8	75
10	105

c)

x	y
1	13
2	43
3	163
4	643
5	2 563
6	10 243

e)

x	y
−2	2000
−1	1000
0	500
1	250
2	125
3	62.5

d)

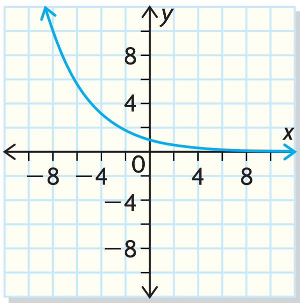
x	y
−2	40
−1	20
0	10
1	5
2	2.5
3	1.25

f)

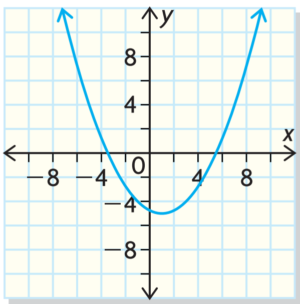
x	y
0.2	−10.8
0.4	−9.6
0.6	−7.2
0.8	−2.4
1	7.2
1.2	26.4

10. Identify each type of function (linear, quadratic, or exponential) from its graph.

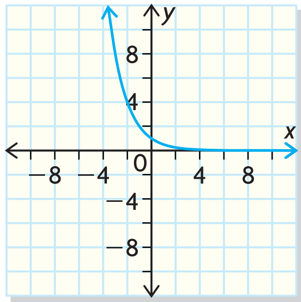
a)



b)



c)



Lesson 4.6

11. For each exponential function, state the base function, $y = b^x$. Then state the transformations that map the base function onto the given function. Use transformations to sketch each graph.

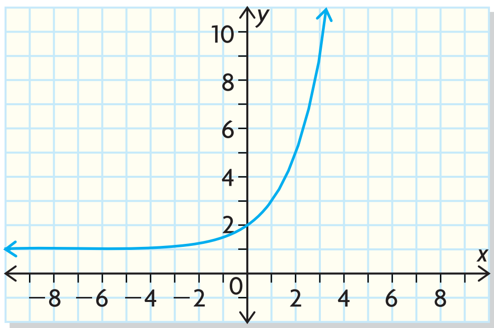
a) $y = \left(\frac{1}{2}\right)^{\frac{x}{2}} - 3$

b) $y = \frac{1}{4}(2)^{-x} + 1$

c) $y = -2(3)^{2x+4}$

d) $y = \frac{-1}{10}(5)^{3x-9} + 10$

12. The exponential function shown has been reflected in the y-axis and translated vertically. State its y-intercept, its asymptote, and a possible equation for it.



Lesson 4.7

13. Complete the table.

	Function	Exponential Growth or Decay?	Initial Value (y-intercept)	Growth or Decay Rate
a)	$V(t) = 100(1.08)^t$			
b)	$P(n) = 32(0.95)^n$			
c)	$A(x) = 5(3)^x$			
d)	$Q(n) = 600\left(\frac{5}{8}\right)^n$			

14. A hot cup of coffee cools according to the equation

$$T(t) = 69\left(\frac{1}{2}\right)^{\frac{t}{30}} + 21$$

where T is the temperature in degrees Celsius and t is the time in minutes.

a) Which part of the equation indicates that this is an example of exponential decay?
b) What was the initial temperature of the coffee?
c) Use your knowledge of transformations to sketch the graph of this function.
d) Determine the temperature of the coffee, to the nearest degree, after 48 min.
e) Explain how the equation would change if the coffee cooled faster.
f) Explain how the graph would change if the coffee cooled faster.

15. The value of a car after it is purchased depreciates according to the formula

$$V(n) = 28\,000(0.875)^n$$

where $V(n)$ is the car's value in the nth year since it was purchased.

a) What is the purchase price of the car?
b) What is the annual rate of depreciation?
c) What is the car's value at the end of 3 years?
d) What is its value at the end of 30 months?
e) How much value does the car lose in its first year?
f) How much value does it lose in its fifth year?

16. Write the equation that models each situation. In each case, describe each part of your equation.
a) the percent of a pond covered by water lilies if they cover one-third of a pond now and each week they increase their coverage by 10%
b) the amount remaining of the radioactive isotope U_{238} if it has a half-life of 4.5×10^9 years
c) the intensity of light if each gel used to change the colour of a spotlight reduces the intensity of the light by 4%

17. The population of a city is growing at an average rate of 3% per year. In 1990, the population was 45 000.
a) Write an equation that models the growth of the city. Explain what each part of the equation represents.
b) Use your equation to determine the population of the city in 2007.
c) Determine the year during which the population will have doubled.
d) Suppose the population took only 10 years to double. What growth rate would be required for this to have happened?

4 Chapter Self-Test

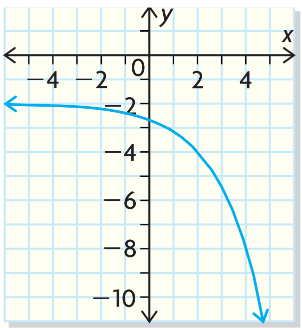
1. The function $f(x) = -\frac{1}{2}(3^{2x+4}) + 5$ is the transformation of the function $g(x) = 3^x$.
   a) Explain how you can tell what type of function $f(x)$ represents just by looking at the equation.
   b) Create a table of values for $f(x)$. Describe how to tell the type of function it is from its table of values.
   c) Describe the transformations necessary (in the proper order) that map $g(x)$ onto $f(x)$. Sketch $f(x)$ and state the equation of its asymptote.

2. Evaluate. Express answers as rational numbers.
   a) $(-5)^{-3}$
   b) $27^{\frac{2}{3}}$

3. Simplify. Use only positive exponents in your final answers.
   a) $(-3x^2y)^3(-3x^{-3}y)^2$
   b) $\dfrac{(5a^{-1}b^2)^{-2}}{125a^5b^{-3}}$
   c) $\sqrt[5]{\dfrac{1024(x^{-1})^{10}}{(2x^{-3})^5}}$
   d) $\dfrac{(8x^6y^{-3})^{\frac{1}{3}}}{(2xy)^3}$

4. A spotlight uses coloured gels to create the different colours of light required for a theatrical production. Each gel reduces the original intensity of the light by 3.6%.
   a) Write an equation that models the intensity of light, I, as a function of the number of gels used.
   b) Use your equation to determine the percent of light left if three gels are used.
   c) Explain why this is an example of exponential decay.

5. A small country that had 2 million inhabitants in 1990 has experienced an average growth in population of 4% per year since then.
   a) Write an equation that models the population, P, of this country as a function of the number of years, n, since 1990.
   b) Use your equation to determine when the population will double (assuming that the growth rate remains stable).



6. Which of these equations correspond to the graph? Explain how you know.
   a) $f(x) = 2(3^{-x}) + 5$
   b) $g(x) = (3^{-2x-4}) - 5$
   c) $h(x) = -0.8(3^{x-3})$
   d) $p(x) = -2\left(3^{\frac{1}{2}x-1}\right) - 2$

7. What are the restrictions on the value of n in $a^{\frac{1}{n}}$ if $a < 0$? Explain.

4 Chapter Task

Modelling Population

Every two years, the United Nations Population Division prepares estimates and projections of world, regional, and national population size and growth.

The estimated population of the world since 1950 is given in the table.

? What is the equation of the function you could use to model the world's population?

Year	Years since 1950	World Population (billions)
1950	0	2.55
1955	5	2.8
1960	10	3
1965	15	3.3
1970	20	3.7
1975	25	4
1980	30	4.5
1985	35	4.85
1990	40	5.3
1995	45	5.7
2000	50	6.1

A. Use the data from the table to create a scatter plot on graph paper.

B. Draw a curve of good fit. What type of function is this? Explain.

C. Using a graphing calculator, create a scatter plot.

D. Use the data in the table to estimate the average growth rate for a 5-year period and the y-intercept of the function.

E. Use the values you found in part D to write an equation for this function. Graph your equation. How well does your graph fit the data?

F. Make any necessary adjustments to your equation. Do so as often as needed, until you are satisfied with the fit.

G. In 2004, the UN predicted that the world's population will peak at 9.2 billion by 2075 and decline slightly to 8.97 billion by 2300. Does the model you found predict these same values?

H. Write a report that summarizes your findings. In your report, include your graph-paper scatter plot and the prediction of the type of function you thought this might be, supported by your reasons. Discuss
- the estimated average 5-year growth rate and the y-intercept of the function
- the original equation you determined
- the changes you made to your original equation and the reasons for those changes
- the calculations you used to check whether your model matches the future population predictions made by the UN
- why the UN predictions may differ from those of your model

Task *Checklist*

- ✔ Did you include all the required elements in your report?
- ✔ Did you properly label the graphs including some values?
- ✔ Did you show your work in your choice of the equation for part E?
- ✔ Did you support your decision in part G?

Chapter

5

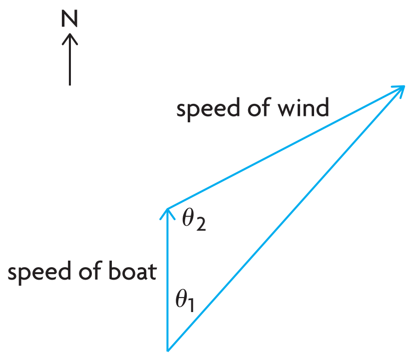
Trigonometric Ratios

GOALS

You will be able to

- Relate the six trigonometric ratios to the unit circle
- Solve real-life problems by using trigonometric ratios, properties of triangles, and the sine and cosine laws
- Prove simple trigonometric identities

? How would changes in the boat's speed and the wind's speed affect the angles in the vector diagram and the speed and direction of the boat?



5 Getting Started

Study | **Aid**

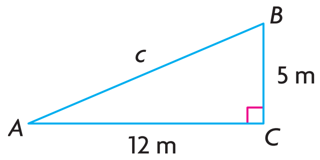
- For help, see Essential Skills Appendix.

Question	Appendix
1	A-4
2–7	A-16
8	A-17

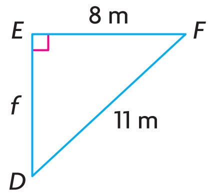
SKILLS AND CONCEPTS You Need

1. Use the Pythagorean theorem to determine each unknown side length.

a)



b)



2. Using the triangles in question 1, determine the sine, cosine, and tangent ratios for each given angle.

a) $\angle A$ **b)** $\angle D$

3. Using the triangles in question 1, determine each given angle to the nearest degree.

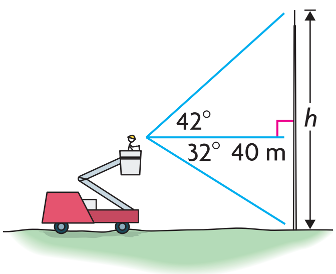
a) $\angle B$ **b)** $\angle F$

4. Use a calculator to evaluate to the nearest thousandth.

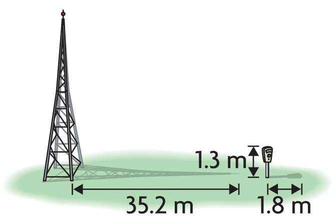
a) $\sin 31°$ **b)** $\cos 70°$

5. Use a calculator to determine θ to the nearest degree.

a) $\cos \theta = 0.3312$ **b)** $\sin \theta = 0.7113$ **c)** $\tan \theta = 1.1145$

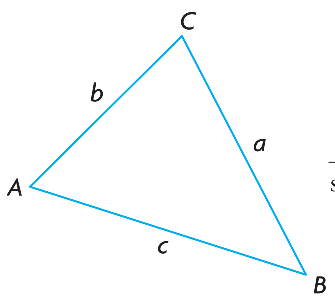


6. Mario is repairing the wires on a radio broadcast tower. He is in the basket of a repair truck 40 m from the tower. When he looks up, he estimates the **angle of elevation** to the top of the tower as 42°. When he looks down, he estimates the **angle of depression** to the bottom of the tower as 32°. How high is the tower to the nearest metre?



7. On a sunny day, a tower casts a shadow 35.2 m long. At the same time, a 1.3 m parking meter that is nearby casts a shadow 1.8 m long. How high is the tower to the nearest tenth of a metre?

8. The **sine law** states that in any triangle, the side lengths are proportional to the sines of the opposite angles.



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use a graphic organizer to show how to use the **sine law** to calculate an unknown angle.

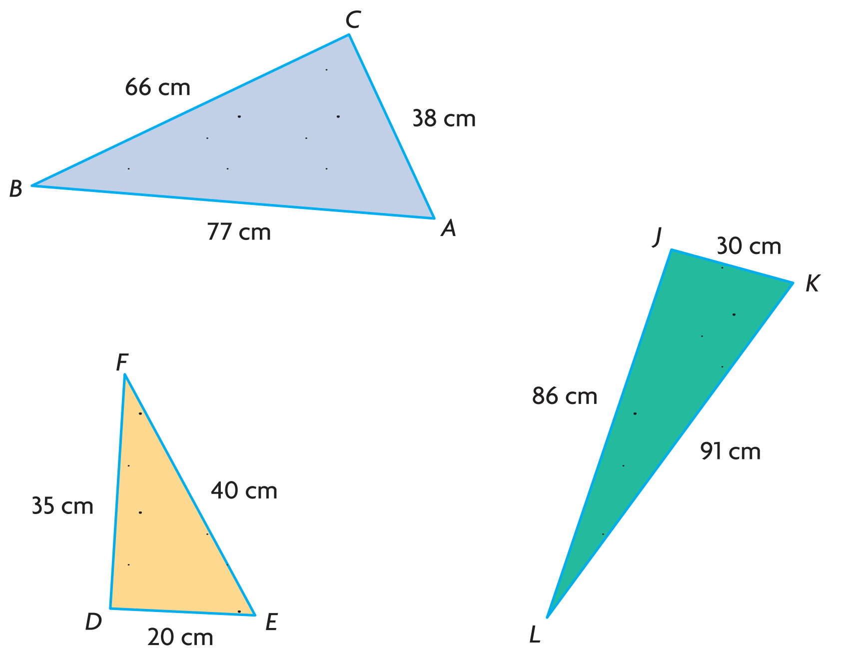
Tech | **Support**

For help using the inverse trigonometric keys on a graphing calculator, see Technical Appendix, B-13.

APPLYING What You Know

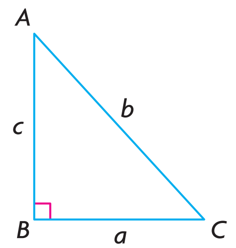
Finding a Right-Angled Triangle

Raymond and Alyssa are covering a patio with triangular pieces of stone tile. They need one tile that has a right angle for the corner of the patio. They don't have a protractor, so they use a tape measure to measure the side lengths of each triangle. The measurements are shown.



Communication *Tip*

It is common practice to label the vertices of a triangle with upper case letters. The side opposite each angle is labelled with the lower case letter corresponding to that angle.



? Which of these triangles can be used for the corner of the patio?

A. In $\triangle ABC$, which angle is most likely a right angle? Justify your decision.

B. Assuming that $\triangle ABC$ is a right triangle, write down the mathematical relationship that relates the three sides.

C. Check to see if $\triangle ABC$ is a right triangle by evaluating each side of the relationship you wrote in part B. Compare both sides.

D. Is $\triangle ABC$ a right triangle? Justify your decision.

E. Repeat parts A to D for the remaining triangles.

F. Which triangular stone would you use for the corner of the patio? Justify your decision.

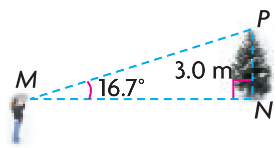
5.1 Trigonometric Ratios of Acute Angles

GOAL

Evaluate reciprocal trigonometric ratios.

LEARN ABOUT the Math

From a position some distance away from the base of a tree, Monique uses a clinometer to determine the angle of elevation to a treetop. Monique estimates that the height of the tree is about 3.0 m.



? How far, to the nearest tenth of a metre, is Monique from the base of the tree?

Communication *Tip*

A clinometer is a device used to measure the angle of elevation (above the horizontal) or the angle of depression (below the horizontal).

EXAMPLE 1 Selecting a strategy to determine a side length in a right triangle

In $\triangle MNP$, determine the length of MN.

Clive's Solution: Using Primary Trigonometric Ratios

$$\tan 16.7° = \frac{3.0}{MN}$$

I knew the opposite side but I needed to calculate the adjacent side *MN*. So I used tangent.

$$MN(\tan 16.7°) = 3.0$$

$$MN = \frac{3.0}{\tan 16.7°}$$

I multiplied both sides of the equation by *MN*, then divided by tan 16.7°.

I used my calculator to evaluate.

$$MN \doteq 10.0 \text{ m}$$

Monique is about 10.0 m away from the base of the tree.

Communication *Tip*

The symbol $\doteq$ means "approximately equal to" and indicates that a result has been rounded.

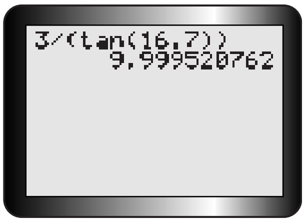
Tony's Solution: Using Reciprocal Trigonometric Ratios

$$\cot 16.7° = \frac{MN}{3.0}$$

NP is opposite the 16.7° angle, and *MN* is adjacent. I used the **reciprocal trigonometric ratio** cot 16.7°. This gave me an equation with the unknown in the numerator, making the equation easier to solve.

$$(3.0)\cot 16.7° = MN$$

To solve for *MN*, I multiplied both sides by 3.0.



I evaluated $\frac{1}{\tan 16.7°}$ to get cot 16.7°.

$$10.0 \text{ m} \doteq MN$$

Monique is about 10.0 m away from the base of the tree.

reciprocal trigonometric ratios

the reciprocal ratios are defined as 1 divided by each of the primary trigonometric ratios

$$\csc \theta = \frac{1}{\sin \theta} = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{\text{adjacent}}{\text{opposite}}$$

Cot θ is the short form for the cotangent of angle θ, sec θ is the short form for the secant of angle θ, and csc θ is the short form for the cosecant of angle θ.

Tech Support

Most calculators do not have buttons for evaluating the reciprocal ratios. For example, to evaluate

- csc 20°, use $\frac{1}{\sin 20°}$
- sec 20°, use $\frac{1}{\cos 20°}$
- cot 20°, use $\frac{1}{\tan 20°}$

Reflecting

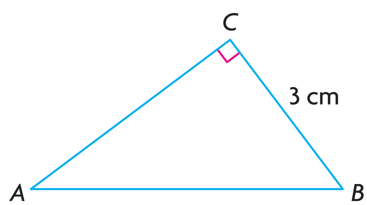
A. What was the advantage of using a reciprocal trigonometric ratio in Tony's solution?

B. Suppose Monique wants to calculate the length of *MP* in $\triangle MNP$. State the two trigonometric ratios that she could use based on the given information. Which one would be better? Explain.

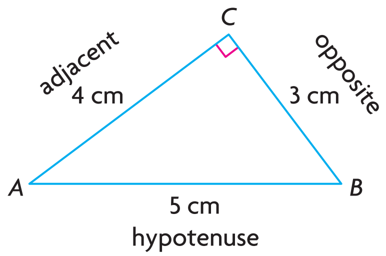
APPLY the Math

EXAMPLE 2 Evaluating the six trigonometric ratios of an angle

$\triangle ABC$ is a right triangle with side lengths of 3 cm, 4 cm, and 5 cm.
If $CB = 3$ cm and $\angle C = 90°$, which trigonometric ratio of $\angle A$ is the greatest?



Sam's Solution



I labelled the sides of the triangle relative to $\angle A$, first in words and then with the side lengths. The hypotenuse is the longest side, so its length must be 5 cm. If the side opposite $\angle A$ is 3 cm, then the side adjacent to $\angle A$ is 4 cm.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} = 0.60 \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} = 0.80 \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4} = 0.75$$

First, I used the definitions of the primary trigonometric ratios to determine the sine, cosine, and tangent of $\angle A$.

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3} \doteq 1.67 \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4} = 1.25 \qquad \cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3} \doteq 1.33$$

Then I evaluated the reciprocal trigonometric ratios for $\angle A$. I wrote the reciprocal of each primary ratio to get the appropriate reciprocal ratio.

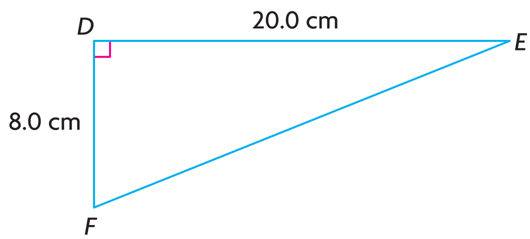
I expressed these ratios as decimals to compare them more easily.

The greatest trigonometric ratio of $\angle A$ is $\csc A$.

EXAMPLE 3 Solving a right triangle by calculating the unknown side and the unknown angles

a) Determine EF in $\triangle DEF$ to the nearest tenth of a centimetre.

b) Express one unknown angle in terms of a primary trigonometric ratio and the other angle in terms of a reciprocal ratio. Then calculate the unknown angles to the nearest degree.



Lina's Solution

a) $EF^2 = (8.0)^2 + (20.0)^2$

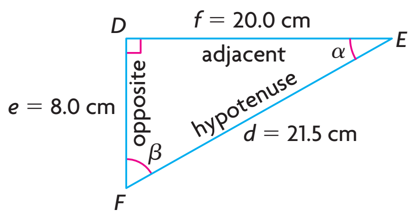
$EF^2 = 464.0 \text{ cm}^2$

$EF = \sqrt{464.0}$

$EF \doteq 21.5 \text{ cm}$

Since $\triangle DEF$ is a right triangle, I used the Pythagorean theorem to calculate the length of *EF*.

b)



I labelled $\angle E$ as α. Side *e* is opposite α and *f* is adjacent to α. So I expressed α in terms of the primary trigonometric ratio $\tan \alpha$.

I labelled $\angle F$ as β. Side *d* is the hypotenuse and *e* is adjacent to β.

$$\tan \alpha = \frac{\text{opposite}}{\text{adjacent}} \qquad \sec \beta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$= \frac{e}{f} \qquad = \frac{d}{e}$$

$$= \frac{8.0}{20.0} \qquad = \frac{21.5}{8.0}$$

$$= 0.40 \qquad \doteq 2.69$$

I expressed β in terms of the reciprocal trigonometric ratio $\sec \beta$.

$\alpha = \tan^{-1}(0.40)$

$\alpha \doteq 22°$

To determine angle α, I used my calculator to evaluate $\tan^{-1}(0.40)$ directly.

$\sec \beta \doteq 2.69$

$\cos \beta \doteq \frac{1}{2.69}$

$\beta \doteq \cos^{-1}\left(\frac{1}{2.69}\right)$

Since my calculator doesn't have a $\sec^{-1}$ key, I wrote $\sec \beta$ in terms of the primary trigonometric ratio $\cos \beta$ before determining β.

$\beta \doteq 68°$

I determined angle β directly by evaluating $\cos^{-1}\left(\frac{1}{2.69}\right)$ with my calculator.

EF is about 21.5 cm long, and $\angle E$ and $\angle F$ are about 22° and 68°, respectively.

Communication *Tip*

Unknown angles are often labelled with the Greek letters θ (theta), α (alpha), and β (beta).

Communication *Tip*

Arcsine ($\sin^{-1}$), arccosine ($\cos^{-1}$), and arctangent ($\tan^{-1}$) are the names given to the inverse trigonometric functions. These are used to determine the angle associated with a given primary ratio.

In Summary

Key Idea

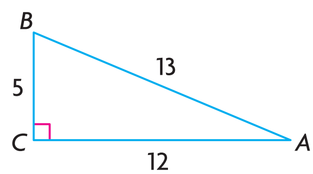
- The reciprocal trigonometric ratios are reciprocals of the primary trigonometric ratios, and are defined as 1 divided by each of the primary trigonometric ratios:
  - $\csc \theta = \dfrac{1}{\sin \theta} = \dfrac{\text{hypotenuse}}{\text{opposite}}$
  - $\sec \theta = \dfrac{1}{\cos \theta} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$
  - $\cot \theta = \dfrac{1}{\tan \theta} = \dfrac{\text{adjacent}}{\text{opposite}}$

Need to Know

- In solving problems, reciprocal trigonometric ratios are sometimes helpful because the unknown variable can be expressed in the numerator, making calculations easier.
- Calculators don't have buttons for cosecant, secant, or cotangent ratios.
- The sine and cosine ratios for an acute angle in a right triangle are less than or equal to 1 so their reciprocal ratios, cosecant and secant, are always greater than or equal to 1.
- The tangent ratio for an acute angle in a right triangle can be less than 1, equal to 1, or greater than 1, so the reciprocal ratio, cotangent, can take on this same range of values.

CHECK Your Understanding

1. Given $\triangle ABC$, state the six trigonometric ratios for $\angle A$.



2. State the reciprocal trigonometric ratios that correspond to $\sin \theta = \frac{8}{17}$, $\cos \theta = \frac{15}{17}$, and $\tan \theta = \frac{8}{15}$.

3. For each primary trigonometric ratio, determine the corresponding reciprocal ratio.

a) $\sin \theta = \dfrac{1}{2}$

b) $\cos \theta = \dfrac{3}{4}$

c) $\tan \theta = \dfrac{3}{2}$

d) $\tan \theta = \dfrac{1}{4}$

4. Evaluate to the nearest hundredth.

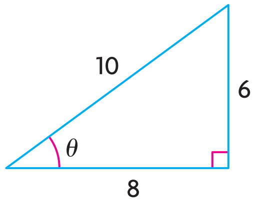
a) $\cos 34°$ **b)** $\sec 10°$ **c)** $\cot 75°$ **d)** $\csc 45°$

PRACTISING

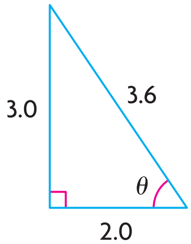
5. **a)** For each triangle, calculate csc θ, sec θ, and cot θ.

K **b)** For each triangle, use one of the reciprocal ratios from part (a) to determine θ to the nearest degree.

i)



iii)



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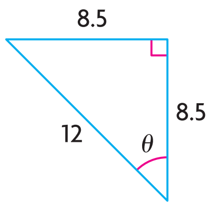

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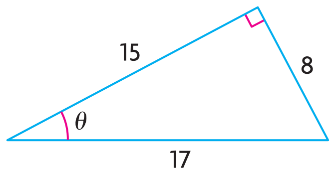


6. Determine the value of θ to the nearest degree.

a) $\cot \theta = 3.2404$

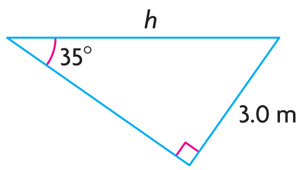
b) $\csc \theta = 1.2711$

c) $\sec \theta = 1.4526$

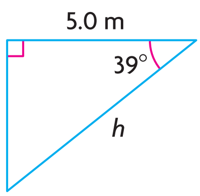
d) $\cot \theta = 0.5814$

7. For each triangle, determine the length of the hypotenuse to the nearest tenth of a metre.

a)

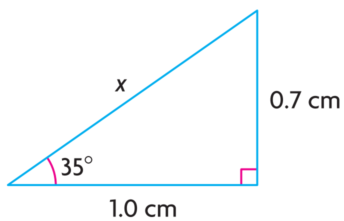


b)

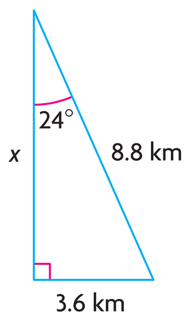


8. For each triangle, use two different methods to determine x to the nearest tenth of a unit.

a)



b)

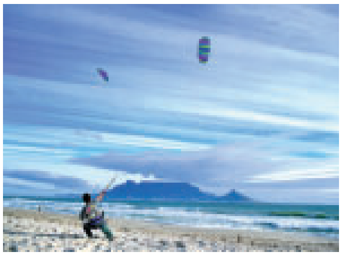


9. Given any right triangle with an acute angle θ,

a) explain why csc θ is always greater than or equal to 1

b) explain why cos θ is always less than or equal to 1

10. Given a right triangle with an acute angle θ, if $\tan \theta = \cot \theta$, describe what this triangle would look like.



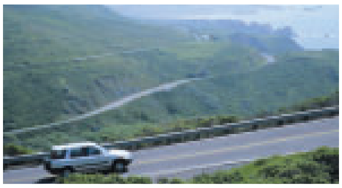
11. **A** A kite is flying 8.6 m above the ground at an angle of elevation of 41°. Calculate the length of string, to the nearest tenth of a metre, needed to fly the kite using
    a) a primary trigonometric ratio
    b) a reciprocal trigonometric ratio

12. A wheelchair ramp near the door of a building has an incline of 15° and a run of 7.11 m from the door. Calculate the length of the ramp to the nearest hundredth of a metre.

13. **T** The hypotenuse, c, of right $\triangle ABC$ is 7.0 cm long. A trigonometric ratio for angle A is given for four different triangles. Which of these triangles has the greatest area? Justify your decision.
    a) $\sec A = 1.7105$
    b) $\cos A = 0.7512$
    c) $\csc A = 2.2703$
    d) $\sin A = 0.1515$

14. The two guy wires supporting an 8.5 m TV antenna each form an angle of 55° with the ground. The wires are attached to the antenna 3.71 m above ground. Using a reciprocal trigonometric ratio, calculate the length of each wire to the nearest tenth of a metre. What assumption did you make?

15. From a position some distance away from the base of a flagpole, Julie estimates that the pole is 5.35 m tall at an angle of elevation of 25°. If Julie is 1.55 m tall, use a reciprocal trigonometric ratio to calculate how far she is from the base of the flagpole, to the nearest hundredth of a metre.



16. The maximum grade (slope) allowed for highways in Ontario is 12%.
    a) Predict the angle θ, to the nearest degree, associated with this slope.
    b) Calculate the value of θ to the nearest degree.
    c) Determine the six trigonometric ratios for angle θ.

17. **C** Organize these terms in a word web, including explanations where appropriate.

sine	cosine	tangent	opposite
cotangent	hypotenuse	cosecant	adjacent
secant	angle of depression	angle	angle of elevation

Extending

18. In right $\triangle PQR$, the hypotenuse, r, is 117 cm and $\tan P = 0.51$. Calculate side lengths p and q to the nearest centimetre and all three interior angles to the nearest degree.

19. Describe the appearance of a triangle that has a secant ratio that is greater than any other trigonometric ratio.

20. The tangent ratio is undefined for angles whose adjacent side is equal to zero. List all the angles between 0° and 90° (if any) for which cosecant, secant, and cotangent are undefined.

5.2 Evaluating Trigonometric Ratios for Special Angles

GOAL

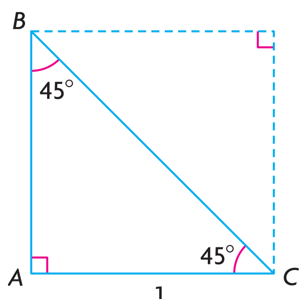
Evaluate exact values of sine, cosine, and tangent for specific angles.

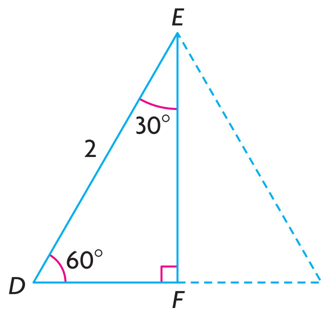
YOU WILL NEED

- ruler
- protractor

LEARN ABOUT the Math

The diagonal of a square of side length 1 unit creates two congruent right isosceles triangles. The height of an equilateral triangle of side length 2 units creates two congruent right scalene triangles.



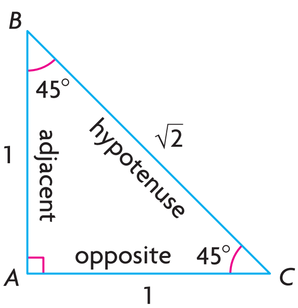


? How can isosceles $\triangle ABC$ and scalene $\triangle DEF$ be used to determine the exact values of the primary trigonometric ratios for 30°, 45°, and 60° angles?

EXAMPLE 1 Evaluating exact values of the trigonometric ratios for a 45° angle

Use $\triangle ABC$ to calculate exact values of sine, cosine, and tangent for 45°.

Carol's Solution



$$BC^2 = AB^2 + AC^2$$
$$BC^2 = 1^2 + 1^2$$
$$BC^2 = 2$$
$$BC = \sqrt{2}$$

I labelled the sides of the triangle relative to $\angle B$. The triangle is isosceles with equal sides of length 1. I used the Pythagorean theorem to calculate the length of the hypotenuse.

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos B = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan B = \frac{\text{opposite}}{\text{adjacent}}$$

I wrote the primary trigonometric ratios for $\angle B$.

$$\sin 45^\circ = \frac{1}{\sqrt{2}} \qquad \cos 45^\circ = \frac{1}{\sqrt{2}} \qquad \tan 45^\circ = \frac{1}{1}$$

$$= \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} \qquad = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} \qquad = 1$$

If I multiplied both the numerator and denominator by $\sqrt{2}$, I would get an equivalent number with a whole-number denominator.

$$= \frac{\sqrt{2}}{2} \qquad = \frac{\sqrt{2}}{2}$$

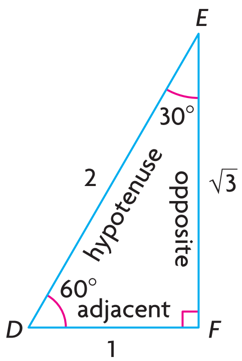
This would be an easier number to use to estimate the size, since I knew that $\sqrt{2}$ is about 1.4, so half of it is about 0.7.

The exact values of sine, cosine, and tangent for 45° are $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{2}}{2}$, and 1, respectively.

EXAMPLE 2 Evaluating exact values of the trigonometric ratios for 30° and 60° angles

Use $\triangle DEF$ to calculate exact values of sine, cosine, and tangent for 30° and 60°.

Trevor's Solution



$$\begin{aligned} DE^2 &= DF^2 + EF^2 \\ 2^2 &= 1^2 + EF^2 \\ 4 &= 1 + EF^2 \\ 3 &= EF^2 \\ \sqrt{3} &= EF \end{aligned}$$

I labelled the sides of the triangle relative to $\angle D$. Since the height of an equilateral triangle divides the triangle into two smaller identical triangles, DF is equal to $\frac{1}{2}DE$. So DF must be 1. I used the Pythagorean theorem to calculate the length of EF.

$$\sin D = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos D = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan D = \frac{\text{opposite}}{\text{adjacent}}$$

I wrote the primary trigonometric ratios for $\angle D$.

$$\sin D = \frac{EF}{DE} \qquad \cos D = \frac{DF}{DE} \qquad \tan D = \frac{EF}{DF}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \frac{1}{2} \qquad \tan 60^\circ = \frac{\sqrt{3}}{1}$$

$$= \sqrt{3}$$

$$\sin E = \frac{DF}{DE} \qquad \cos E = \frac{EF}{DE} \qquad \tan E = \frac{DF}{EF}$$

I wrote the primary trigonometric ratios for $\angle E$ in terms of the sides of the triangle.

$\sin 30° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{1}{\sqrt{3}}$

$$= \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$$

$$= \frac{\sqrt{3}}{3}$$

If I multiplied both the numerator and denominator by $\sqrt{3}$, I would get an equivalent number with a whole-number denominator. This is an easier number to estimate, since $\sqrt{3}$ is about 1.7, so a third of it is about 0.57.

$\sin E = \cos D$ $\cos E = \sin D$ $\tan E = \cot D$

$\sin 30° = \cos 60°$ $\cos 30° = \sin 60°$ $\tan 30° = \cot 60°$

I noticed that sin *E* and cos *E* are equal to cos *D* and sin *D*, respectively. I also noticed that tan *E* is equal to the reciprocal of tan *D*.

The exact values of sine, cosine, and tangent for 30° are $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, and $\frac{\sqrt{3}}{3}$, respectively and for 60° are $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, and $\sqrt{3}$, respectively.

Reflecting

A. In Example 1, would you get the same results if you used $\angle C$ for the 45° angle instead of $\angle B$? Explain.

B. Explain how sin 30° and cos 60° are related.

C. In Example 2, explain why the reciprocal ratios of tan 30° and cot 60° are equal.

D. How can remembering that a 30°–60°–90° triangle is half of an equilateral triangle and that a 45°–45°–90° triangle is isosceles help you recall the exact values of the primary trigonometric ratios for the angles in those triangles?

APPLY the Math

EXAMPLE 3 Determining the exact value of a trigonometric expression

Determine the exact value of $(\sin 45°)(\cos 45°) + (\sin 30°)(\sin 60°)$.

Tina's Solution

$(\sin 45°)(\cos 45°) + (\sin 30°)(\sin 60°)$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

I substituted the exact values of each trigonometric ratio.

$$= \frac{2}{4} + \frac{\sqrt{3}}{4}$$

I evaluated the expression by multiplying, then adding the numerators.

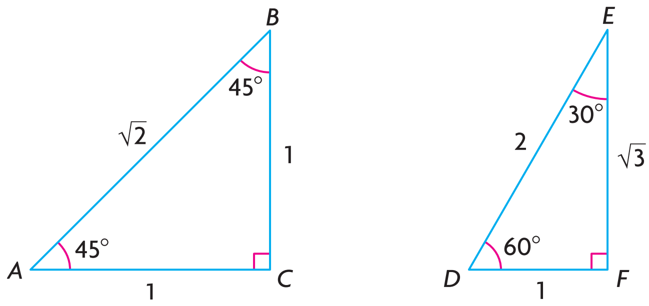
$$= \frac{2 + \sqrt{3}}{4}$$

The exact value is $\frac{2 + \sqrt{3}}{4}$.

In Summary

Key Idea

- The exact values of the primary trigonometric ratios for 30°, 45°, and 60° angles can be found by using the appropriate ratios of sides in isosceles right triangles and half-equilateral triangles with right angles. These are often referred to as "special triangles."



θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
30°	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} \doteq 0.8660$	$\frac{\sqrt{3}}{3} \doteq 0.5774$
45°	$\frac{\sqrt{2}}{2} \doteq 0.7071$	$\frac{\sqrt{2}}{2} \doteq 0.7071$	1
60°	$\frac{\sqrt{3}}{2} \doteq 0.8660$	$\frac{1}{2} = 0.5$	$\sqrt{3} \doteq 1.7321$

Need to Know

- Since tan 45° = 1, angles between 0° and 45° have tangent ratios that are less than 1, and angles between 45° and 90° have tangent ratios greater than 1.
- If a right triangle has one side that is half the length of the hypotenuse, the angle opposite that one side is always 30°.
- If a right triangle has two equal sides, then the angles opposite those sides are always 45°.

CHECK Your Understanding

1. **a)** Draw a right triangle that has one angle measuring 30°. Label the sides using the lengths $\sqrt{3}$, 2, and 1. Explain your reasoning.
b) Identify the adjacent and opposite sides relative to the 30° angle.
c) Identify the adjacent and opposite sides relative to the 60° angle.

2. **a)** Draw a right triangle that has one angle measuring 45°. Label the sides using the lengths 1, 1, and $\sqrt{2}$. Explain your reasoning.
b) Identify the adjacent and opposite sides relative to one of the 45° angles.

3. State the exact values.
a) sin 60° **b)** cos 30° **c)** tan 45° **d)** cos 45°

PRACTISING

4. Determine the exact value of each trigonometric expression.
K
a) $\sin 30° \times \tan 60° - \cos 30°$
b) $2 \cos 45° \times \sin 45°$
c) $\tan^2 30° - \cos^2 45°$
d) $1 - \dfrac{\sin 45°}{\cos 45°}$

5. Using exact values, show that $\sin^2 \theta + \cos^2 \theta = 1$ for each angle.
a) $\theta = 30°$ **b)** $\theta = 45°$ **c)** $\theta = 60°$

6. Using exact values, show that $\dfrac{\sin \theta}{\cos \theta} = \tan \theta$ for each angle.
a) $\theta = 30°$ **b)** $\theta = 45°$ **c)** $\theta = 60°$

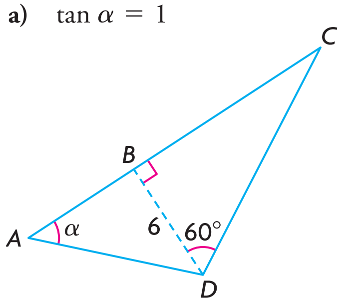
7. Using the appropriate special triangle, determine θ if $0° \leq \theta \leq 90°$.
a) $\sin \theta = \dfrac{\sqrt{3}}{2}$
b) $\sqrt{3} \tan \theta = 1$
c) $2\sqrt{2} \cos \theta = 2$
d) $2 \cos \theta = \sqrt{3}$

8. A 5 m stepladder propped against a classroom wall forms an angle of 30°
A with the wall. Exactly how far is the top of the ladder from the floor? Express your answer in radical form. What assumption did you make?

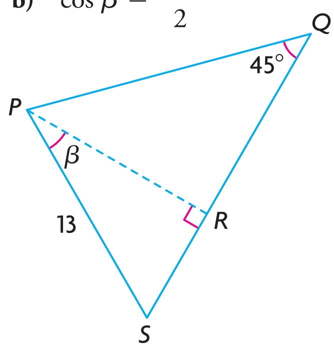
9. Show that $\tan 30° + \dfrac{1}{\tan 30°} = \dfrac{1}{\sin 30° \cos 30°}$.

10. A baseball diamond forms a square of side length 27.4 m. Sarah says that she used a special triangle to calculate the distance between home plate and second base.
a) Describe how Sarah might calculate this distance.
b) Use Sarah's method to calculate this distance to the nearest tenth of a metre.

11. Determine the exact area of each large triangle.
T
a) $\tan \alpha = 1$



b) $\cos \beta = \dfrac{\sqrt{3}}{2}$



12. To claim a prize in a contest, the following skill-testing question was asked:
C Calculate $\sin 45°(1 - \cos 30°) + 5 \tan 60°(\sin 60° - \tan 30°)$.
a) Louise used a calculator to evaluate the expression. Determine her answer to three decimal places.
b) Megan used exact values. Determine her answer in radical form.
c) Only Megan received the prize. Explain why this might have occurred.

Communication Tip

$\tan^2 30° = (\tan 30°)(\tan 30°)$. The expression is squared, not the angle.

Extending

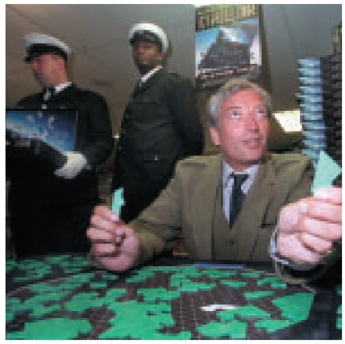
13. If $\cot \alpha = \sqrt{3}$, calculate $(\sin \alpha)(\cot \alpha) - \cos^2 \alpha$ exactly.

14. If $\csc \beta = 2$, calculate $\dfrac{\tan \beta}{\sec \beta} - \sin^2 \beta$ exactly.

15. Using exact values, show that $1 + \cot^2 \theta = \csc^2 \theta$ for each angle.
a) $\theta = 30°$ **b)** $\theta = 45°$ **c)** $\theta = 60°$

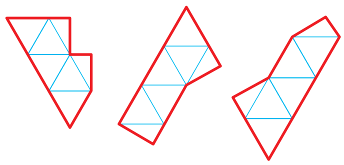
Curious *Math*

The Eternity Puzzle



Eternity, a puzzle created by Christopher Monckton, consists of 209 different pieces. Each piece is made up of twelve $30°-60°-90°$ triangles. The puzzle was introduced in Britain in June 1999, and the goal was to arrange the pieces into the shape of a dodecagon (12-sided polygon). Monckton provided six clues to solve his puzzle, and a £1 000 000 award (about \$2 260 000 Canadian dollars) was offered for the first solution. It turned out that the puzzle didn't take an eternity to solve after all! Alex Selby and Oliver Riordan presented their solution on May 15, 2000, and collected the prize.

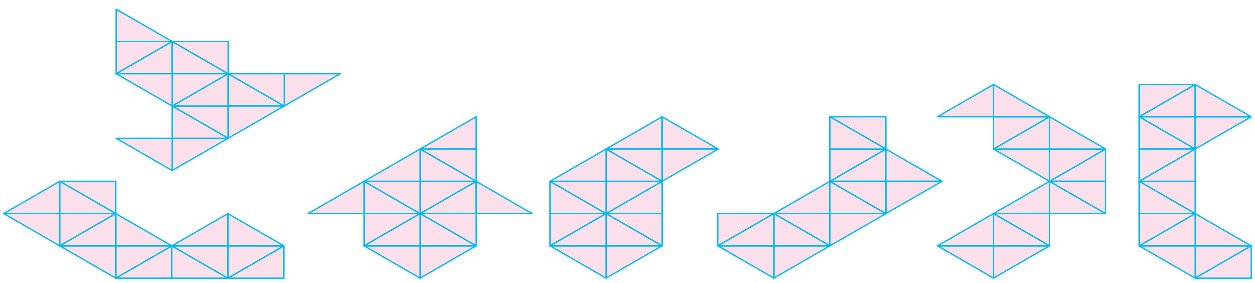
A second solution was found by Guenter Stertenbrink shortly afterwards. Interestingly, all three mathematicians ignored Monckton's clues and found their own answers. Monckton's solution remains unknown.



piece 1 piece 2 piece 3

1. Consider the first three pieces of the Eternity puzzle. Each contains twelve $30°-60°-90°$ triangles. Suppose one such triangle has side lengths of 1, $\sqrt{3}$, and 2, respectively.
   a) For each puzzle piece, determine the perimeter. Write your answer in radical form.
   b) Calculate the area of each puzzle piece. Round your answer to the nearest tenth of a square unit.

2. The seven puzzle pieces shown can be fit together to form a convex shape. Copy these pieces and see if you can find a solution.



5.3 Exploring Trigonometric Ratios for Angles Greater than 90°

GOAL

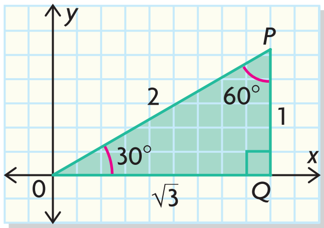
Explore relationships among angles that share related trigonometric ratios.

YOU WILL NEED

- graph paper
- dynamic geometry software (optional)

EXPLORE the Math

Raj is investigating trigonometric ratios of angles greater than 90°. He drew one of the special triangles on a Cartesian grid as shown.

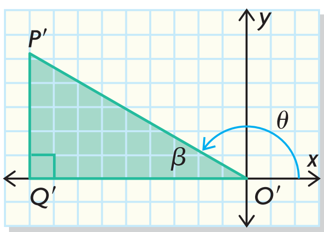


Next he performed a series of reflections in the y- and x-axes.

? Which angles in the Cartesian plane, if any, have primary trigonometric ratios related to those of a 30° angle?

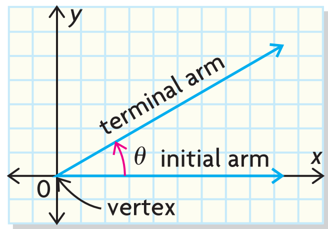
A. Use Raj's sketch of a 30° angle in **standard position** in the Cartesian plane to record the lengths of all sides and the primary trigonometric ratios for 30° to four decimal places.

B. Reflect the triangle from part A in the y-axis. $\angle P'O'Q'$ is now called the **related acute angle** β. What is its angle measure? What is the size of the **principal angle** θ and in which quadrant does the terminal arm lie?



standard position

an angle in the Cartesian plane whose vertex lies at the origin and whose initial arm (the arm that is fixed) lies on the positive x-axis. Angle θ is measured from the initial arm to the terminal arm (the arm that rotates).

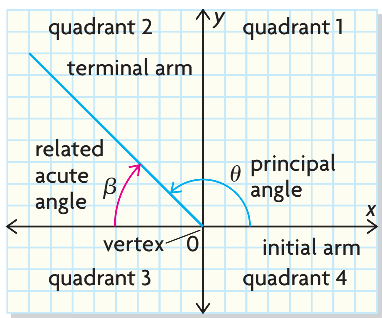


related acute angle

the acute angle between the terminal arm of an angle in standard position and the x-axis when the terminal arm lies in quadrants 2, 3, or 4

principal angle

the counterclockwise angle between the initial arm and the terminal arm of an angle in standard position. Its value is between 0° and 360°.

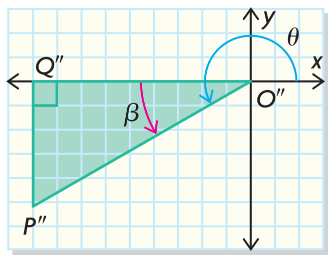


C. Use a calculator to determine the values of the primary trigonometric ratios for the principal angle and the related acute angle. Round your answers to four decimal places and record them in a table similar to the one shown.

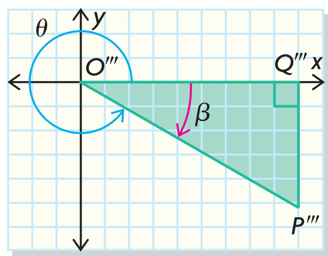
Angles	Quadrant	Sine Ratio	Cosine Ratio	Tangent Ratio
principal angle $\theta =$ _____				
related acute angle $\beta =$ _____				

How are the primary trigonometric ratios for the related acute angle related to the corresponding ratios for the principal angle?

D. Reflect the triangle from part B in the *x*-axis. What is the size of the related acute angle β? What is the size of the principal angle θ, and in which quadrant does the terminal arm lie? Use a calculator to complete your table for each of these angles. How are the primary trigonometric ratios for the related acute angle related to the corresponding ratios for the principal angle?

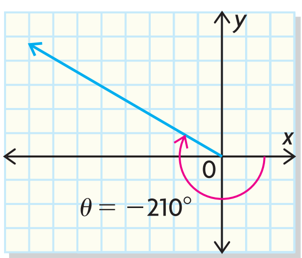


E. Repeat part D, but this time, reflect the triangle from part D in the *y*-axis.



negative angle

an angle measured *clockwise* from the positive *x*-axis



F. Repeat parts A to E, but this time start with a 45° and then a 60° angle in quadrant 1. Use negative angles for some of your trials.

G. Based on your observations, which principal angles and related acute angles in the Cartesian plane have the same primary trigonometric ratio?

Reflecting

H. **i)** When you reflect an acute principal angle θ in the y-axis, why is the resulting principal angle $180° - \theta$?

ii) When you reflect an acute principal angle θ in the y-axis and then in the x-axis, why is the resulting principal angle $180° + \theta$?

iii) When you reflect an acute principal angle θ in the x-axis, why is the resulting principal angle $360° - \theta$ (or $-\theta$)?

I. What does your table tell you about the relationships among the sine, cosine, and tangent of an acute principal angle and the resulting reflected principal angles?

J. How could you have predicted the relationships you described in part I?

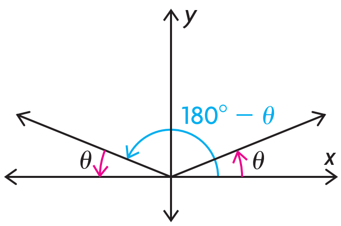
In Summary

Key Idea

- For any principal angle greater than 90°, the values of the primary trigonometric ratios are either the same as, or the negatives of, the ratios for the related acute angle. These relationships are based on angles in standard position in the Cartesian plane and depend on the quadrant in which the terminal arm of the angle lies.

Need to Know

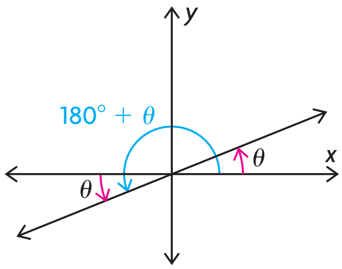
- An angle in the Cartesian plane is in standard position if its vertex lies at the origin and its initial arm lies on the positive x-axis.
- An angle in standard position is determined by a counterclockwise rotation and is always positive. An angle determined by a clockwise rotation is always negative.
- If the terminal arm of an angle in standard position lies in quadrants 2, 3, or 4, there exists a related acute angle and a principal angle.
- If θ is an acute angle in standard position, then
  - the terminal arm of the principal angle $(180° - \theta)$ lies in quadrant 2



$$\sin(180° - \theta) = \sin\theta$$
$$\cos(180° - \theta) = -\cos\theta$$
$$\tan(180° - \theta) = -\tan\theta$$

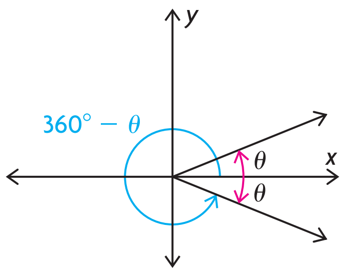
(*continued*)

- the terminal arm of the principal angle $(180° + \theta)$ lies in quadrant 3



$\sin(180° + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$

- the terminal arm of the principal angle $(360° - \theta)$ lies in quadrant 4



$\sin(360° - \theta) = -\sin\theta$
$\cos(360° - \theta) = -\cos\theta$
$\tan(360° - \theta) = \tan\theta$

FURTHER *Your Understanding*

1. State all the angles between 0° and 360° that make each equation true.

a) $\sin 45° = \sin$ ______

b) $\cos$ ______ $= -\cos(-60°)$

c) $\tan 30° = \tan$ ______

d) $\tan 135° = -\tan$ ______

2. Using the special triangles from Lesson 5.2, sketch two angles in the Cartesian plane that have the same value for each given trigonometric ratio.

a) sine **b)** cosine **c)** tangent

3. Sylvie drew a special triangle in quadrant 3 and determined that $\tan(180° + \theta) = 1$.

a) What is the value of angle θ?

b) What would be the exact value of $\tan\theta$, $\cos\theta$, and $\sin\theta$?

4. Based on your observations, copy and complete the table below to summarize the signs of the trigonometric ratios for a principal angle that lies in each of the quadrants.

Trigonometric Ratio	**Quadrant**			
	1	**2**	**3**	**4**
sine	+			
cosine	+			
tangent	+			

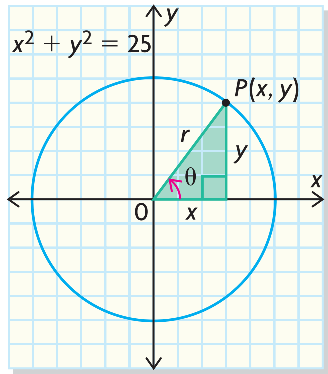
5.4 Evaluating Trigonometric Ratios for Any Angle Between 0° and 360°

GOAL

Use the Cartesian plane to evaluate the primary trigonometric ratios for angles between 0° and 360°.

YOU WILL NEED

- graph paper
- protractor
- dynamic geometry software (optional)



LEARN ABOUT the Math

Miriam knows that the equation of a circle of radius 5 centred at $(0, 0)$ is $x^2 + y^2 = 25$. She also knows that a point $P(x, y)$ on its circumference can rotate from 0° to 360°.

? For any point on the circumference of the circle, how can Miriam determine the size of the corresponding principal angle?

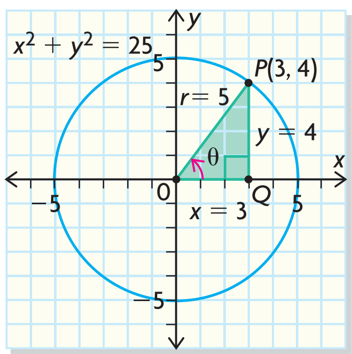
EXAMPLE 1 Relating trigonometric ratios to a point in quadrant 1 of the Cartesian plane

a) If Miriam chooses the point $P(3, 4)$ on the circumference of the circle, determine the primary trigonometric ratios for the principal angle.

b) Determine the principal angle to the nearest degree.

Flavia's Solution

a)



I drew a circle centred about the origin in the Cartesian plane and labelled the point $P(3, 4)$ on the circumference. Then I formed a right triangle with the x-axis. Angle θ is the principal angle and is in standard position. In $\triangle OPQ$, I noticed that the side opposite θ has length $y = 4$ units and the adjacent side has length $x = 3$ units. The hypotenuse is equal to the radius of the circle, so I labelled it r. In this case, $r = 5$ units. From the Pythagorean theorem, I also knew that $r^2 = x^2 + y^2$. Since r is the radius of the circle, it will always be positive.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$= \frac{y}{r} \qquad\qquad = \frac{x}{r} \qquad\qquad = \frac{y}{x}$$

$$= \frac{4}{5} \qquad\qquad = \frac{3}{5} \qquad\qquad = \frac{4}{3}$$

I used the definitions of sine, cosine, and tangent to write each ratio in terms of x, y, and r in the Cartesian plane.

b) $\sin\theta = \frac{4}{5}$

$$\theta = \sin^{-1}\left(\frac{4}{5}\right)$$

I used the inverse sine function on my calculator to determine angle θ.

$$\theta \doteq 53°$$

The principal angle is about 53°.

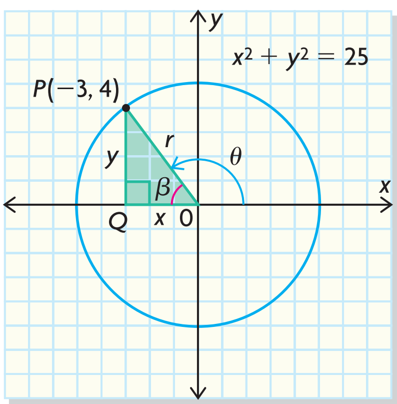
EXAMPLE 2 Relating trigonometric ratios to a point in quadrant 2 of the Cartesian plane

a) If Miriam chooses the point $P(-3, 4)$ on the circumference of the circle, determine the primary trigonometric ratios for the principal angle to the nearest hundredth.

b) Determine the principal angle to the nearest degree.

Gabriel's Solution

a)



I drew a circle centred about the origin in the Cartesian plane and labelled the point $P(-3, 4)$ on the circumference. Then I formed a right triangle with the x-axis. Angle θ is the principal angle and is in standard position. Angle β is the related acute angle.

$$r^2 = x^2 + y^2$$
$$r^2 = 3^2 + 4^2$$
$$r^2 = 9 + 16$$
$$r^2 = 25$$
$$r = 5, \text{ since } r > 0$$

In $\triangle OPQ$, I knew that the lengths of the two perpendicular sides were $|x| = |-3| = 3$ and $y = 4$. The radius of the circle is still 5, so $r = 5$. I used the Pythagorean theorem to confirm this.

$$\sin\beta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\beta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\beta = \frac{\text{opposite}}{\text{adjacent}}$$

$$= \frac{y}{r} \qquad = \frac{|x|}{r} \qquad = \frac{y}{|x|}$$

In $\triangle OPQ$, the side opposite β has length y and the adjacent side has length $|x|$. I used the definitions of sine, cosine, and tangent to write each ratio in terms of x, y, and r in the Cartesian plane.

$$= \frac{4}{5} \qquad = \frac{3}{5} \qquad = \frac{4}{3}$$

$$\sin\theta = \sin\beta \qquad \cos\theta = -\cos\beta \qquad \tan\theta = -\tan\beta$$

$$= \frac{4}{5} \qquad = -\frac{3}{5} \qquad = -\frac{4}{3}$$

Then I took into account the relationship among the trigonometric ratios of the related acute angle β and those of the principal angle θ. Since the terminal arm of angle θ lies in quadrant 2, the sine ratio is positive while the cosine and tangent ratios are negative.

b) $\sin\beta = \frac{4}{5}$

To determine angle β, I used a calculator to evaluate $\sin^{-1}\left(\frac{4}{5}\right)$ directly.

$$\beta = \sin^{-1}\left(\frac{4}{5}\right)$$
$$\doteq 53°$$

$$\theta + \beta = 180°$$

I knew that θ and β add up to 180°. So I subtracted β from 180° to get θ.

$$\theta = 180° - \beta$$
$$= 180° - 53°$$
$$= 127°$$

The principal angle is about 127° because the related acute angle is about 53°.

Reflecting

A. In Example 2, explain why $\sin\theta = \sin\beta$, $\cos\theta \neq \cos\beta$, and $\tan\theta \neq \tan\beta$.

B. If Miriam chose the points $(-3, -4)$ and $(3, -4)$, what would each related acute angle be? How would the primary trigonometric ratios for the corresponding principal angles in these cases compare with those in Examples 1 and 2?

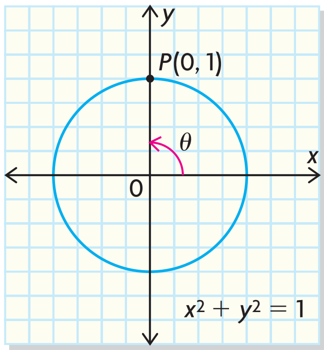
C. Given a point on the terminal arm of an angle in standard position, explain how the coordinates of that point vary from quadrants 1 to 4. How does this variation affect the size of the principal angle (and related acute angle, if it exists) and the values of the primary trigonometric ratios for that angle?

APPLY the Math

EXAMPLE 3 Determining the primary trigonometric ratios for a 90° angle

Use the point $P(0, 1)$ to determine the values of sine, cosine, and tangent for 90°.

Charmaine's Solution



I drew a circle centred about the origin in the Cartesian plane and labelled the point $P(0, 1)$ on the circumference. Angle θ is the principal angle and is 90°.

In this case, I couldn't draw a right triangle by drawing a line perpendicular to the x-axis to P.

This meant that I couldn't use the trigonometric definitions in terms of opposite, adjacent, and hypotenuse.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$= \frac{1}{1} \qquad\qquad = \frac{0}{1} \qquad\qquad = \frac{1}{0}$$

Since $P(0, 1)$, I knew that $x = 0$, $y = 1$, and $r = 1$.

I used the definitions of sine, cosine, and tangent in terms of x, y, and r to write each ratio.

$\sin 90° = 1$ $\quad\cos 90° = 0$ $\quad\tan 90°$ is undefined

Since $x = 0$ and it is in the denominator, $\tan 90°$ is undefined.

The point $P(0, 1)$ defines a principal angle of 90°. The sine and cosine of 90° are 1 and 0, respectively. The tangent of 90° is undefined.

EXAMPLE 4 Determining all possible values of an angle with a specific trigonometric ratio

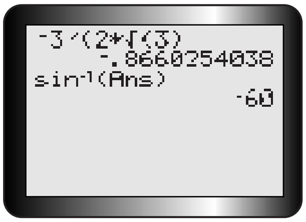
Determine the values of θ if $\csc \theta = -\frac{2\sqrt{3}}{3}$ and $0° \leq \theta \leq 360°$.

Jordan's Solution

$$\csc \theta = -\frac{2\sqrt{3}}{3}$$

$$\sin \theta = -\frac{3}{2\sqrt{3}}$$

Since $0° \leq \theta \leq 360°$, I had to use the Cartesian plane to determine θ. Cosecant is the reciprocal of sine. I found the reciprocal ratio by switching *r* and *y*. Since *r* is always positive, *y* must be -3 in this case. There were two cases where a point on the terminal arm has a negative *y*-coordinate: one in quadrant 3 and the other in quadrant 4.



I used my calculator to evaluate $\frac{-3}{2\sqrt{3}}$. Then I took the inverse sine of the result to determine the angle.

One angle is $-60°$, which is equivalent to $360° + (-60°) = 300°$ in quadrant 4.

The angle $-60°$ corresponds to a related acute angle of 60° of clockwise rotation and has its terminal arm in quadrant 4. I added 360° to $-60°$ to get the equivalent angle using a counterclockwise rotation.

In quadrant 3, the angle is $180° + 60° = 240°$.

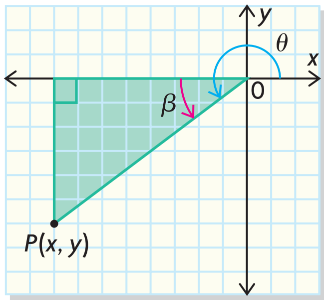
The angle in quadrant 3 must have a related acute angle of 60° as well. So I added 180° to 60° to determine the principal angle.

Given $\csc \theta = -\frac{2\sqrt{3}}{3}$ and $0° \leq \theta \leq 360°$, θ can be either 240° or 300°.

In Summary

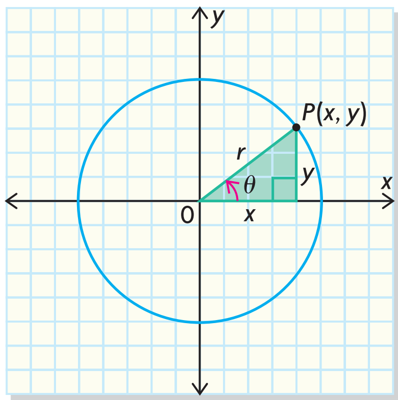
Key Idea

- The trigonometric ratios for any principal angle, θ, in standard position, where $0° \leq \theta \leq 360°$, can be determined by finding the related acute angle, β, using coordinates of any point $P(x, y)$ that lies on the terminal arm of the angle.



Need to Know

- For any point $P(x, y)$ in the Cartesian plane, the trigonometric ratios for angles in standard position can be expressed in terms of x, y, and r.



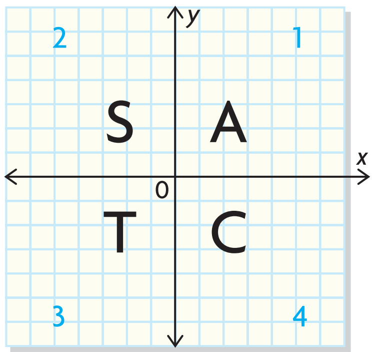
$r^2 = x^2 + y^2$ from the Pythagorean theorem and $r > 0$

$$\sin \theta = \frac{y}{r} \qquad \cos \theta = \frac{x}{r} \qquad \tan \theta = \frac{y}{x}$$

$$\csc \theta = \frac{r}{y} \qquad \sec \theta = \frac{r}{x} \qquad \cot \theta = \frac{x}{y}$$

(continued)

- The CAST rule is an easy way to remember which primary trigonometric ratios are positive in which quadrant. Since *r* is always positive, the sign of each primary ratio depends on the signs of the coordinates of the point.
  - In quadrant 1, **A**ll (A) ratios are positive because both *x* and *y* are positive.
  - In quadrant 2, only **S**ine (S) is positive, since *x* is negative and *y* is positive.
  - In quadrant 3, only **T**angent (T) is positive because both *x* and *y* are negative.
  - In quadrant 4, only **C**osine (C) is positive, since *x* is positive and *y* is negative.



CHECK Your Understanding

1. For each trigonometric ratio, use a sketch to determine in which quadrant the terminal arm of the principal angle lies, the value of the related acute angle β, and the sign of the ratio.

a) $\sin 315°$ **b)** $\tan 110°$ **c)** $\cos 285°$ **d)** $\tan 225°$

2. Each point lies on the terminal arm of angle θ in standard position.

i) Draw a sketch of each angle θ.
ii) Determine the value of r to the nearest tenth.
iii) Determine the primary trigonometric ratios for angle θ.
iv) Calculate the value of θ to the nearest degree.

a) $(5, 11)$ **b)** $(-8, 3)$ **c)** $(-5, -8)$ **d)** $(6, -8)$

3. Use the method in Example 3 to determine the primary trigonometric ratios for each given angle.

a) $180°$ **b)** $270°$ **c)** $360°$

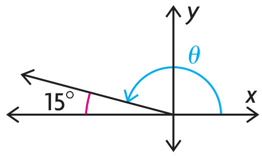
4. Use the related acute angle to state an equivalent expression.

a) $\sin 160°$ **b)** $\cos 300°$ **c)** $\tan 110°$ **d)** $\sin 350°$

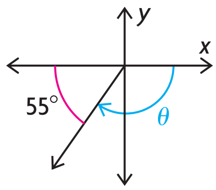
PRACTISING

5. **i)** For each angle θ, predict which primary trigonometric ratios are positive.
ii) Determine the primary trigonometric ratios to the nearest hundredth.

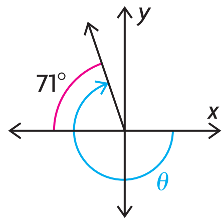
a)



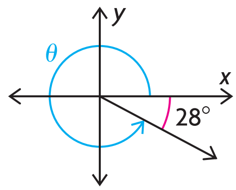
b)



c)



d)



6. K Angle θ is a principal angle that lies in quadrant 2 such that $0° \le \theta \le 360°$. Given each trigonometric ratio,

i) determine the exact values of x, y, and r
ii) sketch angle θ in standard position
iii) determine the principal angle θ and the related acute angle β to the nearest degree

a) $\sin \theta = \frac{1}{3}$

b) $\cot \theta = -\frac{4}{3}$

c) $\cos \theta = -\frac{1}{4}$

d) $\csc \theta = 2.5$

e) $\tan \theta = -1.1$

f) $\sec \theta = -3.5$

7. For each trigonometric ratio in question 6, determine the smallest negative angle that has the same ratio.

8. Use each trigonometric ratio to determine all values of θ, to the nearest degree if $0° \le \theta \le 360°$.

a) $\sin \theta = 0.4815$
b) $\tan \theta = -0.1623$
c) $\cos \theta = -0.8722$
d) $\cot \theta = 8.1516$
e) $\csc \theta = -2.3424$
f) $\sec \theta = 0$

9. Given angle θ, where $0° \leq \theta \leq 360°$, determine two possible values of θ where each ratio would be true. Sketch both principal angles.
   a) $\cos \theta = 0.6951$
   b) $\tan \theta = -0.7571$
   c) $\sin \theta = 0.3154$
   d) $\cos \theta = -0.2882$
   e) $\tan \theta = 2.3151$
   f) $\sin \theta = -0.7503$

10. Given each point $P(x, y)$ lying on the terminal arm of angle θ,
   i) state the value of θ, using both a counterclockwise and a clockwise rotation
   ii) determine the primary trigonometric ratios

   a) $P(-1, -1)$
   b) $P(0, -1)$
   c) $P(-1, 0)$
   d) $P(1, 0)$

11. **A** Dennis doesn't like using x, y, and r to investigate angles. He says that he is going to continue using adjacent, opposite, and hypotenuse to evaluate trigonometric ratios for any angle θ. Explain the weaknesses of his strategy.

12. Given $\cos \theta = -\frac{5}{12}$, where $0° \leq \theta \leq 360°$,
   a) in which quadrant could the terminal arm of θ lie?
   b) determine all possible primary trigonometric ratios for θ.
   c) evaluate all possible values of θ to the nearest degree.

13. **T** Given angle α, where $0° \leq \alpha < 360°$, $\cos \alpha$ is equal to a unique value. Determine the value of α to the nearest degree. Justify your answer.

14. **C** How does knowing the coordinates of a point P in the Cartesian plane help you determine the trigonometric ratios associated with the angle formed by the x-axis and a ray drawn from the origin to P? Use an example in your explanation.

Extending

15. Given angle θ, where $0° \leq \theta \leq 360°$, solve for θ to the nearest degree.
   a) $\cos 2\theta = 0.6420$
   b) $\sin(\theta + 20°) = 0.2045$
   c) $\tan(90° - 2\theta) = 1.6443$

16. When you use the inverse trigonometric functions on a calculator, it is important to interpret the calculator result to avoid inaccurate values of θ. Using these trigonometric ratios, describe what errors might result.
   a) $\sin \theta = -0.8$
   b) $\cos \theta = -0.75$

17. Use sketches to explain why each statement is true.
   a) $2 \sin 32° \neq \sin 64°$
   b) $\sin 20° + \sin 40° \neq \sin 60°$
   c) $\tan 75° \neq 3 \tan 25°$

Mid-Chapter Review

FREQUENTLY ASKED Questions

Study *Aid*

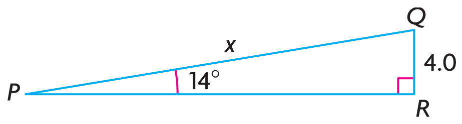
- See Lesson 5.1, Examples 1, 2, and 3.
- Try Mid-Chapter Review Questions 1 to 5.

Q: Given any right triangle, how would you use a trigonometric ratio to determine an unknown side or angle?

A: You can use either a primary trigonometric ratio or a reciprocal trigonometric ratio. The ratio in which the unknown is in the numerator makes the equation easier to solve.

EXAMPLE

Determine x to the nearest tenth of a unit.



$$\csc 14° = \frac{x}{4.0} \qquad \text{or} \qquad \sin 14° = \frac{4.0}{x}$$

$$4.0 \csc 14° = x \qquad\qquad x \sin 14° = 4.0$$

$$16.5 \doteq x \qquad\qquad x = \frac{4.0}{\sin 14°}$$

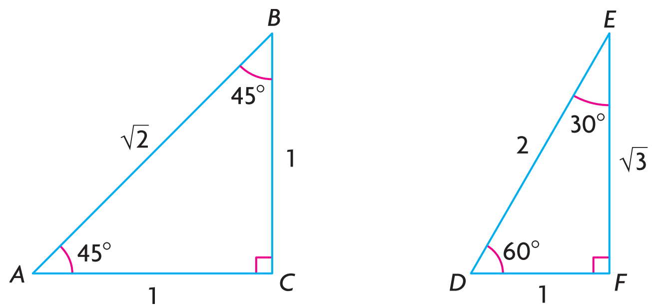
$$x \doteq 16.5$$

Study *Aid*

- See Lesson 5.2, Examples 1, 2, and 3.
- Try Mid-Chapter Review Questions 6 and 7.

Q: What is significant about the trigonometric ratios for 45°–45°–90° and 30°–60°–90° right triangles?

A: The trigonometric ratios for 30°, 45°, and 60° can be determined exactly without using a calculator.



θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
30°	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} \doteq 0.8660$	$\frac{\sqrt{3}}{3} \doteq 0.5774$
45°	$\frac{\sqrt{2}}{2} \doteq 0.7071$	$\frac{\sqrt{2}}{2} \doteq 0.7071$	1
60°	$\frac{\sqrt{3}}{2} \doteq 0.8660$	$\frac{1}{2} = 0.5$	$\sqrt{3} \doteq 1.7321$

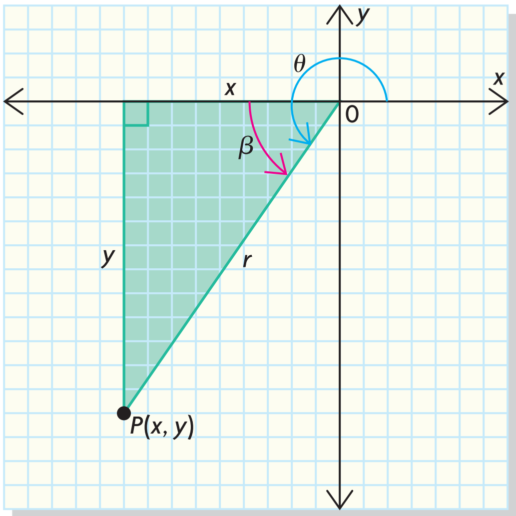
Q: **How can you determine the trigonometric ratios for any angle θ, where $0° \leq \theta \leq 360°$?**

A: Any angle in standard position in the Cartesian plane can be defined using the point $P(x, y)$, provided that P lies on the terminal arm of the angle. The trigonometric ratios can then be expressed in terms of x, y, and r, where r is the distance from the origin to P.

$r^2 = x^2 + y^2$ from the Pythagorean theorem and $r > 0$

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$



Study Aid

- See Lesson 5.4, Examples 1 to 4.
- Try Mid-Chapter Review Questions 9 to 13.

Q: **How can you determine all possible values of the principal angle θ in the Cartesian plane associated with a given trigonometric ratio?**

A: Use the sign of the ratio to help you decide in which quadrant(s) the terminal arm of angle θ could lie. Then sketch the angle(s) in standard position. Use the appropriate inverse trigonometric function on your calculator to determine a value for θ. An angle in standard position is determined by a counterclockwise rotation and is always positive. A negative angle is determined by a clockwise rotation.

Interpret the calculator result in terms of your sketch, and determine the value of any related acute angle β. Use this value of β to determine all possible values of the principal angle θ.

Study Aid

- See Lesson 5.3 and Lesson 5.4, Example 4.
- Try Mid-Chapter Review Questions 10, 11, and 12.

PRACTICE Questions

Lesson 5.1

1. Evaluate each reciprocal trigonometric ratio to four decimal places.

a) csc 20° **b)** sec 75° **c)** cot 10° **d)** csc 81°

2. Determine the value of θ to the nearest degree if $0° \leq \theta \leq 90°$.

a) $\cot \theta = 0.8701$ **b)** $\sec \theta = 4.1011$ **c)** $\csc \theta = 1.6406$ **d)** $\sec \theta = 2.4312$

3. A trigonometric ratio is $\frac{7}{5}$. What ratio could it be, and what angle might it be referring to?

4. Claire is attaching a rope to the top of the mast of her sailboat so that she can lower the sail to the ground to do some repairs. The mast is 8.3 m long, and with her eyes level with the base of the mast, the top forms an angle of 31° with the ground. How much rope does Claire need if 0.5 m of rope is required to tie to the mast? Round your answer to the nearest tenth of a metre.

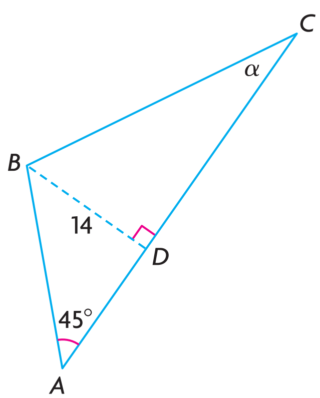
5. If $\csc \theta < \sec \theta$ and θ is acute, what do you know about θ?

Lesson 5.2

6. Determine the exact value of each trigonometric ratio.

a) sin 60° **b)** tan 45° **c)** csc 30° **d)** sec 45°

7. Given $\triangle ABC$ as shown,



a) determine the exact measure of each unknown side if $\sin \alpha = \frac{1}{2}$

b) determine the exact values of the primary trigonometric ratios for $\angle A$ and $\angle DBC$

Lesson 5.3

8. i) Sketch each angle in standard position. Use the sketch to determine the exact value of the given trigonometric ratio.

ii) If $0° \leq \theta \leq 360°$, state all values of θ that have the same given trigonometric ratio.

a) sin 120° **b)** cos 225° **c)** tan 330° **d)** cos 300°

Lesson 5.4

9. $P(-9, 4)$ lies on the terminal arm of an angle in standard position.

a) Sketch the principal angle θ.

b) What is the value of the related acute angle β to the nearest degree?

c) What is the value of the principal angle θ to the nearest degree?

10. Jeff said he found three angles for which $\cos \theta = \frac{4}{5}$. Is that possible if $0° \leq \theta \leq 360°$? Explain.

11. Given $\tan \theta = -\frac{15}{8}$, where $90° \leq \theta \leq 180°$,

a) state the other five trigonometric ratios as fractions

b) determine the value of θ to the nearest degree

12. If $\sin \theta = -0.8190$ and $0° \leq \theta \leq 360°$, determine the value of θ to the nearest degree.

13. Angle θ lies in quadrant 2. Without using a calculator, which ratios must be false? Justify your reasoning.

a) $\cos \theta = 2.3151$ **b)** $\tan \theta = 2.3151$ **c)** $\sec \theta = 2.3151$ **d)** $\csc \theta = 2.3151$ **e)** $\cot \theta = 2.3151$ **f)** $\sin \theta = 2.3151$

5.5 Trigonometric Identities

GOAL

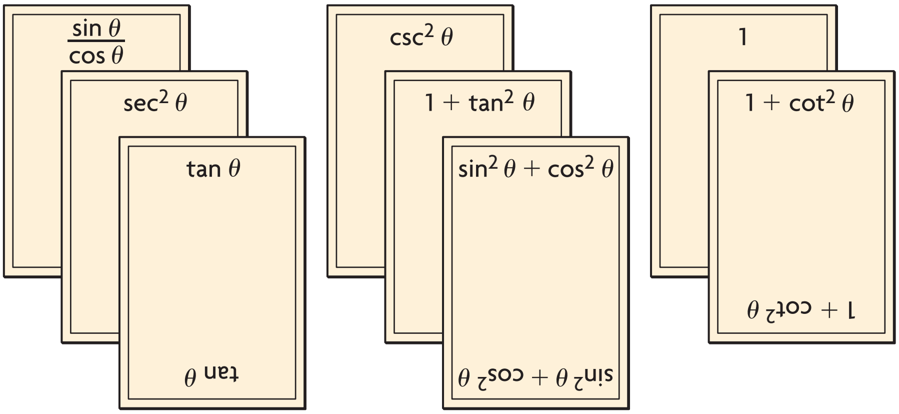
Prove simple trigonometric identities.

LEARN ABOUT the Math

Trident Fish is a game involving a deck of cards, each of which has a mathematical expression written on it. The object of the game is to lay down pairs of equivalent expressions so that each pair forms an identity. Suppose you have the cards shown.

identity

a mathematical statement that is true for all values of the given variables. If the identity involves fractions, the denominators cannot be zero. Any restrictions on a variable must be stated.



? What identities can you form with these cards?

EXAMPLE 1 Proving the quotient identity by rewriting in terms of *x*, *y*, and *r*

Prove the quotient identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ for all angles θ, where $0° \leq \theta \leq 360°$.

Jinji's Solution

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\text{L.S.} = \tan\theta \qquad \text{R.S.} = \frac{\sin\theta}{\cos\theta}$$

I separated the left and the right sides so that I could show that both expressions are equivalent.

$$\text{L.S.} = \frac{y}{x} \qquad \text{R.S.} = \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)}$$

I wrote $\sin \theta$, $\tan \theta$, and $\cos \theta$ in terms of x, y, and r, since θ can be greater than 90°.

$$= \frac{y}{\cancel{r}_1} \times \frac{\cancel{r}^1}{x}$$

I simplified the right side by multiplying the numerator by the reciprocal of the denominator.

$$= \frac{y}{x}$$

$$= \text{L.S.}$$

Since the left side works out to the same expression as the right side, the original equation is an identity.

$\therefore \tan \theta = \dfrac{\sin \theta}{\cos \theta}$ for all angles θ, where $0° \leq \theta \leq 360°$ and $\theta \neq 90°$ or $270°$.

Tan θ is undefined when $\cos \theta = 0$. This occurs when $\theta = 90°$ or $270°$. So θ cannot equal these two values.

EXAMPLE 2 Proving the Pythagorean identity by rewriting in terms of *x*, *y*, and *r*

Prove the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ for all angles θ, where $0° \leq \theta \leq 360°$.

Lisa's Solution

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\text{L.S.} = \sin^2 \theta + \cos^2 \theta \qquad \text{R.S.} = 1$$

I separated the left and the right sides so that I could show that both expressions are equivalent.

$$= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2$$

I wrote $\sin \theta$ and $\cos \theta$ in terms of x, y, and r, since θ can be greater than 90°. Then I simplified.

$$= \frac{y^2}{r^2} + \frac{x^2}{r^2}$$

$$= \frac{y^2 + x^2}{r^2}$$

I knew that $r^2 = x^2 + y^2$ from the Pythagorean theorem. I used this equation to further simplify the left side.

$$= \frac{r^2}{r^2}$$

$$= 1$$

$$= \text{R.S.}$$

Since the left side works out to the same expression as the right side, the original equation is an identity.

$\therefore \sin^2 \theta + \cos^2 \theta = 1$ for all angles θ, where $0° \leq \theta \leq 360°$.

EXAMPLE 3 Proving an identity by using a common denominator

Prove that $1 + \cot^2 \theta = \csc^2 \theta$ for all angles θ between $0°$ and $360°$ except $0°$, $180°$, and $360°$.

Pedro's Solution

$1 + \cot^2 \theta = \csc^2 \theta$

L.S. $= 1 + \cot^2 \theta$ R.S. $= \csc^2 \theta$

I separated the left and the right sides so that I could show that both expressions are equivalent.

$$= 1 + \left(\frac{\cos \theta}{\sin \theta}\right)^2 \qquad = \left(\frac{1}{\sin \theta}\right)^2$$

$$= 1 + \frac{\cos^2 \theta}{\sin^2 \theta} \qquad = \frac{1}{\sin^2 \theta}$$

I expressed the reciprocal trigonometric ratios in terms of the primary ratios $\sin \theta$ and $\cos \theta$. I knew that $\cot \theta = \frac{1}{\tan \theta}$ and $\tan \theta = \frac{\sin \theta}{\cos \theta}$, so $\cot \theta = \frac{\cos \theta}{\sin \theta}$. Since θ can't be $0°$, $180°$, or $360°$, $\sin \theta \neq 0$, I don't have any term that is undefined.

$$= \frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta}$$

On the left side, I expressed 1 as $\frac{\sin^2 \theta}{\sin^2 \theta}$ to get a common denominator of $\sin^2 \theta$.

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta}$$

I used the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ to simplify the numerator.

$$= \frac{1}{\sin^2 \theta}$$

$=$ R.S.

Since the left side works out to the same expression as the right side, the original equation is an identity.

$\therefore 1 + \cot^2 \theta = \csc^2 \theta$ for all angles θ between $0°$ and $360°$ except $0°$, $180°$, and $360°$.

Reflecting

A. What strategy would you use to prove the identity $1 + \tan^2 \theta = \sec^2 \theta$? What restrictions does θ have?

B. When is it important to consider restrictions on θ?

C. How might you create new identities based on Examples 1 and 2?

APPLY the Math

EXAMPLE 4 Proving an identity by factoring

Prove that $\tan \phi = \frac{\sin \phi + \sin^2 \phi}{(\cos \phi)(1 + \sin \phi)}$ for all angles ϕ between 0° and 360°, where $\cos \phi \neq 0$.

Jamal's Solution

$$\tan \phi = \frac{\sin \phi + \sin^2 \phi}{(\cos \phi)(1 + \sin \phi)}$$

I separated the left and the right sides so that I could show that both expressions are equivalent.

$$\text{L.S.} = \tan \phi \qquad \text{R.S.} = \frac{\sin \phi + \sin^2 \phi}{(\cos \phi)(1 + \sin \phi)}$$

$$= \frac{\sin \phi}{\cos \phi} \qquad = \frac{\sin \phi \overset{1}{\cancel{(1 + \sin \phi)}}}{(\cos \phi)\underset{1}{\cancel{(1 + \sin \phi)}}}$$

$$= \frac{\sin \phi}{\cos \phi}$$

I knew that $\tan \phi$ could be written as $\frac{\sin \phi}{\cos \phi}$. The right side is more complicated, so I factored out $\sin \phi$ from the numerator. Since $\cos \phi \neq 0$, the denominator will not be 0. I divided the numerator and denominator by the factor $1 + \sin \phi$.

$$= \text{L.S.}$$

$\therefore \tan \phi = \frac{\sin \phi + \sin^2 \phi}{(\cos \phi)(1 + \sin \phi)}$ for all angles ϕ between 0° and 360°, where $\cos \phi \neq 0$.

Since the left side works out to the same expression as the right side, the original equation is an identity.

In Summary

Key Ideas

- A trigonometric identity is an equation involving trigonometric ratios that is true for all values of the variable.
- Some trigonometric identities are a result of a definition, while others are derived from relationships that exist among trigonometric ratios.

Need to Know

- Some trigonometric identities that are important to remember are shown below $(0° \leq \theta \leq 360°)$.

Identities Based on Definitions	Identities Derived from Relationships	
Reciprocal Identities	**Quotient Identities**	**Pythagorean Identities**
$\csc\theta = \frac{1}{\sin\theta}$, where $\sin\theta \neq 0$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$, where $\cos\theta \neq 0$	$\sin^2\theta + \cos^2\theta = 1$
$\sec\theta = \frac{1}{\cos\theta}$, where $\cos\theta \neq 0$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$, where $\sin\theta \neq 0$	$1 + \tan^2\theta = \sec^2\theta$
$\cot\theta = \frac{1}{\tan\theta}$, where $\tan\theta \neq 0$		$1 + \cot^2\theta = \csc^2\theta$

- To prove that a given trigonometric equation is an identity, both sides of the equation need to be shown to be equivalent. This can be done by
  - simplifying the more complicated side until it is identical to the other side or manipulating both sides to get the same expression
  - rewriting all trigonometric ratios in terms of x, y, and r
  - rewriting all expressions involving tangent and the reciprocal trigonometric ratios in terms of sine and cosine
  - applying the Pythagorean identity where appropriate
  - using a common denominator or factoring as required

CHECK Your Understanding

1. Prove each identity by writing all trigonometric ratios in terms of x, y, and r. State the restrictions on θ.

a) $\cot \theta = \dfrac{\cos \theta}{\sin \theta}$

b) $\tan \theta \cos \theta = \sin \theta$

c) $\csc \theta = \dfrac{1}{\sin \theta}$

d) $\cos \theta \sec \theta = 1$

2. Simplify each expression.

a) $(1 - \sin \alpha)(1 + \sin \alpha)$

b) $\dfrac{\tan \alpha}{\sin \alpha}$

c) $\cos^2 \alpha + \sin^2 \alpha$

d) $\cot \alpha \sin \alpha$

3. Factor each expression.

a) $1 - \cos^2 \theta$

b) $\sin^2 \theta - \cos^2 \theta$

c) $\sin^2 \theta - 2 \sin \theta + 1$

d) $\cos \theta - \cos^2 \theta$

PRACTISING

4. Prove that $\frac{\cos^2\phi}{1 - \sin \phi} = 1 + \sin \phi$, where $\sin \phi \neq 1$, by expressing $\cos^2 \phi$ in terms of $\sin \phi$.

5. Prove each identity. State any restrictions on the variables.

a) $\dfrac{\sin x}{\tan x} = \cos x$

b) $\dfrac{\tan \theta}{\cos \theta} = \dfrac{\sin \theta}{1 - \sin^2 \theta}$

c) $\dfrac{1}{\cos \alpha} + \tan \alpha = \dfrac{1 + \sin \alpha}{\cos \alpha}$

d) $1 - \cos^2 \theta = \sin \theta \cos \theta \tan \theta$

6. Mark claimed that $\frac{1}{\cot \theta} = \tan \theta$ is an identity. Marcia let $\theta = 30°$ and found that both sides of the equation worked out to $\frac{1}{\sqrt{3}}$. She said that this proves that the equation is an identity. Is Marcia's reasoning correct? Explain.

7. Simplify each trigonometric expression.

a) $\sin \theta \cot \theta - \sin \theta \cos \theta$

b) $\cos \theta (1 + \sec \theta)(\cos \theta - 1)$

c) $(\sin x + \cos x)(\sin x - \cos x) + 2 \cos^2 x$

d) $\dfrac{\csc^2 \theta - 3 \csc \theta + 2}{\csc^2 \theta - 1}$

8. Prove each identity. State any restrictions on the variables.

a) $\dfrac{\sin^2 \phi}{1 - \cos \phi} = 1 + \cos \phi$

b) $\dfrac{\tan^2 \alpha}{1 + \tan^2 \alpha} = \sin^2 \alpha$

c) $\cos^2 x = (1 - \sin x)(1 + \sin x)$

d) $\sin^2 \theta + 2 \cos^2 \theta - 1 = \cos^2 \theta$

e) $\sin^4 \alpha - \cos^4 \alpha = \sin^2 \alpha - \cos^2 \alpha$

f) $\tan \theta + \dfrac{1}{\tan \theta} = \dfrac{1}{\sin \theta \cos \theta}$

9. Farah claims that if you separate both sides of an equation into two functions and graph them on the same *xy*-axes on a graphing calculator, you can use the result to prove that the equation is an identity.
A

a) Is her claim correct? Justify your answer.

b) Discuss the limitations of her approach.

10. Is $\csc^2 \theta + \sec^2 \theta = 1$ an identity? Prove that it is true or demonstrate why it is false.

11. Prove that $\sin^2 x\left(1 + \frac{1}{\tan^2 x}\right) = 1$, where $\sin x \neq 0$.
K

12. Prove each identity. State any restrictions on the variables.
T

a) $\dfrac{\sin^2 \theta + 2 \cos \theta - 1}{\sin^2 \theta + 3 \cos \theta - 3} = \dfrac{\cos^2 \theta + \cos \theta}{-\sin^2 \theta}$

b) $\sin^2 \alpha - \cos^2 \alpha - \tan^2 \alpha = \dfrac{2 \sin^2 \alpha - 2 \sin^4 \alpha - 1}{1 - \sin^2 \alpha}$

13. Show how you can create several new identities from the identity $\sin^2 \theta + \cos^2 \theta = 1$ by adding, subtracting, multiplying, or dividing both sides of the equation by the same expression.
C

Extending

14. a) Which equations are not identities? Justify your answers.

b) For those equations that are identities, state any restrictions on the variables.

i) $(1 - \cos^2 x)(1 - \tan^2 x) = \dfrac{\sin^2 x - 2 \sin^4 x}{1 - \sin^2 x}$

ii) $1 - 2 \cos^2 \phi = \sin^4 \phi - \cos^4 \phi$

iii) $\dfrac{\sin \theta \tan \theta}{\sin \theta + \tan \theta} = \sin \theta \tan \theta$

iv) $\dfrac{1 + 2 \sin \beta \cos \beta}{\sin \beta + \cos \beta} = \sin \beta + \cos \beta$

v) $\dfrac{1 - \cos \beta}{\sin \beta} = \dfrac{\sin \beta}{1 + \cos \beta}$

vi) $\dfrac{\sin x}{1 + \cos x} = \csc x - \cot x$

5.6 The Sine Law

YOU WILL NEED

- dynamic geometry software (optional)

Communication | *Tip*

To perform a calculation to a high degree of accuracy, save intermediate answers by using the memory keys of your calculator. Round only after the very last calculation.

GOAL

Solve two-dimensional problems by using the sine law.

LEARN ABOUT *the Math*

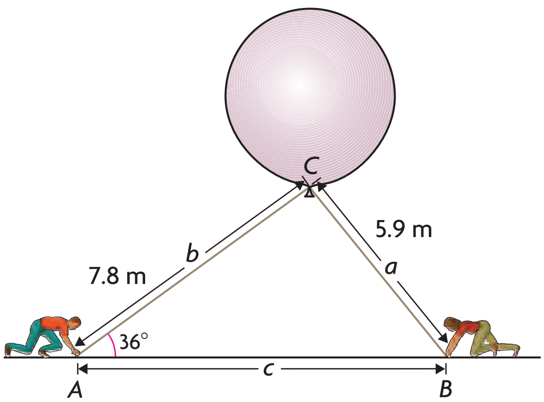
Albert and Belle are part of a scientific team studying thunderclouds. The team is about to launch a weather balloon into an active part of a cloud. Albert's rope is 7.8 m long and makes an angle of 36° with the ground. Belle's rope is 5.9 m long.

? How far, to the nearest tenth of a metre, is Albert from Belle?

EXAMPLE 1 Using the sine law to calculate an unknown length

Determine the distance between Albert and Belle.

Adila's Solution: Assuming that Albert and Belle are on Opposite Sides of the Balloon



From the problem, it is not clear how Albert, Belle, and the balloon are positioned relative to each other. I assumed that Albert and Belle are on opposite sides of the balloon. I drew a sketch of this situation.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

In $\triangle ABC$, I knew one angle and two sides. To determine AB (side c), I needed to know $\angle C$. So I first calculated $\angle B$ using the **sine law**.

$$\frac{5.9}{\sin 36°} = \frac{7.8}{\sin B}$$

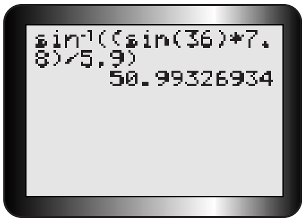
$$(\cancel{\sin 36°}^{1} \times \sin B)\left(\frac{5.9}{\cancel{\sin 36°}_{1}}\right) = (\sin 36° \times \cancel{\sin B}^{1})\left(\frac{7.8}{\cancel{\sin B}_{1}}\right)$$

To solve for $\angle B$, I multiplied both sides of the equation by the lowest common denominator $(\sin 36° \times \sin B)$ to eliminate the fractions.

$$\frac{(\sin B)(\cancel{5.9}^{1})}{\cancel{5.9}_{1}} = \frac{(\sin 36°)(7.8)}{5.9}$$

Then I divided both sides by 5.9 to isolate sin B.

$$\angle B = \sin^{-1}\left(\frac{(\sin 36°)(7.8)}{5.9}\right)$$



I used my calculator to evaluate $\angle B$.

$$\angle B \doteq 51°$$

I rounded to the nearest degree.

$$\angle A + \angle B + \angle C = 180°$$
$$36° + 51° + \angle C = 180°$$
$$\angle C = 180° - (36° + 51°)$$
$$= 93°$$

I calculated $\angle C$ by using the fact that all three interior angles add up to 180°.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Then I used the sine law again to determine c.

$$\frac{5.9}{\sin 36°} = \frac{c}{\sin 93°}$$

$$\frac{5.9}{\sin 36°} \times \sin 93° = \frac{c}{\cancel{\sin 93°}_{1}} \times \cancel{\sin 93°}^{1}$$

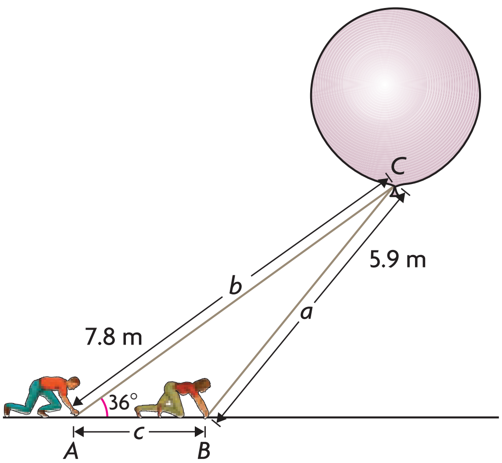
To solve for c, I multiplied both sides of the equation by sin 93°.

$$10.0 \text{ m} \doteq c$$

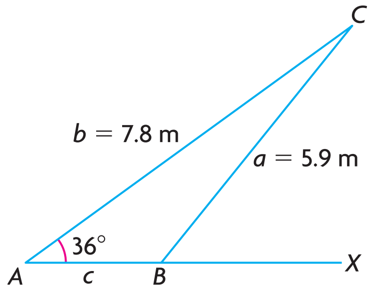
I rounded to the nearest tenth.

If Albert and Belle are on opposite sides of the balloon, they are about 10.0 m apart.

Reuben's Solution: Assuming that Albert and Belle are on the Same Side of the Balloon



The problem did not state how Albert, Belle, and the balloon are positioned relative to each other. I assumed that Albert and Belle are on the same side of the balloon. I drew a sketch of this situation.



In $\triangle ABC$, I knew one angle and two sides. If I knew $\angle C$, I could determine c using the sine law. First I had to determine $\angle B$ in order to get $\angle C$.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

I used the sine law to calculate $\angle B$.

$$\frac{5.9}{\sin 36°} = \frac{7.8}{\sin B}$$

$$(\cancel{\sin 36°}_{1} \times \sin B)\left(\frac{5.9}{\cancel{\sin 36°}_{1}}\right) = (\sin 36° \times \overset{1}{\cancel{\sin B}})\left(\frac{7.8}{\cancel{\sin B}_{1}}\right)$$

To solve for $\angle B$, I first multiplied both sides of the equation by the lowest common denominator ($\sin 36° \times \sin B$) to eliminate the fractions.

$$\frac{(\sin B)(\overset{1}{\cancel{5.9}})}{\cancel{5.9}_{1}} = \frac{(\sin 36°)(7.8)}{5.9}$$

I divided both sides by 5.9 to isolate $\sin B$.

$$\angle B = \sin^{-1}\left(\frac{(\sin 36°)(7.8)}{5.9}\right)$$

I used my calculator to evaluate $\angle B$. I rounded to the nearest degree.

$$\angle CBX \doteq 51°$$

51° is the value of the related acute angle $\angle CBX$, but I wanted the obtuse angle $\angle CBA$ in the triangle.

$$\angle CBA = 180° - 51°$$
$$= 129°$$

$$\angle C = 180° - (\angle A + \angle CBA)$$
$$= 180° - (36° + 129°)$$
$$= 15°$$

I calculated $\angle C$ by using the fact that all three interior angles add up to 180°.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

I used the sine law again to calculate side c.

$$\frac{5.9}{\sin 36°} = \frac{c}{\sin 15°}$$

$$\sin 15° \times \frac{5.9}{\sin 36°} = \cancel{\sin 15°}^{1} \times \frac{c}{\cancel{\sin 15°}_{1}}$$

To solve for c, I multiplied both sides of the equation by sin 15°.

$$2.6 \text{ m} \doteq c$$

I rounded to the nearest tenth.

If Albert and Belle are on the same side of the balloon, they are about 2.6 m apart.

Reflecting

A. Why is the situation in Example 1 called **the ambiguous case of the sine law**?

B. What initial information was given in this problem?

C. What is the relationship between sin B in Adila's solution and sin B in Reuben's solution? Explain why both values of sine are related.

D. Calculate the height of $\triangle ABC$ in both solutions. What do you notice? Compare this value with the length of a and b.

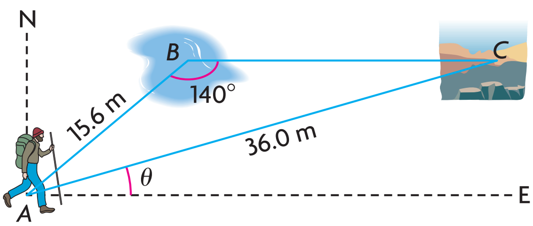
the ambiguous case of the sine law

a situation in which 0, 1, or 2 triangles can be drawn given the information in a problem. This occurs when you know two side lengths and an angle *opposite* one of the sides rather than *between* them (an SSA triangle). If the given angle is acute, 0, 1, or 2 triangles are possible. If the given angle is obtuse, 0 or 1 triangle is possible (see the In Summary box for this lesson).

APPLY the Math

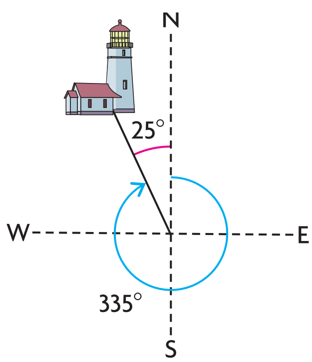
EXAMPLE 2 | **Using the sine law in the ambiguous case to calculate the only possible angle**

Karl's campsite is 15.6 m from a lake and 36.0 m from a scenic lookout as shown. From the lake, the angle formed between the campsite and the lookout is 140°. Karl starts hiking from his campsite to go to the lookout. What is the **bearing** of the lookout from Karl's position ($\angle NAC$)?

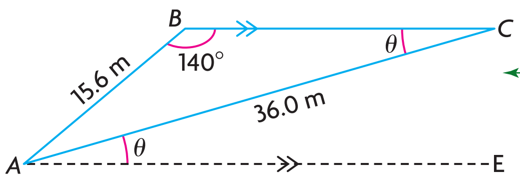


bearing

the direction in which you have to move in order to reach an object. A bearing is a clockwise angle from magnetic north. For example, the bearing of the lighthouse shown is 335°.



Sara's Solution



Based on the given information, this is an SSA triangle. But since the given angle is obtuse, only one situation had to be considered. In $\triangle ABC$, I knew that AE is parallel to BC. Since $\angle C$ and θ are alternate angles between parallel lines, they are equal.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

I needed to calculate an angle, so I set up the sine law equation with the angles in the numerators.

$$\frac{\sin \theta}{15.6} = \frac{\sin 140°}{36.0}$$

$$\overset{1}{\cancel{15.6}} \times \frac{\sin \theta}{\underset{1}{\cancel{15.6}}} = 15.6 \times \frac{\sin 140°}{36.0}$$

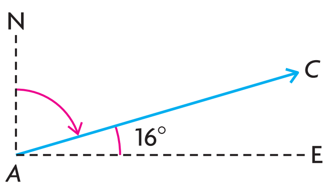
To solve for $\sin \theta$, I multiplied both sides of the equation by 15.6.

$$\theta = \sin^{-1}\left(\frac{(15.6)(\sin 140°)}{36.0}\right)$$

I used the inverse sine function on my calculator to determine $\angle C$ (angle θ).

$$\theta \doteq 16°$$

I rounded to the nearest degree.



In order to state the bearing of the lookout, I needed to know the complementary angle of 16°.

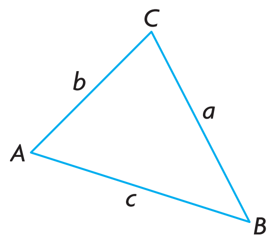
$$\begin{aligned} \angle NAC &= 90° - 16° \\ &= 74° \end{aligned}$$

So I subtracted 16° from 90°.

From Karl's campsite, the lookout has a bearing of about 74°.

In Summary

Key Ideas



- The sine law states that in any $\triangle ABC$, the ratios of each side to the sine of its opposite angle are equal.

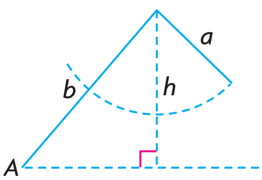
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

- Given any triangle, the sine law can be used if you know
  - two sides and one angle opposite a given side (SSA) or
  - two angles and any side (AAS or ASA)
- The ambiguous case arises in a SSA (side, side, angle) triangle. In this situation, depending on the size of the given angle and the lengths of the given sides, the sine law calculation may lead to 0, 1, or 2 solutions.

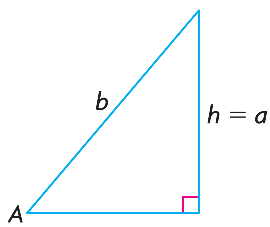
Need to Know

- In the ambiguous case, if $\angle A$, a, and b are given and $\angle A$ is acute, there are four cases to consider. In each case, the height of the triangle is $h = b \sin A$.

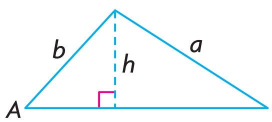
If $\angle A$ is acute and $a < h$, no triangle exists.



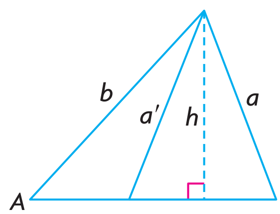
If $\angle A$ is acute and $a = h$, one right triangle exists.



If $\angle A$ is acute and $a > b$, one triangle exists.

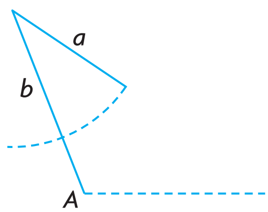


If $\angle A$ is acute and $h < a < b$, two triangles exist.

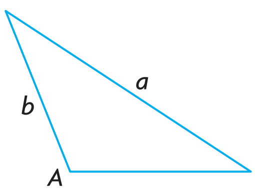


If $\angle A$, a, and b are given and $\angle A$ is obtuse, there are two cases to consider.

If $\angle A$ is obtuse and $a < b$ or $a = b$, no triangle exists.



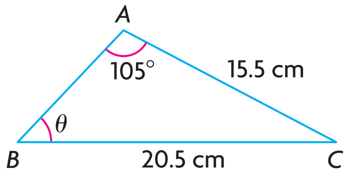
If $\angle A$ is obtuse and $a > b$, one triangle exists.



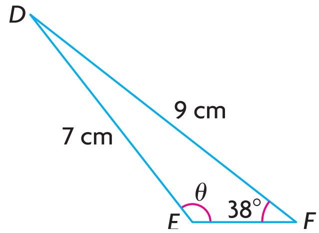
CHECK Your Understanding

1. Determine the measure of angle θ to the nearest degree.

a)



b)



2. A triangular plot of land is enclosed by a fence. Two sides of the fence are 9.8 m and 6.6 m long, respectively. The other side forms an angle of 40° with the 9.8 m side.

a) Draw a sketch of the situation.

b) Calculate the height of the triangle to the nearest tenth. Compare it to the given sides.

c) How many lengths are possible for the third side? Explain.

3. Determine whether it is possible to draw a triangle, given each set of information. Sketch all possible triangles where appropriate. Label all side lengths to the nearest tenth of a centimetre and all angles to the nearest degree.

a) $a = 5.2$ cm, $b = 2.8$ cm, $\angle B = 65°$

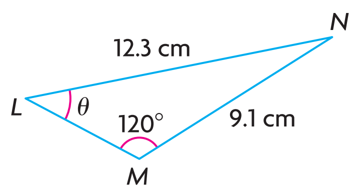
b) $b = 6.7$ cm, $c = 2.1$ cm, $\angle C = 63°$

c) $a = 5.0$ cm, $c = 8.5$ cm, $\angle A = 36°$

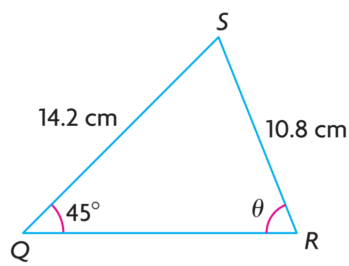
PRACTISING

4. Determine the measure of angle θ to the nearest degree.

a)



b)



5. **K** Where appropriate, sketch all possible triangles, given each set of information. Label all side lengths to the nearest tenth of a centimetre and all angles to the nearest degree.

a) $a = 7.2$ mm, $b = 9.3$ mm, $\angle A = 35°$

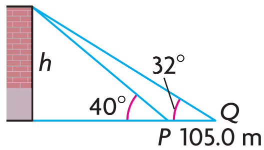
b) $a = 7.3$ m, $b = 14.6$ m, $\angle A = 30°$

c) $a = 1.3$ cm, $b = 2.8$ cm, $\angle A = 33°$

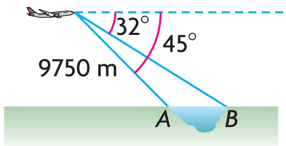
d) $c = 22.2$ cm, $\angle A = 75°$, $\angle B = 43°$

6. The trunk of a leaning tree makes an angle of 12° with the vertical. To prevent the tree from falling over, a 35.0 m rope is attached to the top of the tree and is pegged into level ground some distance away. If the tree is 20.0 m from its base to its top, calculate the angle the rope makes with the ground to the nearest degree.

7. **A** A building of height *h* is observed from two points, *P* and *Q*, that are 105.0 m apart as shown. The angles of elevation at *P* and *Q* are 40° and 32°, respectively. Calculate the height, *h*, to the nearest tenth of a metre.



8. A surveyor in an airplane observes that the angle of depression to two points on the opposite shores of a lake are 32° and 45°, respectively, as shown. What is the width of the lake, to the nearest metre, at those two points?



9. The Pont du Gard near Nîmes, France, is a Roman aqueduct. An observer in a hot-air balloon some distance away from the aqueduct determines that the angle of depression to each end is 54° and 71°, respectively. The closest end of the aqueduct is 270.0 m from the balloon. Calculate the length of the aqueduct to the nearest tenth of a metre.

10. **T** A wind tower at the top of a hill casts a shadow 30 m long along the side of the hill. An observer at the farthest edge of the shadow from the tower estimates the angle of elevation to the top of the tower to be 34°. If the slope of the hill is 13° from the horizontal, how high is the tower to the nearest metre?

11. Carol is flying a kite on level ground, and the string forms an angle of 50° with the ground. Two friends standing some distance from Carol see the kite at angles of elevation of 66° and 35°, respectively. One friend is 11 m from Carol. For each question below, state all possible answers to the nearest metre.
   a) How high is the kite above the ground?
   b) How long is the string?
   c) How far is the other friend from Carol?

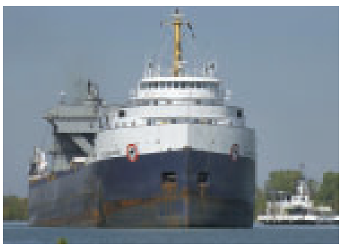
12. The Huqiu Tower in China was built in 961 CE. When the tower was first built, its height was 47 m. Since then it has tilted 2.8°, so it is called China's Leaning Tower. There is a specific point on the ground where you can be equidistant from both the top and the bottom of the tower. How far is this point from the base of the tower? Round your answer to the nearest metre.

13. Your neighbour claims that his lot is triangular, with one side 430 m long and the adjacent side 110 m long. The angle opposite one of these sides is 35°. Determine the other side length of this lot to the nearest metre and the interior angles to the nearest degree.

14. **C** In $\triangle LMN$, $\angle L$ is acute. Using a sketch, explain the relationship between $\angle L$, sides l and m, and the height of the triangle for each situation.
   a) Only one triangle is possible.
   b) Two triangles are possible.
   c) No triangle is possible.

Extending

15. A sailor out in a lake sees two lighthouses 11 km apart along the shore and gets bearings of 285° from his present position for lighthouse A and 237° for lighthouse B. From lighthouse B, lighthouse A has a bearing of 45°.
   a) How far, to the nearest kilometre, is the sailor from both lighthouses?
   b) What is the shortest distance, to the nearest kilometre, from the sailor to the shore?



16. The *Algomarine* is a cargo ship that is 222.5 m long. On the water, small watercraft have the right of way. However, bulk carriers cruise at nearly 30 km/h, so it is best to stay out of their way: If you pass a cargo ship within 40 m, your boat could get swamped! Suppose you spot the *Algomarine* on your starboard (right) side headed your way. The bow and stern of the carrier appear separated by 12°. The captain of the *Algomarine* calls you from the bridge, located at the stern, and says that you are 8° off his bow.
   a) How far, to the nearest metre, are you from the stern?
   b) Are you in danger of being swamped?

17. The Gerbrandy Tower in the Netherlands is an 80 m high concrete tower, on which a 273.5 m guyed mast is mounted. The lower guy wires form an angle of 36° with the ground and attach to the tower 155 m above ground. The upper guy wires form an angle of 59° with the ground and attach to the mast 350 m above ground. How long are the upper and lower guy wires? Round your answers to the nearest metre.

5.7 The Cosine Law

GOAL

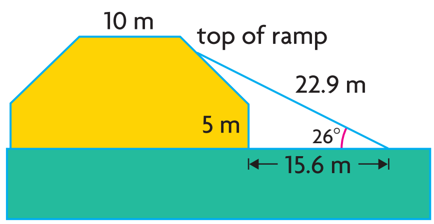
Solve two-dimensional problems by using the cosine law.

YOU WILL NEED

- dynamic geometry software (optional)

LEARN ABOUT the Math

A barn whose cross-section resembles half a regular octagon with a side length of 10 m needs some repairs to its roof. The roofers place a 22.9 m ramp against the side of the building, forming an angle of 26° with the ground. The ramp will be used to transport the materials needed for the repair. The base of the ramp is 15.6 m from the side of the building.

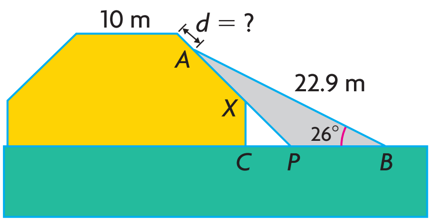


? How far, to the nearest tenth of a metre, is the top of the ramp from the flat roof of the building?

EXAMPLE 1 Using the cosine law to calculate an unknown length

Determine the distance from the top of the ramp to the roof by using the **cosine law**.

Tina's Solution

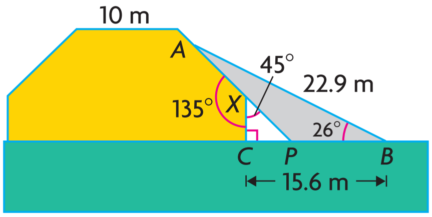


I labelled the top of the ramp *A* and the bottom of the ramp *B*. Then I drew a line from *A* along the sloped part of the building to *X* and extended the line to the ground at *P*. I labelled the point where the side of the building touches the ground *C*.

$$\angle AXC + \angle CXP = 180°$$
$$135° + \angle CXP = 180°$$
$$\angle CXP = 180° - 135°$$
$$= 45°$$

In $\triangle XCP$, $\angle C$ is 90°. Since the octagon is regular, each interior angle is 135°. So $\angle AXC$ is 135°. To determine $\angle CXP$, I subtracted 135° from 180°.

$\therefore$ $\triangle XCP$ is a $45° - 45° - 90°$ special triangle.



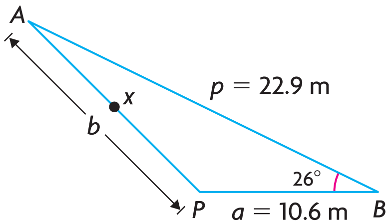
$$CP + PB = 15.6$$

$$5 + PB = 15.6$$

From the given information, I knew that $XC = 5$ m, so $CP = 5$ m, since the triangle is isosceles.

$$PB = 15.6 - 5$$
$$= 10.6 \text{ m}$$

I then subtracted CP from CB to determine the length of PB.



In $\triangle APB$, I knew two side lengths and the contained angle formed by those sides. So I couldn't use the sine law to determine AP. I used the cosine law instead.

$$b^2 = a^2 + p^2 - 2ap \cos B$$
$$b^2 = (10.6)^2 + (22.9)^2 - 2(10.6)(22.9)\cos 26°$$
$$b^2 = 200.42 \text{ m}^2$$
$$b = \sqrt{200.42}$$
$$b \doteq 14.16 \text{ m}$$

I substituted the values of a, p, and $\angle B$ into the formula. I calculated b by evaluating the right side of the equation and determining its square root.

$$XP = 5\sqrt{2}$$
$$AX + XP = b$$
$$AX + 5\sqrt{2} = 14.16$$
$$AX = 14.16 - 5\sqrt{2}$$
$$\doteq 7.09 \text{ m}$$

To determine the distance from the top of the ramp to the roof, I needed to calculate AX first. I knew that XP is a multiple of $\sqrt{2}$ because $\triangle XCP$ is a $45° - 45° - 90°$ special triangle. So I subtracted XP from b to determine AX.

$$\text{required distance} = 10 - AX$$
$$= 10 - 7.09$$
$$\doteq 2.9 \text{ m}$$

Then I subtracted AX from 10 m to get the distance from the top of the ramp to the roof.

The top of the ramp is about 2.9 m from the flat roof of the building.

Reflecting

A. Why did Tina draw line AP on her sketch as part of her solution?

B. Could Tina have used the sine law, instead of the cosine law, to solve the problem? Explain your reasoning.

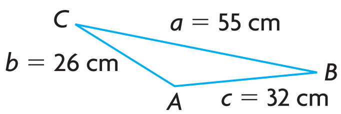
C. The Pythagorean theorem is a special case of the cosine law. What conditions would have to exist in a triangle in order for the cosine law to simplify to the Pythagorean theorem?

APPLY the Math

EXAMPLE 2 Using the cosine law to determine an angle

In $\triangle ABC$, determine $\angle A$ to the nearest degree if $a = 55$ cm, $b = 26$ cm, and $c = 32$ cm.

Claudio's Solution



I drew a diagram of the triangle and labelled all sides.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Since I knew all three sides (SSS) but no angles, I couldn't use the sine law. So I used the cosine law to determine $\angle A$.

$$55^2 = 26^2 + 32^2 - 2(26)(32)\cos A$$

I substituted the values of a, b, and c into the formula.

$$\cos A = \frac{55^2 - (26^2 + 32^2)}{-2(26)(32)}$$

$$\angle A = \cos^{-1}\left(\frac{55^2 - (26^2 + 32^2)}{-2(26)(32)}\right)$$

I used the inverse cosine function on my calculator to determine $\angle A$.

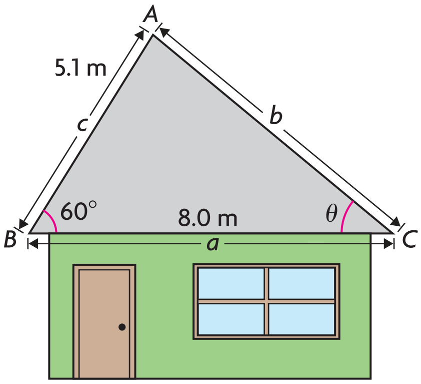
$$\angle A \doteq 143°$$

Given $\triangle ABC$, $\angle A$ is about 143°.

EXAMPLE 3 Solving a problem by using the cosine and the sine laws

Mitchell wants his 8.0 wide house to be heated with a solar hot-water system. The tubes form an array that is 5.1 m long. In order for the system to be effective, the array must be installed on the south side of the roof and the roof needs to be inclined by 60°. If the north side of the roof is inclined more than 40°, the roof will be too steep for Mitchell to install the system himself. Will Mitchell be able to install this system by himself?

Serina's Solution



I drew a sketch of the situation. I wanted to use the sine law to determine angle θ to solve the problem. But before I could do that, I needed to determine the length of side *b*.

$$b^2 = a^2 + c^2 - 2ac\cos B$$

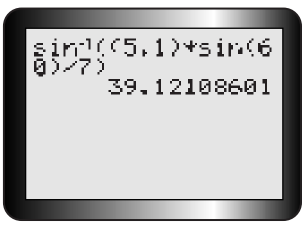
Since I knew two sides and the angle between them, I couldn't use the sine law to determine *b*. So I used the cosine law.

$$b^2 = (8.0)^2 + (5.1)^2 - 2(8.0)(5.1)\cos 60°$$
$$b^2 = 49.21 \text{ m}^2$$
$$b = \sqrt{49.21}$$
$$b \doteq 7.0 \text{ m}$$

I substituted the values of *a*, *c*, and $\angle B$ into the formula. I calculated *b* by evaluating the right side of the equation and determining its square root.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$
$$\frac{\sin \theta}{5.1} = \frac{\sin 60°}{7.0}$$
$$\overset{1}{\cancel{5.1}} \times \frac{\sin \theta}{\underset{1}{\cancel{5.1}}} = 5.1 \times \frac{\sin 60°}{7.0}$$
$$\theta = 5.1 \times \frac{\sin 60°}{7.0}$$

I determined $\angle C$ (θ) by using the sine law. Since I needed to solve for an angle, I wrote the sine law with the angles in the numerators. I multiplied both sides of the equation by 5.1 to solve for sin θ.



I used the inverse sine function on my calculator to determine angle θ.

$$\theta \doteq 39°$$

Since Mitchell's roof is inclined about 39° on the north side, he will be able to install the solar hot-water system by himself.

In Summary

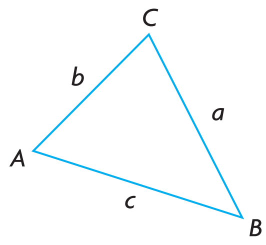
Key Idea

- Given any triangle, the cosine law can be used if you know
  - two sides and the angle contained between those sides (SAS) or
  - all three sides (SSS)

Need to Know

- The cosine law states that in any $\triangle ABC$,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



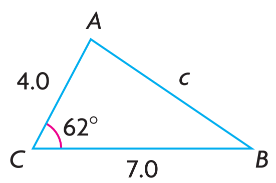
- If $\angle A = 90°$ and $\angle A$ is the contained angle, then the cosine law simplifies to the Pythagorean theorem:

$$a^2 = b^2 + c^2 - 2bc\cos 90°$$
$$a^2 = b^2 + c^2 - 2bc(0)$$
$$a^2 = b^2 + c^2$$

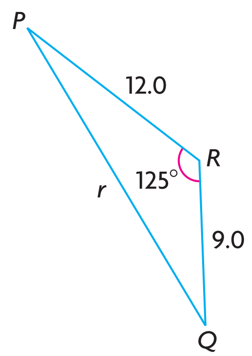
CHECK Your Understanding

1. Determine each unknown side length to the nearest tenth.

a)

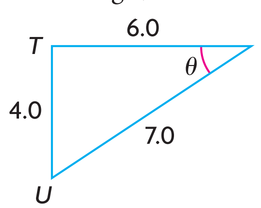


b)

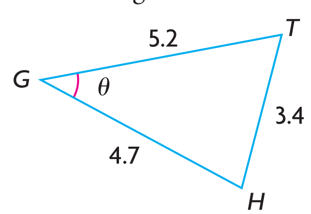


2. For each triangle, determine the value of θ to the nearest degree.

a)

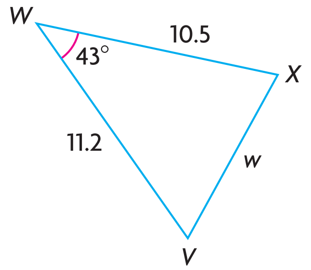


b)

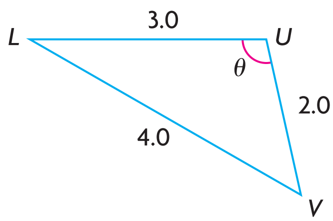


PRACTISING

3. **a)** Determine w to the nearest tenth.



b) Determine the value of θ to the nearest degree.

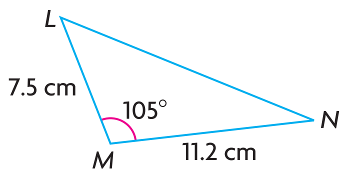


c) In $\triangle ABC$, $a = 11.5$, $b = 8.3$, and $c = 6.6$. Calculate $\angle A$ to the nearest degree.

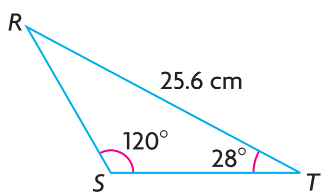
d) In $\triangle PQR$, $q = 25.1$, $r = 71.3$, and $\cos P = \frac{1}{4}$. Calculate p to the nearest tenth.

4. **K** Calculate each unknown angle to the nearest degree and each unknown length to the nearest tenth of a centimetre.

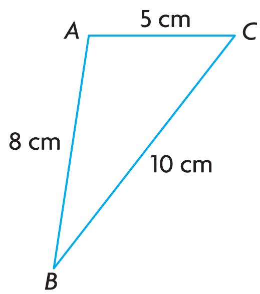
a)



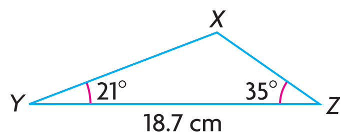
b)

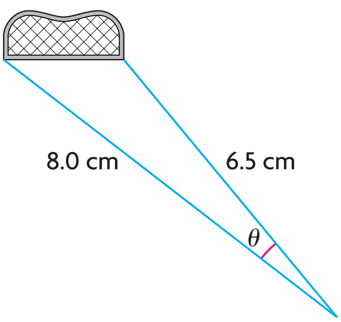


c)

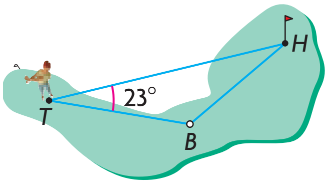


d)



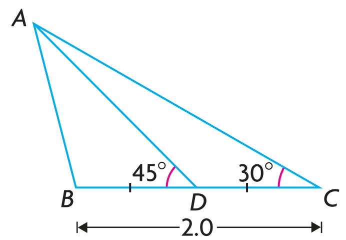


5. **A** The posts of a hockey goal are 2.0 m apart. A player attempts to score by shooting the puck along the ice from a point 6.5 m from one post and 8.0 m from the other. Within what angle θ must the shot be made? Round your answer to the nearest degree.



6. While golfing, Sahar hits a tee shot from T toward a hole at H, but the ball veers 23° and lands at B. The scorecard says that H is 270 m from T. If Sahar walks 160 m to the ball (B), how far, to the nearest metre, is the ball from the hole?

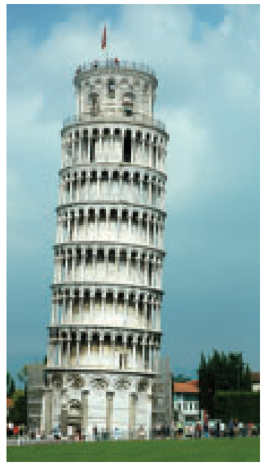
7. Given $\triangle ABC$ at the right, $BC = 2.0$ and D is the midpoint of BC. Determine AB, to the nearest tenth, if $\angle ADB = 45°$ and $\angle ACB = 30°$.



8. Two forest fire towers, A and B, are 20.3 km apart. From tower A, the bearing of tower B is 70°. The ranger in each tower observes a fire and radios the bearing of the fire from the tower. The bearing from tower A is 25° and from tower B is 345°. How far, to the nearest tenth of a kilometre, is the fire from each tower?

9. Two roads intersect at an angle of 15°. Darryl is standing on one of the roads 270 m from the intersection.
T
   a) Create a question that requires using the sine law to solve it. Include a complete solution and a sketch.
   b) Create a question that requires using the cosine law to solve it. Include a complete solution and a sketch.

10. The Leaning Tower of Pisa is 55.9 m tall and leans 5.5° from the vertical. If its shadow is 90.0 m long, what is the distance from the top of the tower to the top edge of its shadow? Assume that the ground around the tower is level. Round your answer to the nearest metre.



11. The side lengths and the interior angles of any triangle can be determined by using the cosine law, the sine law, or a combination of both. Sketch a triangle and state the minimum information required to use
C
   a) the cosine law
   b) both laws

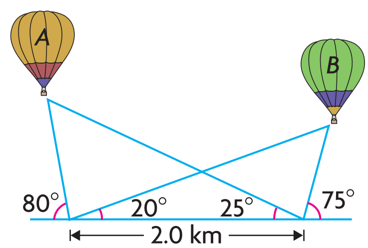
   Under each sketch, use the algebraic representation of the law to show how to determine all unknown quantities.

Extending

12. The interior angles of a triangle are 120°, 40°, and 20°. The longest side is 10 cm longer than the shortest side. Determine the perimeter of the triangle to the nearest centimetre.

13. For each situation, determine all unknown side lengths to the nearest tenth of a centimetre and/or all unknown interior angles to the nearest degree. If more than one solution is possible, state all possible answers.
   a) A triangle has exactly one angle measuring 45° and sides measuring 5.0 cm, 7.4 cm, and 10.0 cm.
   b) An isosceles triangle has at least one interior angle of 70° and at least one side of length 11.5 cm.

14. Two hot-air balloons are moored to level ground below, each at a different location. An observer at each location determines the angle of elevation to the opposite balloon as shown at the right. The observers are 2.0 km apart.
   a) What is the distance separating the balloons, to the nearest tenth of a kilometre?
   b) Determine the difference in height (above the ground) between the two balloons. Round your answer to the nearest metre.



5.8 Solving Three-Dimensional Problems by Using Trigonometry

YOU WILL NEED

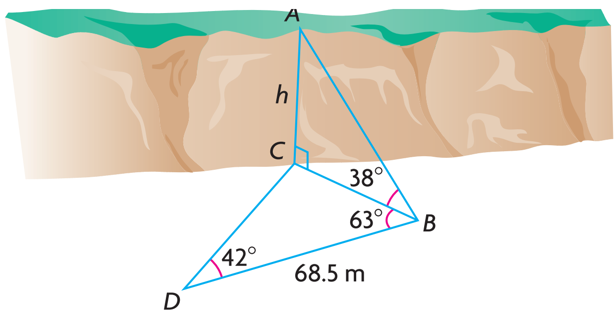
- dynamic geometry software (optional)

GOAL

Solve three-dimensional problems by using trigonometry.

LEARN ABOUT the Math

From point B, Manny uses a clinometer to determine the angle of elevation to the top of a cliff as 38°. From point D, 68.5 m away from Manny, Joe estimates the angle between the base of the cliff, himself, and Manny to be 42°, while Manny estimates the angle between the base of the cliff, himself, and his friend Joe to be 63°.



? What is the height of the cliff to the nearest tenth of a metre?

EXAMPLE 1 Solving a three-dimensional problem by using the sine law

Calculate the height of the cliff to the nearest tenth of a metre.

Matt's Solution

In $\triangle DBC$:

$$\angle C = 180° - (63° + 42°)$$
$$= 75°$$

BC is in $\triangle ABC$. In $\triangle ABC$, I don't have enough information to calculate *h*, but *BC* is also in $\triangle DBC$.

In $\triangle DBC$, I knew two angles and a side length. Before I could calculate *BC*, I needed to determine $\angle C$. I used the fact that the sum of all three interior angles is 180°.

$$\frac{BC}{\sin D} = \frac{BD}{\sin C}$$

Using $\triangle DBC$ and the value of $\angle C$, I used the sine law to calculate BC.

$$\frac{BC}{\sin 42°} = \frac{68.5}{\sin 75°}$$

$$\cancel{\sin 42°}^{1} \times \frac{BC}{\cancel{\sin 42°}_{1}} = \sin 42° \times \frac{68.5}{\sin 75°}$$

To solve for BC, I multiplied both sides of the equation by sin 42°.

$$BC = \sin 42° \times \frac{68.5}{\sin 75°}$$

I used my calculator to evaluate.

$$BC \doteq 47.45 \text{ m}$$

$$\tan 38° = \frac{h}{BC}$$

$$\tan 38° = \frac{h}{47.45}$$

Then I used $\triangle ABC$ to calculate h. I knew that $\triangle ABC$ is a right triangle and that h is opposite the 38° angle while BC is adjacent to it. So I used tangent.

$$\tan 38° \times 47.45 = \frac{h}{\cancel{47.45}_{1}} \times \cancel{47.45}^{1}$$

$$37.1 \text{ m} \doteq h$$

To evaluate h, I multiplied both sides of the equation by 47.45.

The height of the cliff is about 37.1 m.

Reflecting

A. Was the given diagram necessary to help Matt solve the problem? Explain.

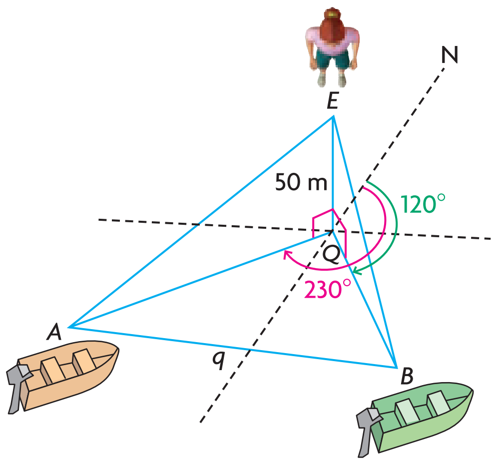
B. Why did Matt begin working with $\triangle DBC$ instead of $\triangle ABC$?

C. What strategies might Matt use to check whether his answer is reasonable?

APPLY the Math

EXAMPLE 2 Solving a three-dimensional problem by using the sine law

Emma is on a 50 m high bridge and sees two boats anchored below. From her position, boat A has a bearing of 230° and boat B has a bearing of 120°. Emma estimates the angles of depression to be 38° for boat A and 35° for boat B. How far apart are the boats to the nearest metre?



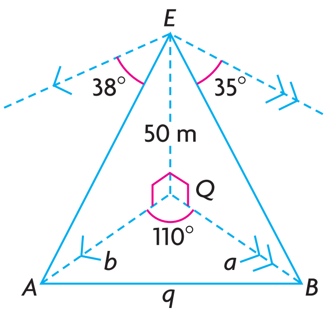
Kelly's Solution

In $\triangle AQB$:

$\angle Q = 230° - 120°$

$= 110°$

In $\triangle AQB$, I knew that the value of $\angle Q$ is equal to the difference of the bearings of boats A and B. So I subtracted 120° from 230°.

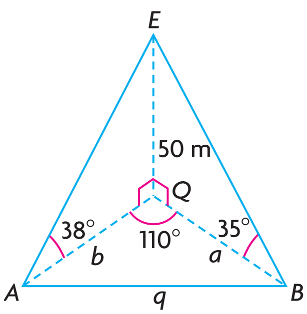


In $\triangle AQB$, I knew only one angle and no side lengths. In order to calculate q, I needed to determine AQ (side b) and BQ (side a) first.

I drew a sketch that included the angles of depression. I used these angles to determine the values of $\angle EAQ$ and $\angle EBQ$.

$\angle EAQ = 38°$ $\angle EBQ = 35°$

The angle of depression to A is measured from a line parallel to AQ. So $\angle EAQ$ is equal to 38°. Using the same reasoning, I determined that $\angle EBQ$ is equal to 35°.



I included the values of $\angle EAQ$ and $\angle EBQ$ in my sketch.

In $\triangle AEQ$:

$$\tan 38^\circ = \frac{50}{b}$$

$$b = \frac{50}{\tan 38^\circ}$$

$$b \doteq 64.0 \text{ m}$$

In $\triangle BEQ$:

$$\tan 35^\circ = \frac{50}{a}$$

$$a = \frac{50}{\tan 35^\circ}$$

$$a \doteq 71.4 \text{ m}$$

Since $\triangle AEQ$ and $\triangle BEQ$ are right triangles, I expressed AQ in terms of tan 38° and BQ in terms of tan 35°. Then I solved for b and a.

$$q^2 = b^2 + a^2 - 2ba \cos 110^\circ$$

In $\triangle AQB$, I now knew two side lengths and the angle between those sides. So I used the cosine law to calculate q.

$$q^2 = (64.0)^2 + (71.4)^2 - 2(64.0)(71.4)\cos 110^\circ$$

$$q = \sqrt{12\,320.6}$$

$$q \doteq 111 \text{ m}$$

I substituted the values of b and a into the equation and evaluated q.

The boats are about 111 m apart.

In Summary

Key Ideas

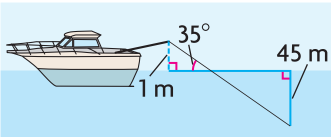
- Three-dimensional problems involving triangles can be solved using some combination of these approaches:
  - trigonometric ratios
  - the Pythagorean theorem
  - the sine law
  - the cosine law
- The approach you use depends on the given information and what you are required to find.

Need to Know

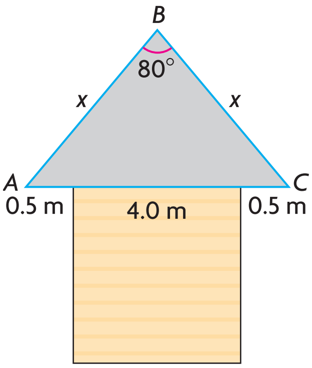
- When solving problems, always start with a sketch of the given information. Determine any unknown angles by using any geometric facts that apply, such as facts about parallel lines, interior angles in a triangle, and so on. Revise your sketch so that it includes any new information that you determined. Then use trigonometry to solve the original problem.
- In right triangles, use the primary or reciprocal trigonometric ratios.
- In all other triangles, use the sine law and/or the cosine law.

Given Information	Required to Find	Use
SSA	angle	sine law
ASA or AAS	side	sine law
SAS	side	cosine law
SSS	side	cosine law

CHECK Your Understanding



1. Morana is trolling for salmon in Lake Ontario. She sets the fishing rod so that its tip is 1 m above water and the line forms an angle of 35° with the water's surface. She knows that there are fish at a depth of 45 m. Describe the steps you would use to calculate the length of line she must let out.



2. Josh is building a garden shed that is 4.0 m wide. The two sides of the roof are equal in length and must meet at an angle of 80°. There will be a 0.5 m overhang on each side of the shed. Josh wants to determine the length of each side of the roof.

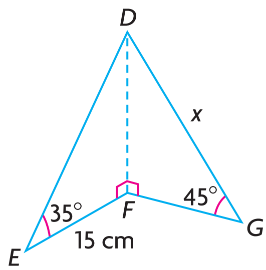
a) Should he use the sine law or the cosine law? Explain.

b) How could Josh use the primary trigonometric ratios to calculate x? Explain.

PRACTISING

3. **K** Determine the value of x to the nearest centimetre and θ to the nearest degree. Explain your reasoning for each step of your solution.

a)



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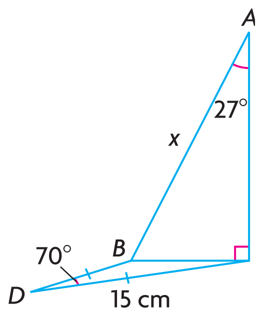

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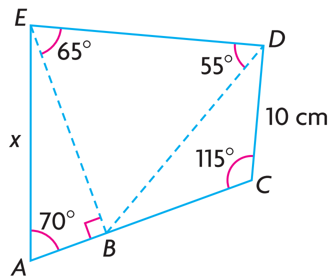

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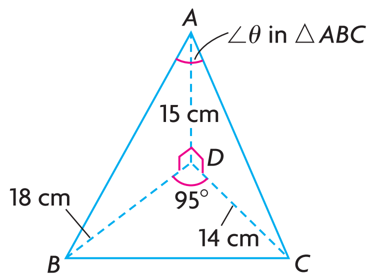
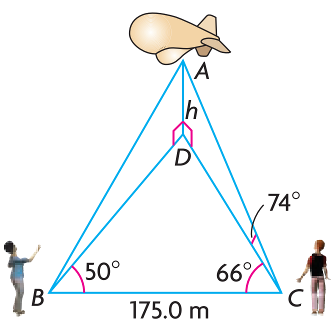




4. As a project, a group of students was asked to determine the altitude, h, of a promotional blimp. The students' measurements are shown in the sketch at the left.

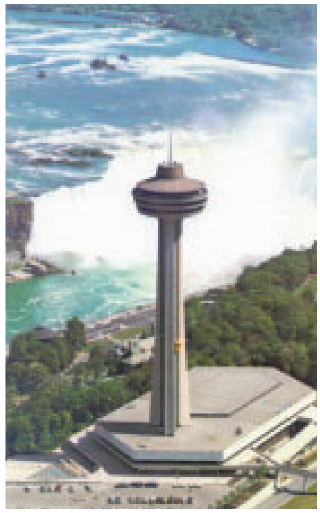
a) Determine h to the nearest tenth of a metre. Explain each of your steps.

b) Is there another way to solve this problem? Explain.

5. While Travis and Bob were flying a hot-air balloon from Beamsville to Vineland in southwestern Ontario, they decided to calculate the straight-line distance, to the nearest metre, between the two towns.

- From an altitude of 226 m, they simultaneously measured the angle of depression to Beamsville as 2° and to Vineland as 3°.
- They measured the angle between the lines of sight to the two towns as 80°.

Is there enough information to calculate the distance between the two towns? Justify your reasoning with calculations.

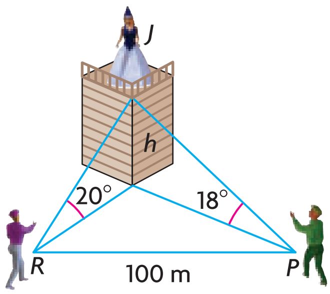


6. **A** The observation deck of the Skylon Tower in Niagara Falls, Ontario, is 166 m above the Niagara River. A tourist in the observation deck notices two boats on the water. From the tourist's position,

- the bearing of boat A is 180° at an angle of depression of 40°
- the bearing of boat B is 250° at an angle of depression of 34°

Calculate the distance between the two boats to the nearest metre.

7. Suppose Romeo is serenading Juliet while she is on her balcony. Romeo is facing north and sees the balcony at an angle of elevation of 20°. Paris, Juliet's other suitor, is observing the situation and is facing west. Paris sees the balcony at an angle of elevation of 18°. Romeo and Paris are 100 m apart as shown. Determine the height of Juliet's balcony above the ground, to the nearest metre.



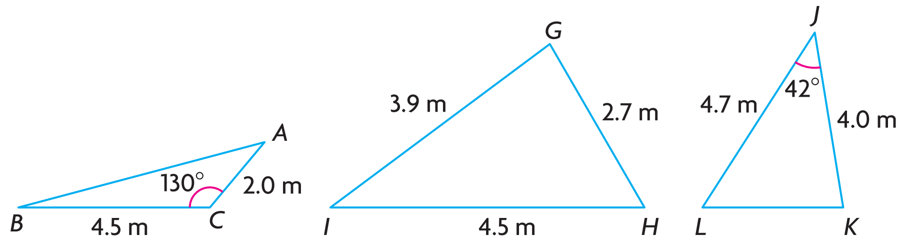
8. A coast guard helicopter hovers between an island and a damaged sailboat.

- From the island, the angle of elevation to the helicopter is 73°.
- From the helicopter, the island and the sailboat are 40° apart.
- A police rescue boat heading toward the sailboat is 800 m away from the scene of the accident. From this position, the angle between the island and the sailboat is 35°.
- At the same moment, an observer on the island notices that the sailboat and police rescue boat are 68° apart.

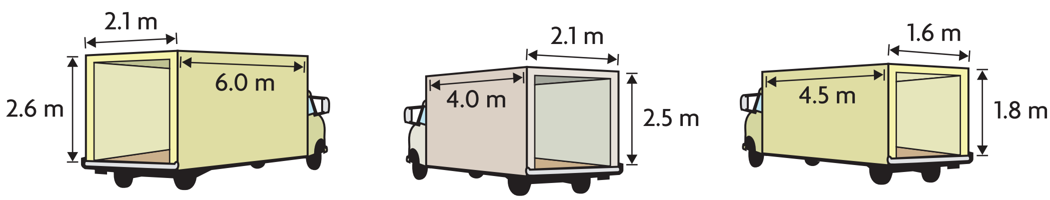
Explain how you would calculate the straight-line distance, to the nearest metre, from the helicopter to the sailboat. Justify your reasoning with calculations.

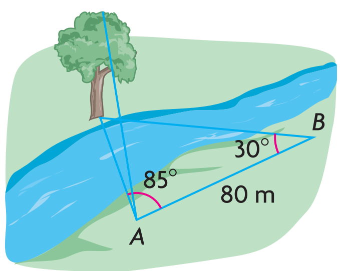
9. Brit and Tara are standing 13.5 m apart on a dock when they observe a sailboat moving parallel to the dock. When the boat is equidistant between both girls, the angle of elevation to the top of its 8.0 m mast is 51° for both observers. Describe how you would calculate the angle, to the nearest degree, between Tara and the boat as viewed from Brit's position. Justify your reasoning with calculations.

10. **T** In setting up for an outdoor concert, a stage platform has been dismantled into three triangular pieces as shown.



There are three vehicles available to transport the pieces. In order to prevent damaging the platform, each piece must fit exactly inside the vehicle. Explain how you would match each piece of the platform to the best-suited vehicle. Justify your reasoning with calculations.





11. Bert wants to calculate the height of a tree on the opposite bank of a river. To do this, he lays out a baseline 80 m long and measures the angles as shown at the left. The angle of elevation from A to the top of the tree is 28°. Explain if this information helps Bert to calculate the height of the tree to the nearest metre. Justify your reasoning with calculations.

12. **C** Chandra's homework question reads like this:
Bill and Chris live at different intersections on the same street, which runs north to south. When both of them stand at their front doors, they see a hot-air balloon toward the east at angles of elevation of 41° and 55°, respectively. Calculate the distance between the two friends.
    a) Chandra says she doesn't have enough information to answer the question. Evaluate Chandra's statement. Justify your reasoning with calculations.
    b) What additional information, if any, would you need to solve the problem? Justify your answer.

Extending

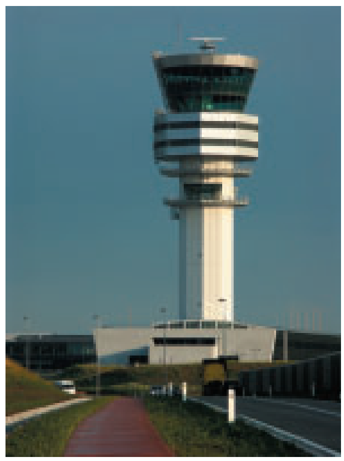
13. Two roads intersect at 34°. Two cars leave the intersection on different roads at speeds of 80 km/h and 100 km/h. After 2 h, a traffic helicopter that is above and between the two cars takes readings on them. The angle of depression to the slower car is 20°, and the straight-line distance from the helicopter to that car is 100 km. Assume that both cars are travelling at constant speed.

a) Calculate the straight-line distance, to the nearest kilometre, from the helicopter to the faster car. Explain your reasoning for each step of your solution.

b) Determine the altitude of the helicopter to the nearest kilometre.

14. Simone is facing north at the entrance of a tunnel through a mountain. She notices that a 1515 m high mountain in the distance has a bearing of 270° and its peak appears at an angle of elevation of 35°. After she exits the tunnel, the same mountain has a bearing of 258° and its peak appears at an angle of elevation of 31°. Assuming that the tunnel is perfectly level and straight, how long is it to the nearest metre?

15. An airport radar operator locates two planes flying toward the airport. The first plane, P, is 120 km from the airport, A, at a bearing of 70° and with an altitude of 2.7 km. The other plane, Q, is 180 km away on a bearing of 125° and with an altitude of 1.8 km. Calculate the distance between the two planes to the nearest tenth of a kilometre.



16. Mario is standing at ground level exactly at the corner where two exterior walls of his apartment building meet. From Mario's position, his apartment window on the north side of the building appears 44.5 m away at an angle of elevation of 55°. Mario notices that his friend Thomas's window on the west side of the building appears 71.0 m away at an angle of elevation of 34°.

a) If a rope were pulled taut from one window to the other, around the outside of the building, how long, to the nearest tenth of a metre, would the rope need to be? Explain your reasoning.

b) What is the straight-line distance through the building between the two windows? Round your answer to the nearest tenth of a metre.

5 Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 5.5, Examples 1 to 4.
- Try Chapter Review Questions 6 and 7.

Q: What steps would you follow to prove a trigonometric identity?

A: A trigonometric identity is an equation involving trigonometric ratios that is true for all values of the variable. You may rewrite the trigonometric ratios in terms of x, y, and r and then simplify, or you may rewrite each side of the equation in terms of sine and cosine and then use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, where appropriate. If a trigonometric ratio is in the denominator of a fraction, there are restrictions on the variable because the denominator cannot equal zero.

For example, the solution below is one way to prove that $\tan^2\theta + 1 = \sec^2\theta$ is an identity.

EXAMPLE

$\tan^2\theta + 1 = \sec^2\theta$

$$\begin{aligned}
\text{L.S.} &= \tan^2\theta + 1 & \qquad \text{R.S.} &= \sec^2\theta \\
&= \left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 & &= \left(\frac{1}{\cos\theta}\right)^2 \\
&= \frac{\sin^2\theta}{\cos^2\theta} + 1 & &= \frac{1}{\cos^2\theta} \\
&= \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} \\
&= \frac{1}{\cos^2\theta} \\
&= \text{R.S.}
\end{aligned}$$

First separate both sides of the equation.

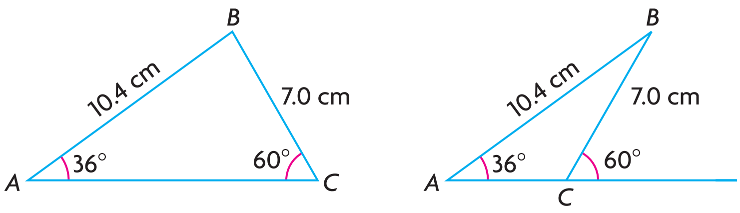
Write $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$. The left side of the equation is more complicated, so simplify it. Find a common denominator. Then use the Pythagorean identity. Since the denominator cannot equal 0, there is a restriction on θ, so $\cos\theta \neq 0$.

$\therefore \tan^2\theta + 1 = \sec^2\theta$ for all angles θ, where $\cos\theta \neq 0$.

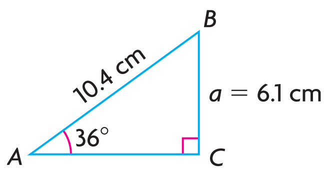
Q: How do you know when you are dealing with the ambiguous case of the sine law?

A: The ambiguous case of the sine law refers to the situation where 0, 1, or 2 triangles are possible given the information in a problem. This situation occurs when you know two side lengths and an angle (SSA).

For example, given $\triangle ABC$, where $\angle A = 36°$, $a = 7.0$ cm, and $c = 10.4$ cm, there are two possible triangles:



If $a = 6.1$ cm, then $\triangle ABC$ is a right triangle and 6.1 cm is the shortest possible length for a:



If $a < 6.1$ cm, a triangle cannot be drawn.

Study Aid

- See Lesson 5.6, Examples 1 and 2.
- Try Chapter Review Questions 8 and 9.

Q: How do you decide when to use the sine law or the cosine law to solve a problem?

A: Given any triangle, if you know two sides and the angle between those sides, or all three sides, use the cosine law. If you know an angle opposite a side, use the sine law.

Study Aid

- See Lesson 5.7, Examples 1 and 2.
- Try Chapter Review Questions 10 and 11.

Q: What approaches are helpful in solving two- and three-dimensional trigonometric problems?

A: Always start with a sketch of the given information because the sketch will help you determine whether the Pythagorean theorem, the sine law, or the cosine law is the best method to use. If you have right triangles, use the Pythagorean theorem and/or trigonometric ratios. If you know three sides or two sides and the contained angle in an oblique triangle, use the cosine law. For all other cases, use the sine law.

Study Aid

- See Lesson 5.8, Examples 1, 2, and 3.
- Try Chapter Review Questions 12 and 13.

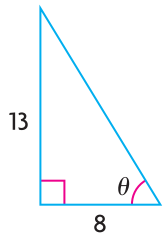
PRACTICE Questions

Lesson 5.1

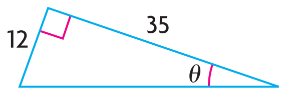
1. i) For each triangle, state the reciprocal trigonometric ratios for angle θ.

ii) Calculate the value of θ to the nearest degree.

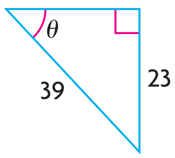
a)



b)



c)



Lesson 5.2

2. Determine the exact value of each trigonometric expression. Express your answers in simplified radical form.

a) $(\sin 45°)(\cos 45°) + (\sin 30°)(\cos 60°)$

b) $(1 - \tan 45°)(\sin 30°)(\cos 30°)(\tan 60°)$

c) $\tan 30° + 2(\sin 45°)(\cos 60°)$

Lesson 5.3

3. i) State the sign of each trigonometric ratio. Use a calculator to determine the value of each ratio.

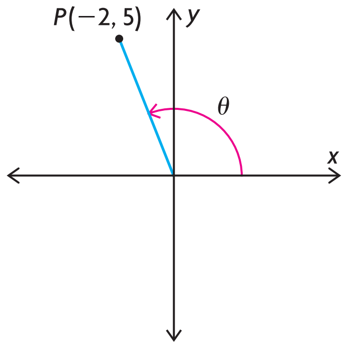
ii) For each trigonometric ratio, determine the principal angle and, where appropriate, the related acute angle. Then sketch another angle that has the equivalent ratio. Label the principal angle and the related acute angle on your sketch.

a) $\tan 18°$ **b)** $\sin 205°$ **c)** $\cos(-55°)$

Lesson 5.4

4. For each sketch, state the primary trigonometric ratios associated with angle θ. Express your answers in simplified radical form.

a)



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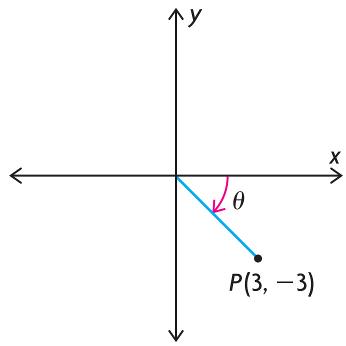

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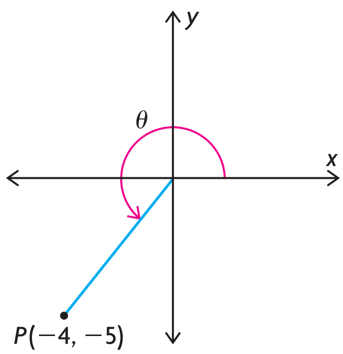


5. Given $\cos \phi = \frac{-7}{\sqrt{53}}$, where $0° \le \phi \le 360°$,

a) in which quadrant(s) does the terminal arm of angle ϕ lie? Justify your answer.

b) state the other five trigonometric ratios for angle ϕ.

c) calculate the value of the principal angle ϕ to the nearest degree.

Lesson 5.5

6. Determine whether the equation $\cos \beta \cot \beta = \frac{1}{\sin \beta} - \sin \beta$ is an identity. State any restrictions on angle β.

7. Prove each identity. State any restrictions on the variables if all angles vary from 0° to 360°.

a) $\tan \alpha \cos \alpha = \sin \alpha$

b) $\frac{1}{\cot \phi} = \sin \phi \sec \phi$

c) $1 - \cos^2 x = \frac{\sin x \cos x}{\cot x}$

d) $\sec \theta \cos \theta + \sec \theta \sin \theta = 1 + \tan \theta$

Lesson 5.6

8. Determine whether it is possible to draw a triangle given each set of information. Sketch all possible triangles where appropriate. Calculate, then label, all side lengths to the nearest tenth of a centimetre and all angles to the nearest degree.

a) $b = 3.0$ cm, $c = 5.5$ cm, $\angle B = 30°$

b) $b = 12.2$ cm, $c = 8.2$ cm, $\angle C = 34°$

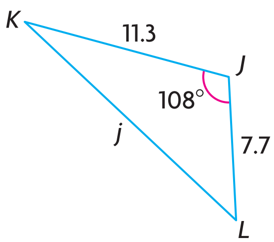
c) $a = 11.1$ cm, $c = 5.2$ cm, $\angle C = 33°$

9. Two forest fire stations, P and Q, are 20.0 km apart. A ranger at station Q sees a fire 15.0 km away. If the angle between the line PQ and the line from P to the fire is 25°, how far, to the nearest tenth of a kilometre, is station P from the fire?

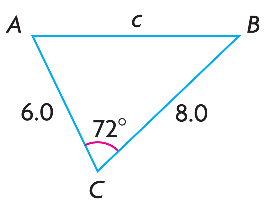
Lesson 5.7

10. Determine each unknown side length to the nearest tenth.

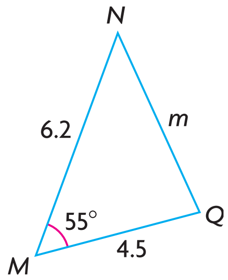
a)



b)



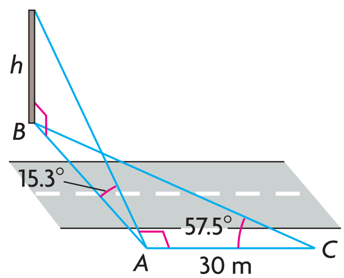
c)



11. Two spotlights, one blue and the other white, are placed 6.0 m apart on a track on the ceiling of a ballroom. A stationary observer standing on the ballroom floor notices that the angle of elevation is 45° to the blue spotlight and 70° to the white one. How high, to the nearest tenth of a metre, is the ceiling of the ballroom?

Lesson 5.8

12. To determine the height of a pole across a road, Justin takes two measurements. He stands at point A directly across from the base of the pole and determines that the angle of elevation to the top of the pole is 15.3°. He then walks 30 m parallel to the freeway to point C, where he sees that the base of the pole and point A are 57.5° apart. From point A, the base of the pole and point C are 90.0° apart. Calculate the height of the pole to the nearest metre.



13. While standing at the left corner of the schoolyard in front of her school, Suzie estimates that the front face is 8.9 m wide and 4.7 m high. From her position, Suzie is 12.0 m from the base of the right exterior wall. She determines that the left and right exterior walls appear to be 39° apart. From her position, what is the angle of elevation, to the nearest degree, to the top of the left exterior wall?

5 Chapter Self-Test

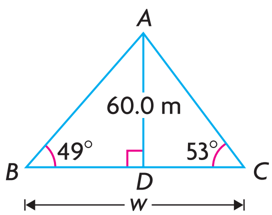
1. **i)** For each point, sketch the angle in standard position to determine all six trigonometric ratios.
ii) Determine the value of the principal angle and the related acute angle, where appropriate, to the nearest degree.
a) $P(-3, 0)$ **b)** $S(-8, -6)$

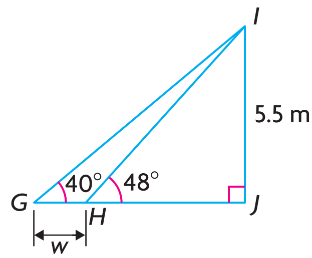
2. Given angle θ, where $0° \leq \theta \leq 360°$, determine all possible angles for θ.
a) $\sin \theta = -\frac{1}{2}$
b) $\cos \theta = \frac{\sqrt{3}}{2}$
c) $\cot \theta = -1$
d) $\sec \theta = -2$

3. Given $\cos \theta = -\frac{5}{13}$, where the terminal arm of angle θ lies in quadrant 2, evaluate each trigonometric expression.
a) $\sin \theta \cos \theta$ **b)** $\cot \theta \tan \theta$

4. **i)** Prove each identity. Use a different method for parts (a) and (b). State any restrictions on the variables.
ii) Explain why these identities are called Pythagorean identities.
a) $\tan^2 \phi + 1 = \sec^2 \phi$ **b)** $1 + \cot^2 \alpha = \csc^2 \alpha$

5. **a)** Sketch a triangle of your own choice and label the sides and angles.
b) State all forms of the cosine law that apply to your triangle.
c) State all forms of the sine law that apply to your triangle.

6. For each triangle, calculate the value of w to the nearest tenth of a metre.
a)


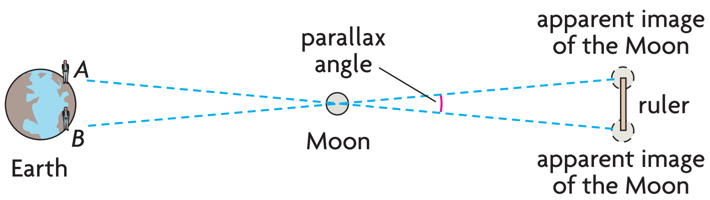
b)


7. Given each set of information, determine how many triangles can be drawn. Calculate, then label, all side lengths to the nearest tenth and all interior angles to the nearest degree, where appropriate.
a) $a = 1.5$ cm, $b = 2.8$ cm, and $\angle A = 41°$
b) $a = 2.1$ cm, $c = 6.1$ cm, and $\angle A = 20°$

8. To estimate the amount of usable lumber in a tree, Chitra must first estimate the height of the tree. From points A and B on the ground, she determined that the angles of elevation for a certain tree were 41° and 52°, respectively. The angle formed at the base of the tree between points A and B is 90°, and A and B are 30 m apart. If the tree is perpendicular to the ground, what is its height to the nearest metre?

Parallax

Parallax is the apparent displacement of an object when it is viewed from two different positions.



Astronomers measure the parallax of celestial bodies to determine how far those bodies are from Earth.

On October 28, 2004, three astronomers (Peter Cleary, Pete Lawrence, and Gerardo Addiègo) each at a different location on Earth, took a digital photo of the Moon during a lunar eclipse at exactly the same time. The data related to these photos is shown.

	Shortest Distance on Earth's Surface Between Two Locations	Parallax Angle
AB (Montréal, Canada to Selsey, UK)	5 220 km	0.7153°
AC (Montréal, Canada to Montevideo, Uruguay)	9 121 km	1.189°
BC (Selsey, UK to Montevideo, Uruguay)	10 967 km	1.384°

? What is the most accurate method to determine the distance between the Moon and Earth, from the given data?

A. Sketch a triangle with the Moon and locations *A* and *B* as the vertices. Label all the given angles and distances. What kind of triangle do you have?

B. Determine all unknown sides to the nearest kilometre and angles to the nearest thousandth of a degree. How far, to the nearest kilometre, is the Moon from either Montréal or Selsey?

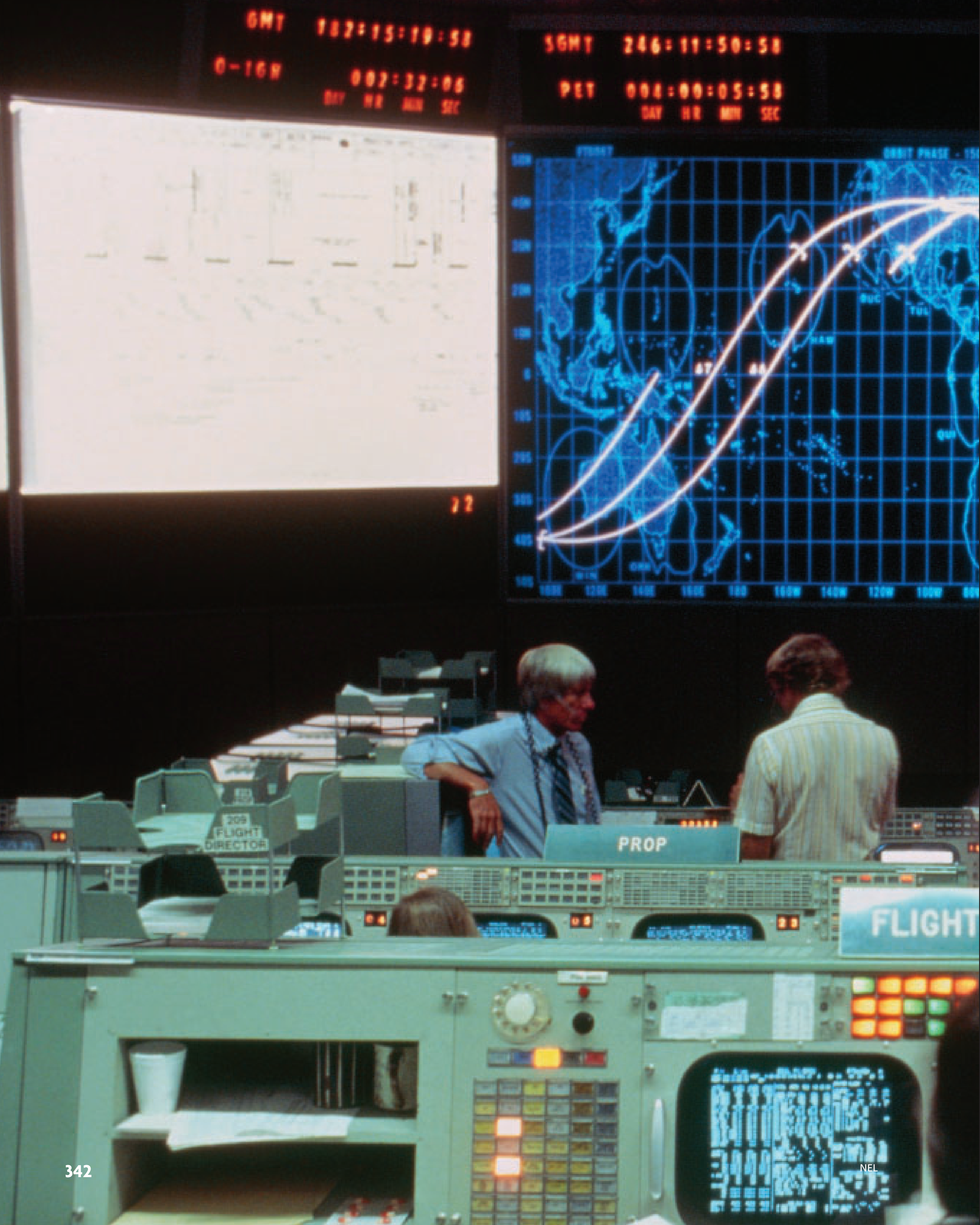
C. Repeat parts A and B for locations *B* and *C*, and for *A* and *C*.

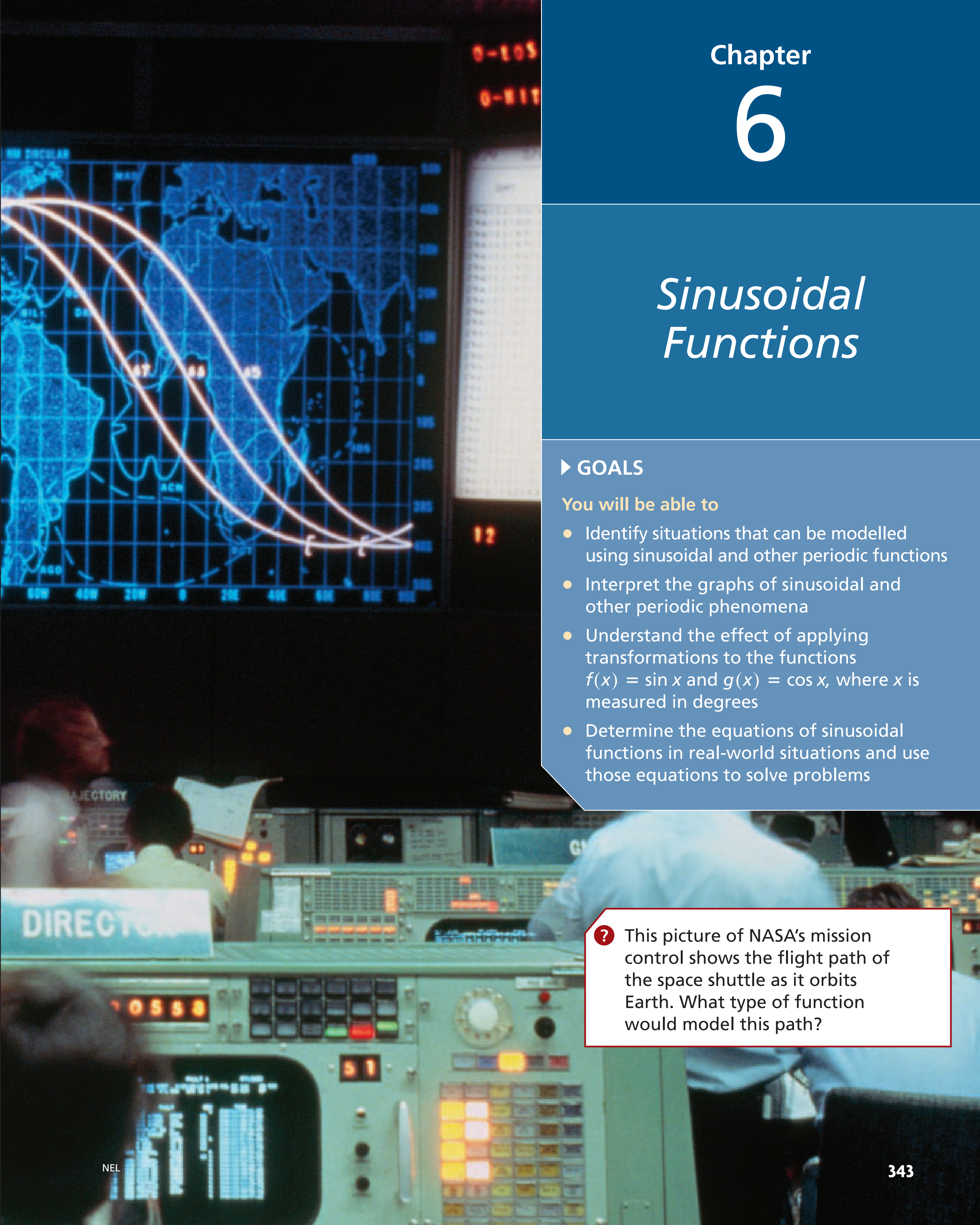
D. On October 28, 2004, the Moon was about 391 811 km from Earth (surface to surface). Calculate the relative error, to the nearest tenth of a percent, for all three distances you calculated.

E. Which of your results is most accurate? What factors contribute most to the error in this experiment?

Task *Checklist*

- ✔ Did you draw the correct sketches?
- ✔ Did you show your work?
- ✔ Did you provide appropriate reasoning?
- ✔ Did you explain your thinking clearly?


GMT
SGMT 246:11:50:58
PET
DAY HR MIN SEC
209
FLIGHT
DIRECTOR
PROP
FLIGHT



Chapter 6

Sinusoidal Functions

GOALS

You will be able to

- Identify situations that can be modelled using sinusoidal and other periodic functions
- Interpret the graphs of sinusoidal and other periodic phenomena
- Understand the effect of applying transformations to the functions $f(x) = \sin x$ and $g(x) = \cos x$, where x is measured in degrees
- Determine the equations of sinusoidal functions in real-world situations and use those equations to solve problems

? This picture of NASA's mission control shows the flight path of the space shuttle as it orbits Earth. What type of function would model this path?

6 Getting Started

Study | Aid

- For help, see Essential Skills Appendix.

Question	Appendix
3, 4, 5	A-16
6, 7	A-14

SKILLS YOU NEED

1. Marcus sells 100 T-shirts per week at a price of \$30 per shirt. A survey indicates that if he reduces the price of each shirt by \$2, he will sell 20 more shirts per week. If x represents the number of times the price is reduced by \$2, then the revenue generated from T-shirt sales can be modelled by the function $R(x) = (30 - 2x)(100 + 20x)$.

a) Explain what the factors $(30 - 2x)$ and $(100 + 20x)$ represent in $R(x)$.

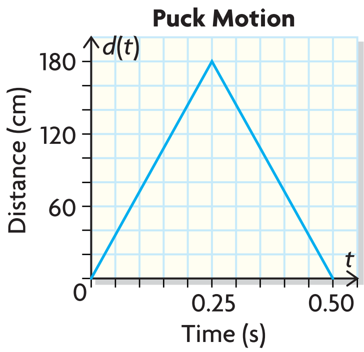
b) How many times will the price have to be dropped for the total revenue to be 0?

c) How many times will the price have to be dropped to reach the maximum revenue?

d) What is the maximum revenue?

e) What price will the T-shirts sell for to obtain the maximum revenue?

f) How many T-shirts will be sold to obtain the maximum revenue?



2. An air hockey puck is shot to the opposite end of the table and ricochets back. The puck's distance in centimetres from where it was shot in terms of time in seconds can be modelled by the graph shown at the left.

a) How far did the puck travel?

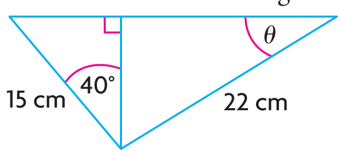
b) When was the puck farthest away from where it was shot?

c) How fast was the puck travelling in the first 0.25 s?

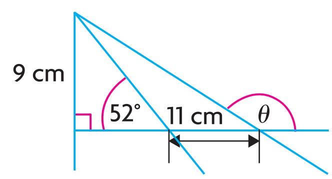
d) State the domain and range of the function.

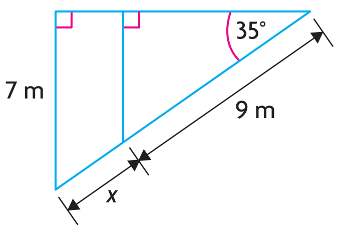
3. Determine θ to the nearest degree.

a)



b)



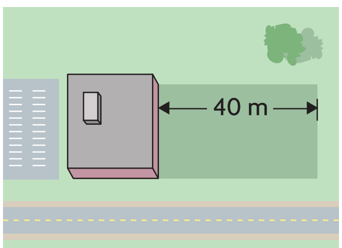


4. Determine the value of x in the triangle at the left to the nearest tenth of a metre.

5. Use transformations of the graph $f(x) = 2^x$ to sketch the graphs of the following:

a) $y = -f(x)$ **c)** $y = f(x) + 4$

b) $y = 3f(x)$ **d)** $y = -2f(x - 3)$



6. An aerial photograph shows that a building casts a shadow 40 m long when the angle of elevation of the Sun was 32°. How tall is the building?

7. List all the different types of transformations that you know. For each one, describe how a graph of $f(x) = x^2$ would change if the transformation is applied to it.

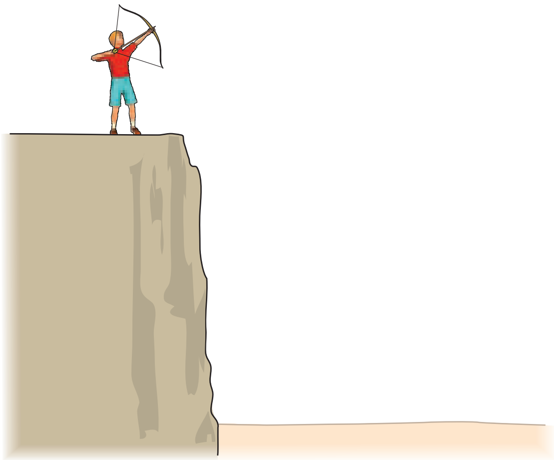
APPLYING What You Know

Flying Arrows

An arrow is shot into the air from the edge of a cliff. The height of the arrow above the ground is a function of time and can be modelled by

$$h(t) = -5t^2 + 20t + 25,$$

where the height, $h(t)$, is measured in metres at time, t, measured in seconds.



? How can you describe the flight of the arrow using this function?

A. What is the initial height of the arrow?

B. Calculate $h(2)$. Explain what this value represents in this situation.

C. When will the arrow strike the ground?

D. When will the arrow reach its maximum height?

E. What is the maximum height reached by the arrow?

F. State the domain and range of the function in this situation.

G. Summarize what you determined about the relationship between the height of the arrow and time.

6.1 Periodic Functions and Their Properties

YOU WILL NEED

- graph paper

GOAL

Interpret and describe graphs that repeat at regular intervals.

LEARN ABOUT the Math

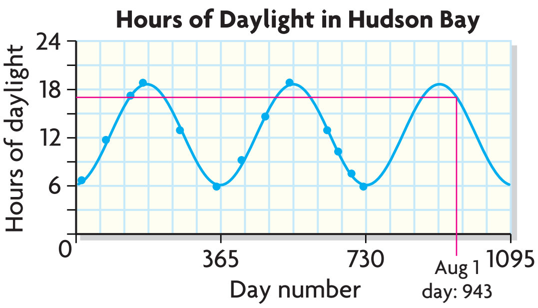
The number of hours of daylight at any particular location changes with the time of year. The table shows the average number of hours of daylight for approximately a two-year period at Hudson Bay, Nunavut. *Note*: Day 15 is January 15 of year 1. Day 74 is March 15 of year 1. Day 411 is February 15 of year 2.

Day	15	74	135	166	258	349	411	470	531	561	623	653	684	714
Hours of Daylight	6.7	11.7	17.2	18.8	12.9	5.9	9.2	14.6	18.8	18.1	12.9	10.2	7.5	5.9

? How many hours of daylight will there be on August 1 of year 3?

EXAMPLE 1 Representing data in a graph to make predictions

Jacob's Solution



I drew a scatter plot with the day as the independent variable and the hours of daylight as the dependent variable. I drew a smooth curve to connect the points.

The data and graph repeat every 365 days. I can tell because the greatest number of hours of daylight occurs on days 166 and 531, and $531 - 166 = 365$.

The least number of hours of daylight occurs on days 349 and 714, also 365 days apart.

I used the pattern to extend the graph to year 3. That would be 1095 days.

The number of hours of daylight for day 943 is about 17 h.

I used the graph to estimate the number of hours of daylight for day 943.

Reflecting

A. Why does it make sense to call the graph of the hours of daylight a **periodic function**?

B. How does the table help you predict the **period** of the graph?

C. Which points on the graph could you use to determine the range of this function?

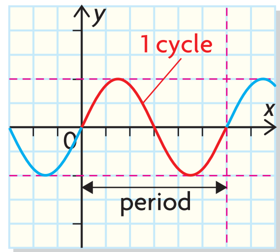
D. How does knowing the period of a periodic graph help you predict future events?

periodic function

a function whose graph repeats at regular intervals; the *y*-values in the table of values show a repetitive pattern when the *x*-values change by the same increment

period

the change in the independent variable (typically *x*) corresponding to one cycle; a cycle of a periodic function is a portion of the graph that repeats

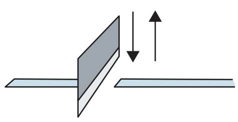


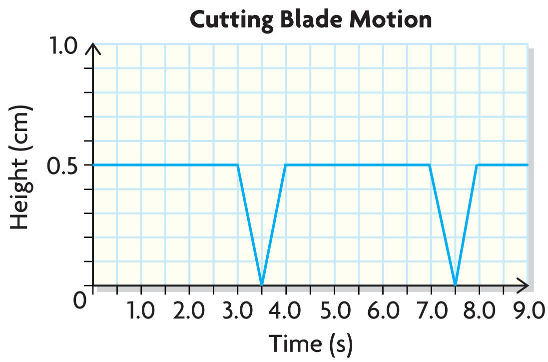
APPLY the Math

EXAMPLE 2 Interpreting periodic graphs and connecting them to real-world situations

Part A: Analyzing a Cutting Blade's Motion

Tanya's mother works in a factory that produces tape measures. One day, Tanya and her brother Norman accompany their mother to work. During manufacturing, a metal strip is cut into 6 m lengths and is coiled within the tape measure holder. A cutting machine chops the strips into their appropriate lengths. Tanya's mother shows a graph that models the motion of the cutting blade on the machine in terms of time.





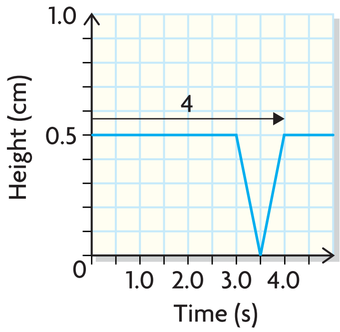
How can Norman interpret the graph and relate its characteristics to the manufacturing process?

Norman's Solution

This is a periodic function. ← It's a periodic function because the graph repeats in exactly the same way at regular intervals.

The period of this function is 4 s.

The cutting blade cuts a new section of metal strip every 4 s because the graph has a pattern that repeats every 4 s.



The maximum height of the blade is 0.5 cm. The minimum height is 0 cm.

The y-value is always 0.5 cm or less, so the blade can't be higher than this. When the height is 0 cm, the blade is hitting the cutting surface.

The blade stops for 3 s intervals.

Flat sections, like the ones from 0 to 3.0 and 4.0 to 7.0, must mean that the blade stops for these intervals. The machine is probably pulling the next 6 m section of metal strip through before it's cut.

The blade takes 1 s to go up and down.

Parts of the graph, like from $t = 3.0$ to $t = 3.5$, show that the blade takes 0.5 s to go down.

Other parts of the graph, like from $t = 3.5$ to $t = 4.0$, show that the blade takes 0.5 s to go up.

The blade will strike the cutting surface again at 11.5 s and every 4 s after that.

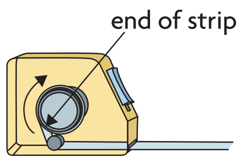
Since the graph repeats every 4 s and the blade hits the surface at 3.5 s and 7.5 s, I can figure out the next time it will hit the surface.

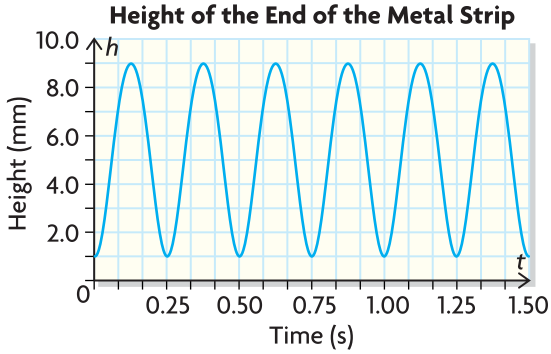
Part B: Analyzing the Motion of the Tape as It Is Spooled

Farther down the assembly line, the metal strip is raised and spooled onto a rotating cylinder contained within the tape measure.

Tanya notices that the height of the end of the metal strip that attaches to the spool goes up and down as the rest of the strip is pulled onto the cylinder.

Tanya's mother shows them a graph that models the height of the end of the strip in terms of time.





How can Tanya interpret the graph and relate its characteristics to the manufacturing process?

Tanya's Solution

This is a periodic function.

It's a periodic function because the graph repeats in exactly the same way at regular intervals. This time the action is smooth.

The range for this function is $\{h \in \mathbf{R} \mid 1 \leq h \leq 9\}$.

The highest the graph goes is 9 mm, and the lowest is 1 mm. The heights are always at or between these two values.

The period of this function is 0.25 s.

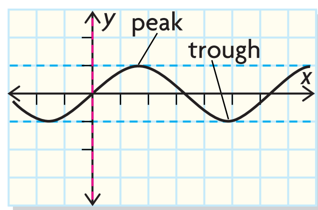
The first **trough** is at $t = 0$. The next trough is at $t = 0.25$. The distance between the two troughs gives the period.

I could also have measured the distance between the first two **peaks** to get that value.

The period represents the time it takes for the rotating cylinder to make one complete revolution.

trough
the minimum point on a graph

peak
the maximum point on a graph



equation of the axis

the equation of the horizontal line halfway between the maximum and the minimum; it is determined by

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$

amplitude

half the difference between the maximum and minimum values; it is also the vertical distance from the function's axis to the maximum or minimum value

$$\frac{9 + 1}{2} = 5$$

I calculated the halfway point between the maximum and minimum values of the graph, giving me the **equation of the axis**.

The equation of the axis for this function is $h = 5$.

$$9 - 5 = 4$$

The **amplitude** of a function is the vertical distance from its axis ($h = 5$) to its maximum value (9 mm).

The amplitude of this function is 4 mm.

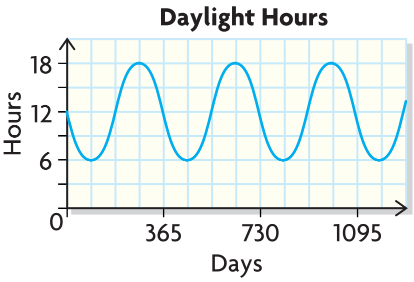
EXAMPLE 3 Identifying a periodic function from its graph

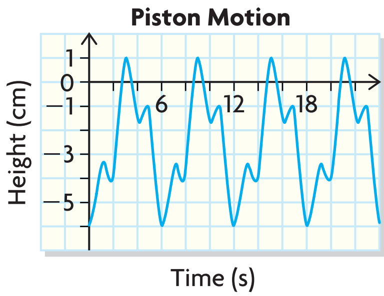
Determine whether the term *periodic* can be used to describe the graph for each situation. If so, state the period, equation of the axis, and amplitude.

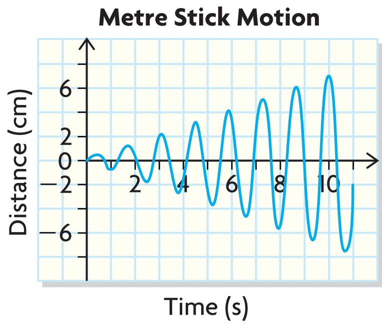
a) the average number of hours of daylight over a three-year period

b) the motion of a piston on an automated assembly line

c) a student is moving a metre stick back and forth with progressively larger movements







Tina's Solution

a) periodic

The graph looks like a series of waves that are the same size and shape. The waves repeat at regular intervals, so the function is periodic.

period = 1 year

The graph repeats its pattern every 365 days. That is the period of the function.

$$\frac{18 + 6}{2} = 12$$

equation of the axis: $h = 12$

To get the equation of the axis, I calculated the halfway point between the maximum and minimum values of the height.

$$18 - 12 = 6$$

amplitude = 6 h

The amplitude is the vertical distance from its axis ($h = 12$) to the maximum value (18 h) or minimum value (6 h).

b) periodic

The shape of the graph repeats over the same interval, so the function is periodic.

period = 6 s

The graph repeats every 6 s, so that's the period of the function.

$$\frac{1 + (-6)}{2} = -2.5$$

The equation of the axis is halfway between the maximum of 1 and the minimum of −6.

equation of the axis: $h = -2.5$

$$1 - (-2.5) = 3.5$$

The distance between the maximum and the axis is 3.5.

amplitude = 3.5 cm

c) nonperiodic

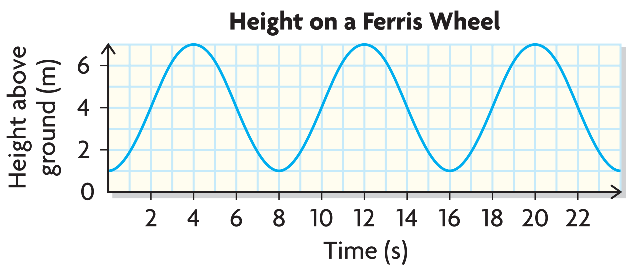
The shape of the graph does not repeat over the same interval, so the function is not periodic.

This means that the function does not have a period, amplitude, or equation of the axis.

In Summary

Key Ideas

- A function that produces a graph that has a regular repeating pattern over a constant interval is called a periodic function. It describes something that happens in a cycle, repeating in the same way over and over.



- A function that produces a graph that does not have a regular repeating pattern over a constant interval is called a nonperiodic function.

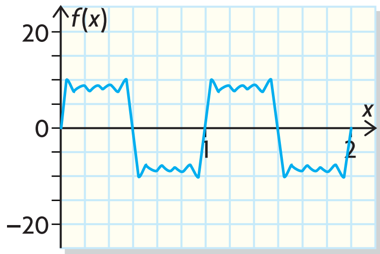
Need to Know

- Extending the graph of a periodic function by using the repeating pattern allows you to make reasonable predictions by extrapolating.
- The graph of a periodic function permits you to figure out the key features of the repeating pattern it represents, such as the period, amplitude, and equation of the axis.

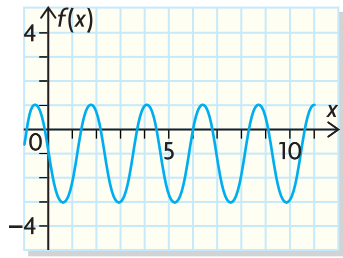
CHECK Your Understanding

1. Which of the following graphs are periodic? Explain why or why not.

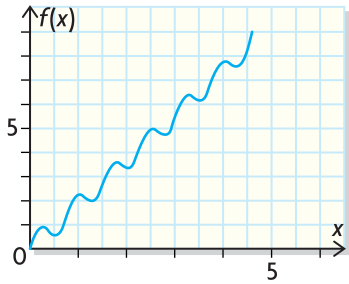
a)



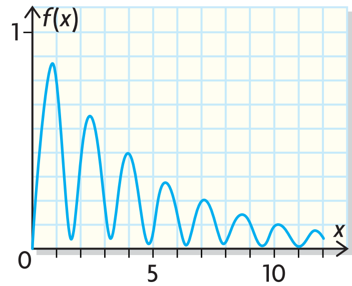
c)



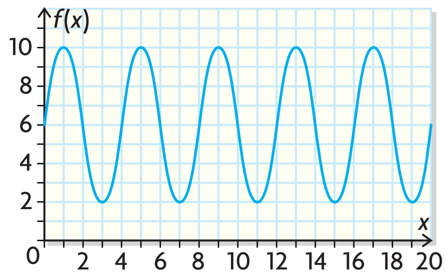
b)

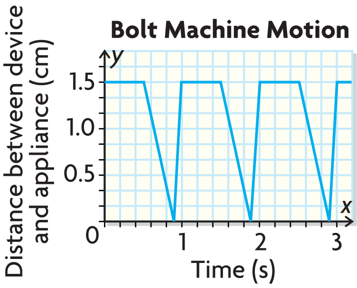


d)



2. Determine the range, period, equation of the axis, and amplitude of the function shown.





3. The motion of an automated device for attaching bolts to a household appliance on an assembly line can be modelled by the graph shown at the left.

a) What is the period of one complete cycle?

b) What is the maximum distance between the device and the appliance?

c) What is the range of this function?

d) If the device can run for five complete cycles only before it must be turned off, determine the domain of the function.

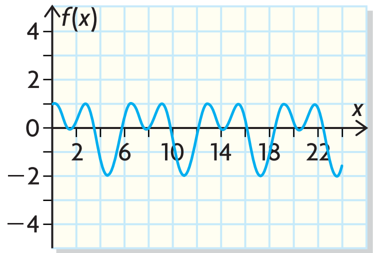
e) Determine the equation of the axis.

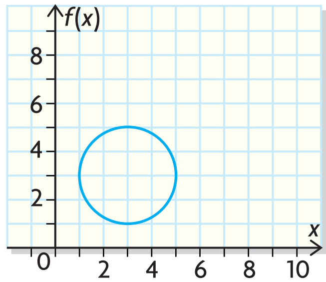
f) Determine the amplitude.

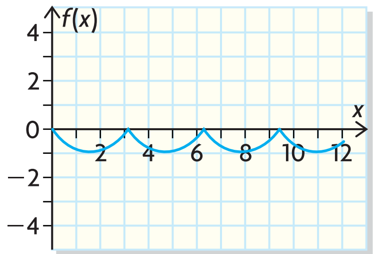
g) There are several parts to each complete cycle of the graph. Explain what each part could mean in the context of "attaching the bolt."

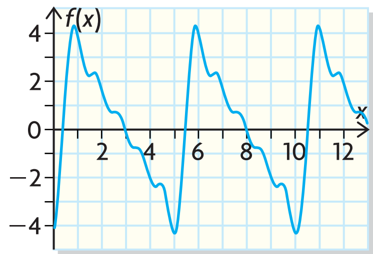
PRACTISING

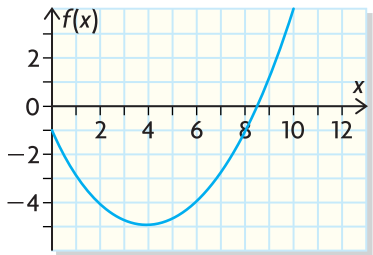
4. Identify which graphs are periodic. Estimate the period of the functions that
K you identify as periodic.

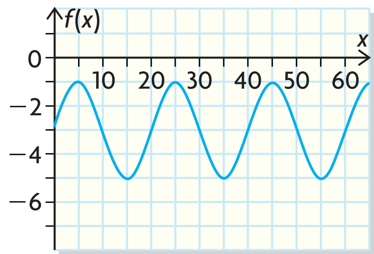
a)


d)


b)


e)


c)


f)


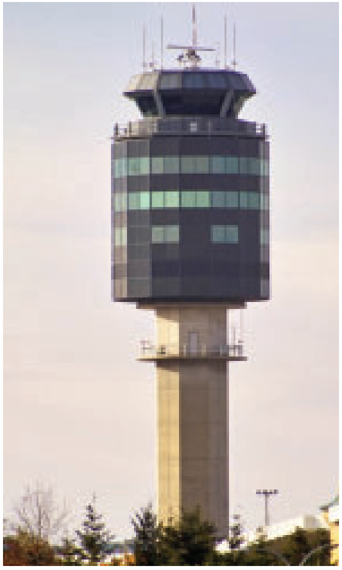
5. Which of the following situations would produce periodic graphs?

a) Sasha is monitoring the height of one of the cutting teeth on a chainsaw. The saw is on the ground, and the chain is spinning.
- independent variable: time
- dependent variable: height of tooth above the ground

b) Alex is doing jumping jacks.
- independent variable: time
- dependent variable: Alex's height above the ground

c) The cost of riding in a taxi varies, depending on how far you travel.
- independent variable: distance travelled
- dependent variable: cost

d) Brittany invested her money in a Guaranteed Investment Certificate whose return was 4% per year.
- independent variable: time
- dependent variable: value of the certificate



e) You throw a basketball to a friend, but she is so far away that the ball bounces on the ground four times.
   - independent variable: distance
   - dependent variable: bounce height

f) The antenna on a radar tower is rotating and emitting a signal to track incoming planes.
   - independent variable: time
   - dependent variable: intensity of the signal

6. Which of the tables of values might represent periodic functions? Justify.

a)

x	y
−5	9
−4	4
−3	1
−2	0
−1	1
0	4
1	9
2	16

b)

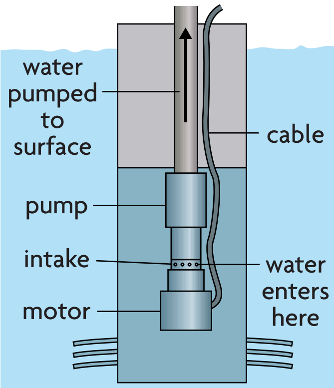
x	y
0.7	5
0.9	6
1.1	7
1.3	5
1.5	6
1.7	7
1.9	5
2.1	6

c)

x	y
23	−6
26	−6.5
29	−7
32	−7.5
35	−8
38	−8.5
41	−9
44	−9.5

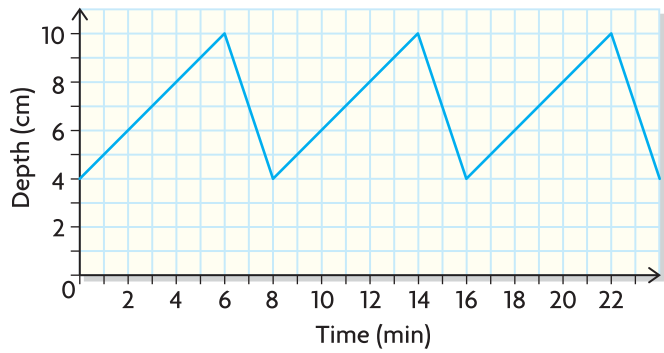
d)

x	y
1	5
2	6
4	5
7	6
11	5
16	6
22	5
29	6



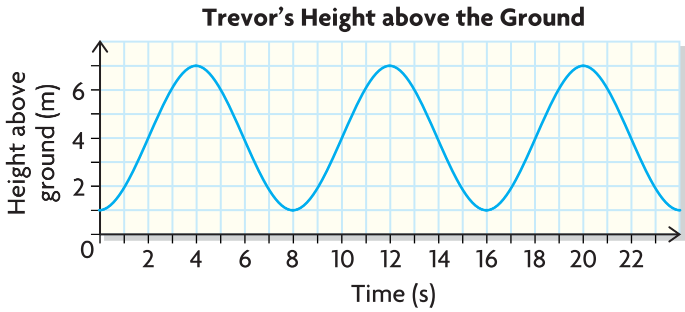
7. Chantelle has a submersible pump in her basement. During a heavy rain, the pump turned off and on to drain water collecting under her house's foundation. The graph models the depth of the water below her basement floor in terms of time. The depth of the water decreased when the pump was on and increased when the pump was off.

Depth of Water under Basement Floor



a) Is the function periodic?
b) At what depth does the pump turn on?
c) How long does the pump remain on?
d) What is the period of the function? Include the units of measure.
e) What is the range of the function?
f) What will the depth of the water be at 3 min?
g) When will the depth of the water be 10 cm?
h) What will the depth of the water be at 62 min?

8. **A** While riding on a Ferris wheel, Trevor's height above the ground in terms of time can be represented by the graph shown.



a) What is the period of this function, and what does it represent?
b) What is the equation of the axis?
c) What is the amplitude?
d) What is the range of the function?
e) After 24 s, when will Trevor be at the lowest height again?
f) At what times is Trevor at the top of the wheel?
g) When will his height be 4 m between 24 s and 30 s?

9. Sketch the graph of a periodic function with a period of 20, an amplitude of 6, and whose equation of the axis is $y = 7$.

10. Sketch the graph of a periodic function whose period is 4 and whose range is $\{y \in \mathbf{R} \mid -2 \leq y \leq 5\}$.

11. **T** Maria's bicycle wheel has a diameter of 64 cm. As she rides at a speed of 21.6 km/h, she picks up a stone in her tire. Draw a graph that shows the stone's height above the ground as she continues to ride at this speed for 2 s more.

12. A spacecraft is in an elliptical orbit around Earth. The spacecraft's distance above Earth's surface in terms of time is recorded in the table.

Time (min)	0	6	12	18	24	30	36	42	48	54	60	66	72	78
Distance (km)	550	869	1000	869	550	232	100	232	550	869	1000	869	550	232

a) Plot the data, and draw the resulting curve.
b) Is the graph periodic?
c) What is the period of the function, and what does it represent?
d) What is the approximate distance between the spacecraft and Earth at 8 min?
e) At what times is the spacecraft farthest from Earth?
f) If the spacecraft completes only six orbits before descending to Earth, what is the domain of the function?

13. Water is stored in a cylindrical container. Sometimes water is removed from the container, and other times water is added. The table records the depth of the water at specific times.

Time (min)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Depth (cm)	10	20	30	40	40	40	25	10	20	30	40	40	40	25	10	20	30	40	40	40

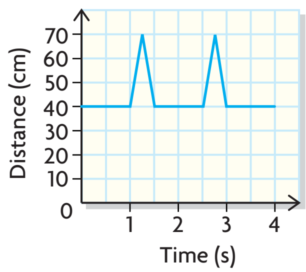
a) Plot the data, and draw the resulting curve.
b) Is the graph periodic?
c) Determine the period, the equation of the axis, and the amplitude of the function.
d) How fast is the depth of the water increasing when the container is being filled?
e) How fast is the depth of the water decreasing when the container is being drained?
f) Is the container ever empty? Explain.

14. Write a definition of a periodic function. Include an example, and use your definition to explain why it is periodic.
C

Extension

15. A Calculator-Based Ranger (CBR) is a motion detector that can attach to a graphing calculator. When the CBR is activated, it records the distance an object is in front of the detector in terms of time. The data are stored in the calculator. A scatter plot based on those recorded distances and times can then be drawn using the graphing calculator. Distance is the dependent variable, and time is the independent variable. Denis holds the paddle of the CBR at 60 cm for 3 s and then, within 0.5 s, moves the paddle so that it is 30 cm from the detector. He holds the paddle there for 2 s and then, within 0.5 s, moves the paddle back to the 60 cm location. Denis repeats this process three times.
a) Draw a sketch of the resulting graph. Include a scale.
b) What is the period of the function?
c) Determine the range and domain of the function.

16. Describe the motion of the paddle in front of a CBR that would have produced the graph shown.



6.2 Investigating the Properties of Sinusoidal Functions

YOU WILL NEED

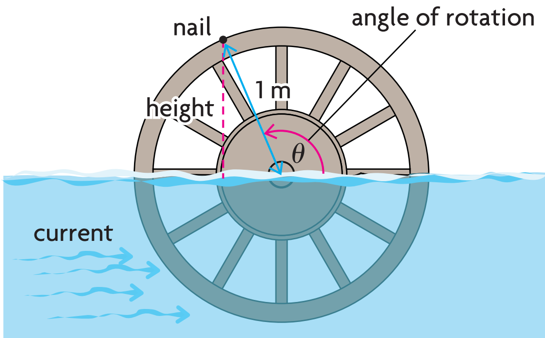
- cardboard
- ruler
- protractor
- metre stick
- thumbtack
- graphing calculator

GOAL

Examine the two functions that are associated with all sinusoidal functions.

INVESTIGATE the Math

Paul uses a generator powered by a water wheel to produce electricity. Half the water wheel is submerged below the surface of a river. The wheel has a radius of 1 m. A nail on the circumference of the wheel starts at water level. As the current flows down the river, the wheel rotates counterclockwise to power the generator. The height of the nail changes as the wheel rotates.

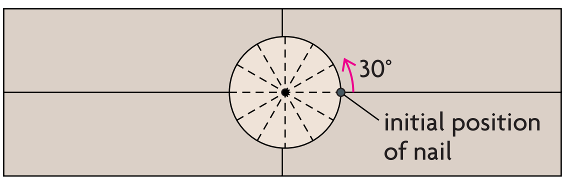


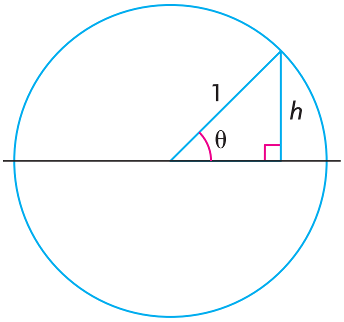
? How can you describe the position of the nail using an equation?

A. Construct a scale model of the water wheel. On a piece of cardboard, cut out a circle with a radius of 10 cm to represent the water wheel's 1 m radius.

B. Locate the centre of the circle. Use a protractor to divide your cardboard wheel into 30° increments through the centre. Draw a dot to represent the nail on the circumference of the circle at one of the lines you drew to divide the wheel.

C. On a rectangular piece of cardboard about 100 cm long and 30 cm wide, draw a horizontal line to represent the water level and a vertical line both through the centre. Attach the cardboard wheel to the centre of the rectangular piece of cardboard with a thumbtack, with the rectangle behind the wheel.



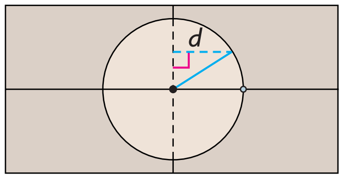


D. Rotate the cardboard wheel 30° counterclockwise. Measure the height, *h*, of the nail: the perpendicular distance from the nail to the horizontal line. Copy the table, and record the *actual* distance the nail is above the horizontal line at 30° by multiplying the scale height by 10 and converting to metres. Continue to rotate the wheel in 30° increments, measuring *h* and recording the actual heights. If the nail goes below the horizontal line, record the height as a negative value. Continue until the nail has rotated 720°.

Angle of Rotation, θ (°)	0	30	60	90	120	• • •	690	720
Actual Height of Nail, *h* (m)	0			1				

E. Use your data to graph height versus angle of rotation.

F. Use your model of the water wheel to examine the horizontal distance, *d*, the nail is from a vertical line that passes through the centre of the water wheel. Start with the nail initially positioned at water level.



Rotate the cardboard wheel 30° counterclockwise, and measure the distance the nail is from the vertical line. Copy the table, and record the *actual* distance the nail is from the vertical line at 30°, again adjusting for the scale factor. Continue to rotate the wheel in 30° increments, and record the actual distances. If the nail goes to the left of the vertical line, record the distance as a negative value. Continue until the nail has rotated 720°.

Angle of Rotation, θ (°)	0	30	60	90	120	• • •	690	720
Actual Distance from Vertical Line, *d* (m)	0			1				

G. Use your data to graph horizontal distance versus angle of rotation.

H. Use your graphing calculator to determine the cosine and sine of each rotation angle. Make sure your calculator is in DEGREE mode and evaluate to the nearest hundredth.

Tech | *Support*

You can generate the tables using the List feature on your graphing calculator. Try putting degrees in L1, replacing L2 with "cos(L1)" and L3 with "sin(L1)."

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos θ													
θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin θ													

I. Based on the tables you created in parts D, F, and H, select the appropriate equation that describes the height, h, of the nail on the water wheel in terms of the rotation. Also, identify another equation that describes the distance, d, the nail is from the vertical line in terms of the rotation.

$$d(\theta) = \sin\theta \qquad d(\theta) = 0.5\,\theta \qquad \theta(d) = \sin d \qquad d(\theta) = \cos\theta$$

$$h(\theta) = \sin\theta \qquad h(\theta) = 0.5\,\theta \qquad \theta(d) = \sin h \qquad h(\theta) = \cos\theta$$

Reflecting

J. Use your graphing calculator to graph $y = \sin x$ and $y = \cos x$, where $0° \leq x \leq 360°$, and compare these graphs to the graphs from parts E and G. Use words such as *amplitude*, *period*, *equation of the axis*, *increasing intervals*, *decreasing intervals*, *domain*, and *range* in your comparison.

K. State the coordinates of five key points that would allow you to draw the sinusoidal function $y = \sin x$ quickly over the interval $0°$ to $360°$.

L. State the coordinates of five key points that would allow you to draw the sinusoidal function $y = \cos x$ quickly over the interval $0°$ to $360°$.

M. What transformation can you apply to the cosine curve that will result in the sine curve?

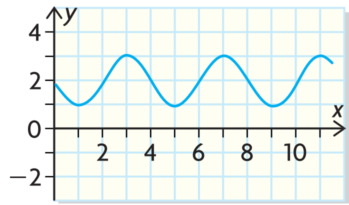
N. What ordered pair could you use to represent the point on the wheel that corresponds to the nail's location in terms of θ, the angle of rotation?

Tech Support

For help graphing trigonometric functions on your graphing calculator, see Technical Appendix, B-14.

sinusoidal function

a periodic function whose graph looks like smooth symmetrical waves, where any portion of the wave can be horizontally translated onto another portion of the curve; graphs of sinusoidal functions can be created by transforming the graph of the function $y = \sin x$ or $y = \cos x$

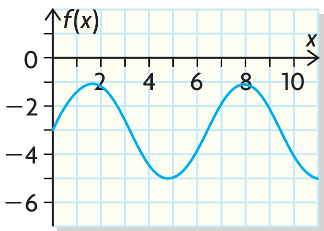


APPLY the Math

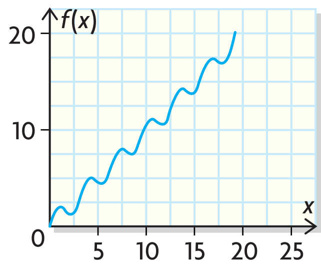
EXAMPLE 1 Identifying the function

Determine whether the graph represents a periodic function. If it does, determine whether it represents a sinusoidal function.

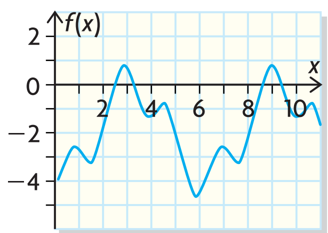
a)



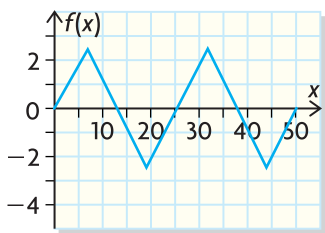
c)



b)



d)



Bridget's Solution

a) periodic and sinusoidal

The function repeats, so it's periodic. It looks like smooth symmetrical waves, where any portion of the wave can be horizontally translated onto another portion of the curve.

b) periodic

The pattern repeats but the waves aren't symmetrical.

c) neither periodic nor sinusoidal

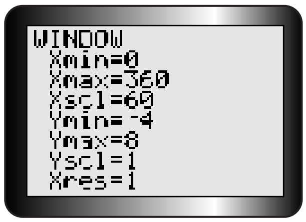
It looks like smooth symmetrical waves; however, I can't horizontally translate any portion of the wave onto another portion of the curve.

d) periodic

The pattern repeats but the waves aren't smooth curves.

EXAMPLE 2 Identifying the properties of a sinusoidal function

Graph the function $f(x) = 4\sin(3x) + 2$ on a graphing calculator using the WINDOW settings shown in DEGREE mode.



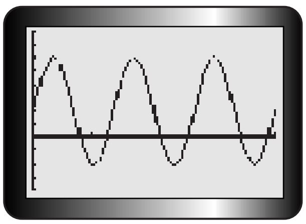
a) Is the function periodic? If it is, is it sinusoidal?

b) From the graph, determine the period, the equation of the axis, the amplitude, and the range.

c) Calculate $f(20°)$.

Beth's Solution

a)



Because it repeats, the graph is periodic.

Since it forms a series of identical, symmetrical smooth waves, it is sinusoidal.

b) period $= 120°$

The graph completes three cycles in 360°, so one cycle, which is the period, must be 120°.

equation of the axis: $y = \frac{-2 + 6}{2}$

The axis is halfway between the minimum of -2 and the maximum of 6.

$y = 2$

$6 - 2 = 4$

amplitude $= 4$

To get the amplitude, I calculated the vertical distance between a maximum and the axis. It's 4.

range: $\{y \in \mathbf{R} \mid -2 \leq y \leq 6\}$

For the range, the greatest *y*-value on the graph (the maximum) is 6, and the least *y*-value (the minimum) is -2.

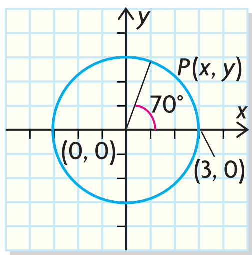
c)

$$\begin{aligned} f(x) &= 4\sin(3x) + 2 \\ f(20°) &= 4\sin(3(20°)) + 2 \\ &= 4\sin(60°) + 2 \\ &\doteq 4(0.866) + 2 \\ &= 5.464 \end{aligned}$$

$f(20°)$ means find *y* when $x = 20°$. I substituted 20 for *x* and then calculated *y*.

EXAMPLE 3 Determining the coordinates of a point from a rotation angle

Determine the coordinates of the point $P(x, y)$ resulting from a rotation of 70° centred at the origin and starting from the point (3, 0).



Anne's Solution

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}$$

$$\sin \theta = \frac{y}{r}$$

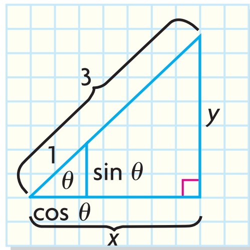
$$\frac{3}{1} = \frac{x}{\cos \theta} \quad \text{and} \quad \frac{3}{1} = \frac{y}{\sin \theta}$$

$$x = 3 \cos \theta \qquad y = 3 \sin \theta$$

$$P(x, y) = (3 \cos \theta, 3 \sin \theta)$$

The water wheel solution was based on a circle of radius 1. The coordinates of the nail after a rotation of θ were $(\cos \theta, \sin \theta)$. But this circle doesn't have a radius of 1. Its radius is 3.

I used similar triangles to figure out the coordinates of the larger triangle.



The coordinates for any point $P(x, y)$ on a circle of radius r are $P(x, y) = (r \cos \theta, r \sin \theta)$.

This means that the coordinates of the new point after a rotation of θ from the point $(r, 0)$ about $(0, 0)$ can be determined from $(r \cos \theta, r \sin \theta)$.

$$P(x, y) = (3 \cos 70°, 3 \sin 70°)$$
$$\doteq (1.03, 2.82)$$

I substituted the radius and angle of rotation into the ordered pair $(r \cos \theta, r \sin \theta)$ and got the coordinates of the image point.

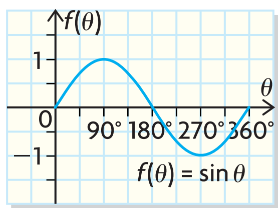
In Summary

Key Idea

- The function $f(\theta) = \sin \theta$ is a periodic function that represents the height (vertical distance) of a point from the x-axis as it rotates $\theta°$ about a circle with radius 1.
- The function $f(\theta) = \cos \theta$ is a periodic function that represents the horizontal distance of a point from the y-axis as it rotates $\theta°$ about a circle with radius 1.

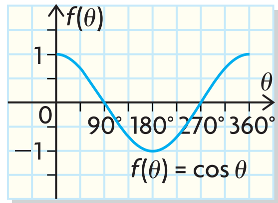
Need to Know

- The graph of $f(\theta) = \sin \theta$ has these characteristics:
  - The period is 360°.
  - The amplitude is 1, the maximum value is 1, and the minimum value is -1.
  - The domain is $\{\theta \in \mathbf{R}\}$, and the range is $-1 \leq f(\theta) \leq 1$.
  - The zeros are located at 0°, 180°, 360°, ...

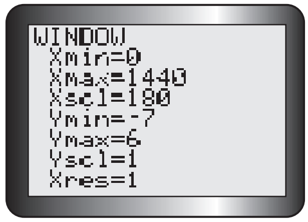


(continued)

- The graph of $f(\theta) = \cos\theta$ has these characteristics:
  - The period is 360°.
  - The amplitude is 1, the maximum value is 1, and the minimum value is -1.
  - The domain is $\{\theta \in \mathbf{R}\}$, and the range is $-1 \le f(\theta) \le 1$.
  - The zeros are located at 90°, 270°, 450°, ...
- The sine function and cosine function are congruent sinusoidal curves; the cosine curve is the sine curve translated 90° to the left.
- Any point $P(x, y)$ on a circle centred at (0, 0) with radius r and rotated through an angle θ can be expressed as an ordered pair $(r\cos\theta, r\sin\theta)$.



CHECK Your Understanding



1. Using a graphing calculator in DEGREE mode, graph each sinusoidal function. Use the WINDOW settings shown. From the graph, state the amplitude, period, and equation of the axis for each.

a) $y = 3\sin(2x) + 1$ **b)** $y = 4\cos(0.5x) - 2$

2. a) If $h(x) = \sin(5x) - 1$, calculate $h(25°)$.

b) If $f(x) = \cos x$ and $f(x) = 0$, list the values of x where $0° \le x \le 360°$.

3. A buoy rises and falls as it rides the waves. The equation $h(t) = \cos(36t)°$ models the displacement of the buoy, $h(t)$, in metres at t seconds.

a) Graph the displacement from 0 s to 20 s, in 2.5 s intervals.

b) Determine the period of the function from the graph.

c) What is the displacement at 35 s?

d) At what time, to the nearest second, does the displacement first reach -0.8 m?

4. Determine the coordinates of the new point after a rotation of 50° about (0, 0) from the point (2, 0).

PRACTISING

5. Using a graphing calculator and the WINDOW settings shown, graph each function. Use DEGREE mode. State whether the resulting functions are periodic. If so, state whether they are sinusoidal.

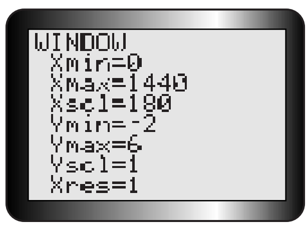
a) $y = 3\sin x + 1$ **c)** $y = \cos(2x) - \sin x$ **e)** $y = 0.5\cos x - 1$

b) $y = (0.004x)\sin x$ **d)** $y = 0.005x + \sin x$ **f)** $y = \sin 90°$

6. Based on your observations in question 5, what can you conclude about any function that possesses sine or cosine in its equation?

7. If $g(x) = \sin x$ and $h(x) = \cos x$, where $0° \le x \le 360°$, calculate each and explain what it means.

a) $g(90°)$ **b)** $h(90°)$



8. **K** Using a graphing calculator in DEGREE mode, graph each sinusoidal function. Use the WINDOW settings shown. From the graph, state the amplitude, period, increasing intervals, decreasing intervals, and equation of the axis for each.

a) $y = 2\sin x + 3$ **c)** $y = \sin(0.5x) + 2$ **e)** $y = 2\sin(0.25x)$

b) $y = 3\sin x + 1$ **d)** $y = \sin(2x) - 1$ **f)** $y = 3\sin(0.5x) + 2$

9. a) If $f(x) = \cos x$, calculate $f(35°)$.
   b) If $g(x) = \sin(2x)$, calculate $g(10°)$.
   c) If $h(x) = \cos(3x) + 1$, calculate $h(20°)$.
   d) If $f(x) = \cos x$ and $f(x) = -1$, calculate x for $0° \leq x \leq 360°$.
   e) If $f(x) = \sin x$ and $f(x) = -1$, calculate x for $0° \leq x \leq 360°$.

10. Determine all values where $\sin x = \cos x$ for $-360° \leq x \leq 360°$.

T

11. a) Determine the coordinates of the new point after a rotation of 25° about (0, 0) from the point (1, 0).
    b) Determine the coordinates of the new point after a rotation of 80° about (0, 0) from the point (5, 0).
    c) Determine the coordinates of the new point after a rotation of 120° about (0, 0) from the point (4, 0).
    d) Determine the coordinates of the new point after a rotation of 230° about (0, 0) from the point (3, 0).

12. Sketch the sinusoidal graphs that satisfy the properties in the table.

	Period	Amplitude	Equation of the Axis	Number of Cycles
a)	4	3	$y = 5$	2
b)	20	6	$y = 4$	3
c)	80	5	$y = -2$	2

13. Jim is riding a Ferris wheel, where t is time in seconds. Explain what each of
A the following represents.
    a) $h(10)$, where $h(t) = 5\cos(18t)°$
    b) $h(10)$, where $h(t) = 5\sin(18t)°$

14. Compare the graphs for $y = \sin x$ and $y = \cos x$, where $0° \leq x \leq 360°$.
C How are they the same, and how are they different?

Extending

15. If the water level in the original water wheel situation was lowered so that three-quarters of the wheel was exposed, determine the equation of the sinusoidal function that describes the height of the nail in terms of the rotation.

16. A spring bounces up and down according to the model $d(t) = 0.5\cos(120t)°$, where $d(t)$ is the displacement in centimetres from the rest position and t is time in seconds. The model does not consider the effects of gravity.
    a) Make a table for $0 \leq t \leq 9$. Use 0.5 s intervals.
    b) Draw the graph.
    c) Explain why the function models periodic behaviour.
    d) What is the relationship between the amplitude of the function and the displacement of the spring from its rest position?

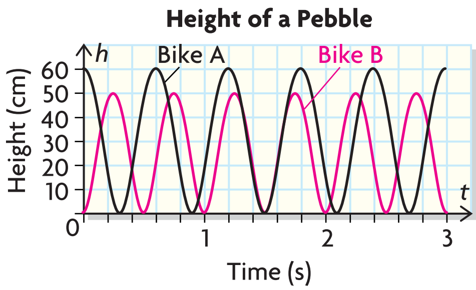
6.3 Interpreting Sinusoidal Functions

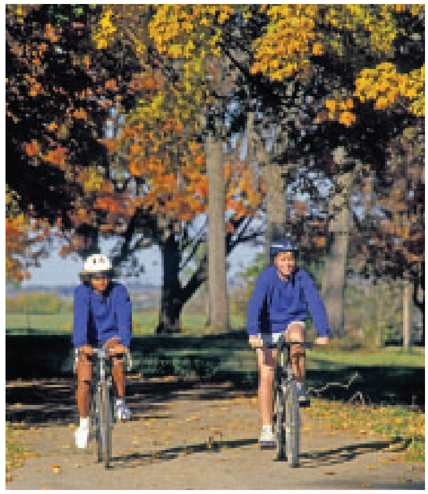
GOAL

Relate details of sinusoidal phenomena to their graphs.

LEARN ABOUT the Math

Two students are riding their bikes. A pebble is stuck in the tire of each bike. The two graphs show the heights of the pebbles above the ground in terms of time.





? What information about the bikes can you gather from the graphs of these functions?

EXAMPLE 1 Connecting the graph of a sinusoidal function to the situation

Joanne's Solution: Comparing Peaks of a Sinusoidal Function

For Bike A, the pebble was initially at its highest height of 60 cm. For Bike B, the pebble was initially at its lowest height of 0 cm.

For Bike A, the graph starts at a peak. For Bike B, the graph starts at a trough.

The wheels have different diameters. The diameter of the wheel on Bike A is 60 cm. The diameter of the wheel on Bike B is 50 cm.

I noticed that the peaks on the graph are different. The peak for Bike A is at $h = 60$, which is greater than the peak for Bike B, which is at $h = 50$. The troughs, however, are the same, $h = 0$.

Glen's Solution: Comparing Periods

The wheel on Bike A takes 0.6 s to complete one revolution. The wheel on Bike B takes 0.5 s to complete one revolution.

The period of Bike A is 0.6 s. The period of Bike B is 0.5 s.

The graph for Bike A completes 5 cycles in 3 s, so the period, or length of one cycle, is 0.6 s.

The graph for Bike B completes 2 cycles in 1 s, so the period is 0.5 s.

Scott's Solution: Comparing Equations of the Axes in Sinusoidal Functions

Bike A: $\frac{60 + 0}{2} = 30$

Bike B: $\frac{50 + 0}{2} = 25$

The axis is halfway between a peak (or maximum) and a trough (or minimum). I added the maximum and the minimum and then divided by 2.

The equation of the axis for Bike A is $h = 30$.

The equation of the axis for Bike B is $h = 25$.

The axle for the wheel on Bike A is 30 cm above the ground. The axle for the wheel on Bike B is 25 cm above the ground.

Karen's Solution: Comparing Speeds

Circumference:

Speed is equal to distance divided by time, so first I had to figure out how far each bike travels when the wheel completes one revolution. This distance is the circumference. I calculated the two circumferences.

Bike A	Bike B
$C_A = 2\pi r_A$	$C_B = 2\pi r_B$
$C_A = 2\pi(30)$	$C_B = 2\pi(25)$
$C_A = 60\pi$	$C_B = 50\pi$
$C_A \doteq 188.5$ cm	$C_B \doteq 157.1$ cm
$C_A \doteq 1.885$ m	$C_B \doteq 1.571$ m
$s_A = \frac{d}{t}$	$s_B = \frac{d}{t}$
$s_A = \frac{1.885}{0.6}$	$s_B = \frac{1.571}{0.5}$
$s_A \doteq 3.14$ m/s	$s_B \doteq 3.14$ m/s

To calculate the speed, I divided each circumference by the time taken to complete one revolution.

The bikes are travelling at the same speed.

Reflecting

A. How would changing the speed of the bike affect the sinusoidal graph?

B. For a third rider travelling at the same speed but on a bike with a larger wheel than that on Bike A, how would the graph of the resulting sinusoidal function compare with that for Bike A and Bike B?

C. What type of information can you learn by examining the graph modelling the height of a pebble stuck on a tire in terms of time?

APPLY the Math

EXAMPLE 2 Comparing graphs and situations

Annette's shop teacher was discussing table saws. The teacher produced two different graphs for two different types of saw. In each case, the graphs show the height of one tooth on the circular blade relative to the cutting surface of the saw in terms of time.

Table Saw A

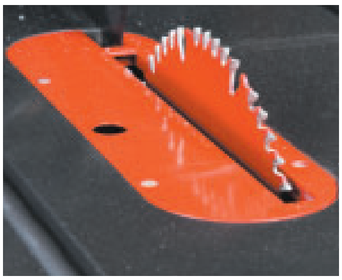
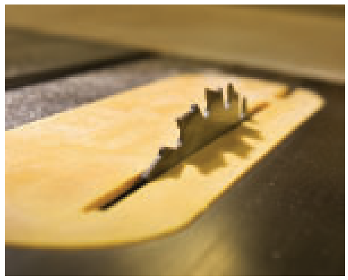
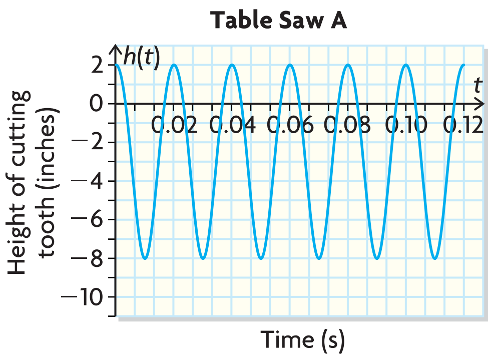
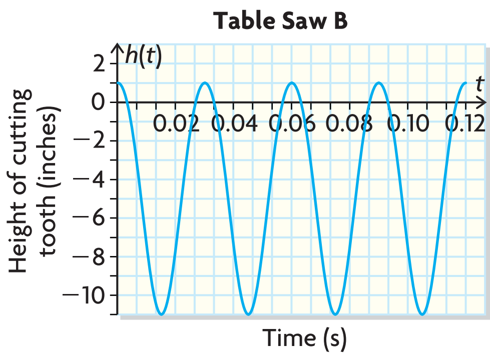


Table Saw B







What information about the table saws can Annette gather from the graphs?

Repko's Solution

The blade on Table Saw A is set higher than the blade on Table Saw B.

The peaks on the graph are different. The peak for A is at $h = 2$; the peak for B is at $h = 1$.

The blade on Table Saw A takes 0.02 s to complete one revolution.

One of the easiest ways to find the period is to figure out how long it takes to go from one peak on the graph to the next.

On graph A, the first peak is at 0 s, and the next is at 0.02 s. This means that the period of graph A is 0.02 s.

The blade on Table Saw B takes 0.03 s to complete one revolution.

On graph B, the first peak is at 0 s, and the next is at 0.03 s. The period of graph B is 0.03 s.

The axle for the blade on Table Saw A is 3 in. below the cutting surface.

For graph A, I found the equation of the axis by adding 2 and -8 and then dividing by 2. That gave me -3. The equation of the axis for graph A is $h = -3$.

The axle for the blade on Table Saw B is 5 in. below the cutting surface.

For graph B, I added 1 and -11 and then divided by 2. That gave me -5. The equation of the axis for graph B is $h = -5$.

The radius of the circular cutting blade on Table Saw A is 5 in.

The radius of the circular cutting blade on Table Saw B is 6 in.

For graph A, I got the amplitude by taking the difference between 2 and -3. The amplitude for graph A is 5.

For graph B, the amplitude is the difference between 1 and -5. The amplitude for graph B is 6.

In both cases, the distance from the axis to a peak represents the radius of the circular cutting blade.

EXAMPLE 3 Using technology to understand a situation

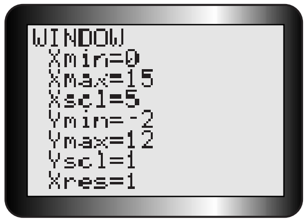
The function $j(t) = 4.1 \sin(64.7t)° + 5.8$, where t is time in years since May 1992 and $j(t)$ is the number of applications for jobs each week (in hundreds), models demand for employment in a particular city.

a) Using graphing technology in DEGREE mode and the WINDOW settings shown, graph the function and then sketch the graph.

b) How long is the employment cycle? Explain how you know.

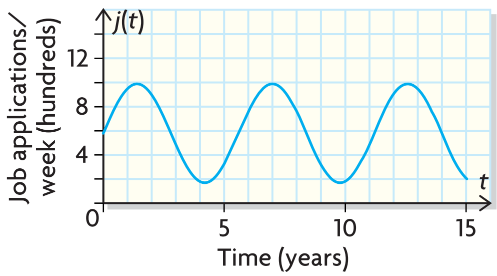
c) What is the minimum number of applications per week in this city?

d) Calculate $j(10)$, and explain what it represents in terms of the situation.



Karl's Solution

a)



I sketched all the cycles the window showed.

b) $6.9 - 1.34 = 5.56$

The employment cycle is 5.56 years, the distance between peaks or troughs.

To calculate the cycle, I calculated the *x*-interval between the first and second peak.

c) The minimum number of applications per week is 170.

I looked for a trough on the graph and read the *j*-coordinate.

d) $j(10) = 1.88$

There were 188 applications in May 2002.

I looked for the place where the *t*-coordinate was 10.

In Summary

Key Idea

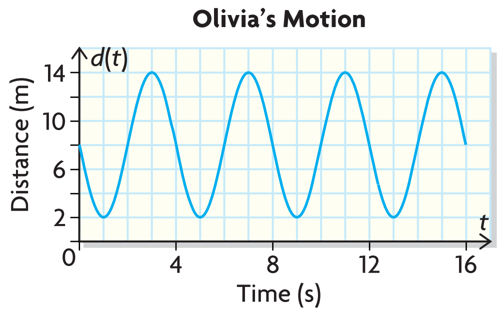
- The sine and cosine functions can be used as models to solve problems that involve many types of repetitive motions and trends.

Need to Know

- If a situation can be described by a sinusoidal function, the graph of the data should form a series of symmetrical waves that repeat at regular intervals. The amplitude of the sine or cosine function depends on the situation being modelled.
- One cycle of motion corresponds to one period of the sine function.
- The distance of a circular path is calculated from the circumference of the path. The speed of an object following a circular path can be calculated by dividing the distance by the period, the time to complete one rotation.

CHECK Your Understanding

1. Olivia was swinging back and forth in front of a motion detector when the detector was activated. Her distance from the detector in terms of time can be modelled by the graph shown.



a) What is the equation of the axis, and what does it represent in this situation?

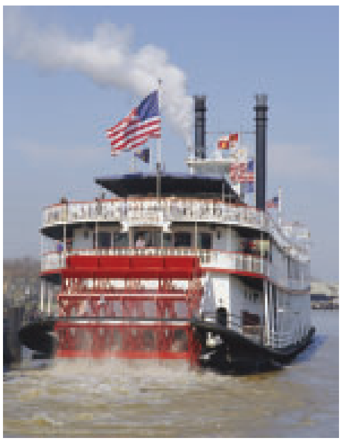
b) What is the amplitude of this function?

c) What is the period of this function, and what does it represent in this situation?

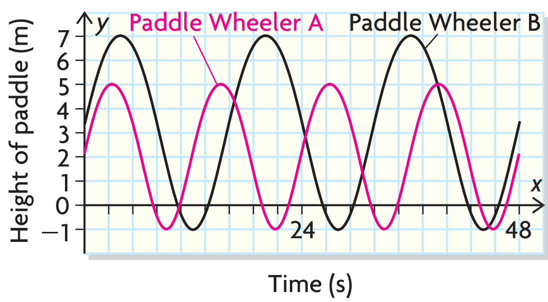
d) How close did Olivia get to the motion detector?

e) At $t = 7$ s, would it be safe to run between Olivia and the motion detector? Explain your reasoning.

f) If the motion detector was activated as soon as Olivia started to swing from at rest, how would the graph change? (You may draw a diagram or a sketch.) Would the resulting graph be sinusoidal? Why or why not?

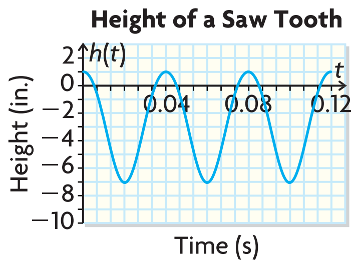


2. Marianna collected some data on two paddle wheels on two different boats and constructed two graphs. Analyze the graphs, and explain how the wheels differ. Refer to the radius of each wheel, the height of the axle relative to the water, the time taken to complete one revolution, and the speed of each wheel.

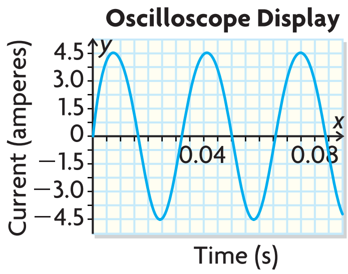


3. Draw two sinusoidal functions that have the same period and axes but have different amplitudes.

PRACTISING



4. Evan's teacher gave him a graph to help him understand the speed at which a tooth on a saw blade travels. The graph shows the height of one tooth on the circular blade relative to the cutting surface relative to time.
   K
   - **a)** How high above the cutting surface is the blade set?
   - **b)** What is the period of the function, and what does it represent in this situation?
   - **c)** What is the amplitude of the function, and what does it represent in this situation?
   - **d)** How fast is a tooth on the circular cutting blade travelling in inches per second?



5. An oscilloscope hooked up to an alternating current (AC) circuit shows a sine curve on its display.
   - **a)** What is the period of the function? Include the units of measure.
   - **b)** What is the equation of the axis of the function? Include the units of measure.
   - **c)** What is the amplitude of the function? Include the units of measure.

6. Sketch a height-versus-time graph of the sinusoidal function that models each situation. Draw at least three cycles. Assume that the first point plotted on each graph is at the lowest possible height.
   - **a)** A Ferris wheel with a radius of 7 m, whose axle is 8 m above the ground, and that rotates once every 40 s
   - **b)** A water wheel with a radius of 3 m, whose centre is at water level, and that rotates once every 15 s
   - **c)** A bicycle tire with a radius of 40 cm and that rotates once every 2 s
   - **d)** A girl lying on an air mattress in a wave pool that is 3 m deep, with waves 0.5 m in height that occur at 7 s intervals

7. The tables show the varying length of daylight for Timmins, Ontario, located at a latitude of 48°, and Miami, Florida, located at a latitude of 25°. The length of the day is calculated as the interval between sunrise and sunset.

Timmins, at latitude 48°

Day of Year	15	46	74	105	135	165	196	227	258	288	319	349
Hours of Daylight	8.8	10.2	11.9	13.7	15.2	16.1	15.7	14.4	12.6	10.9	9.2	8.3

Miami, at latitude 25°

Day of Year	15	46	74	105	135	165	196	227	258	288	319	349
Hours of Daylight	10.7	11.3	12.0	12.8	13.6	13.8	13.6	13.1	12.3	11.6	10.9	10.5

a) Plot the data on separate coordinate systems, and draw a smooth curve through each set of points.

b) Compare the two curves. Refer to the periods, amplitudes, and equations of the axes.

c) What might you infer about the relationship between hours of daylight and the latitude at which you live?

8. The diameter of a car's tire is 52 cm. While the car is being driven, the tire picks up a nail.

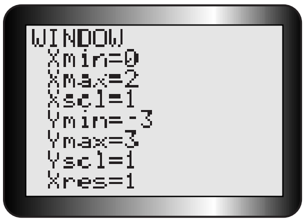
T

a) Draw a graph of the height of the nail above the ground in terms of the distance the car has travelled since the tire picked up the nail.

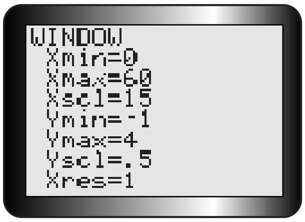
b) How high above the ground will the nail be after the car has travelled 0.1 km?

c) How far will the car have travelled when the nail reaches a height of 20 cm above the ground for the fifth time?

d) What assumption must you make concerning the driver's habits for the function to give an accurate height?



9. In high winds, the top of a signpost vibrates back and forth. The distance the tip of the post vibrates to the left and right of its resting position can be defined by the function $d(t) = 3 \sin(1080t)°$, where $d(t)$ represents the distance in centimetres at time t seconds. If the wind speed decreases by 20 km/h, the vibration of the tip can be modelled by the function $d(t) = 2 \sin(1080t)°$. Using graphing technology in DEGREE mode and the WINDOW settings shown, produce the two graphs. How does the reduced wind speed affect the period, amplitude, and equation of the axis?



10. The height, $h(t)$, of a basket on a water wheel at time t can be modelled by $h(t) = 2 \sin(12t) + 1.5°$, where t is in seconds and $h(t)$ is in metres.

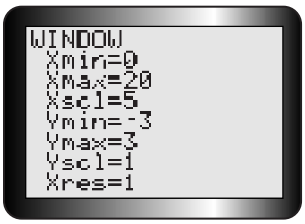
a) Using graphing technology in DEGREE mode and the WINDOW settings shown, graph $h(t)$ and sketch the graph.

b) How long does it take for the wheel to make a complete revolution? Explain how you know.

c) What is the radius of the wheel? Explain how you know.

d) Where is the centre of the wheel located in terms of the water level? Explain how you know.

e) Calculate $h(10)$, and explain what it represents in terms of the situation.



11. The equation $h(t) = 2.5 \sin(72t)°$ models the displacement of a buoy in metres at t seconds.

A

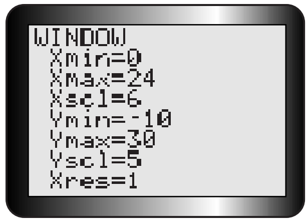
a) Using graphing technology in DEGREE mode and the WINDOW settings shown, graph $h(t)$ and sketch the graph.

b) How long does it take for the buoy to travel from the peak of a wave to the next peak? Explain how you know.

c) How many waves will cause the buoy to rise and fall in 1 min? Explain how you know.

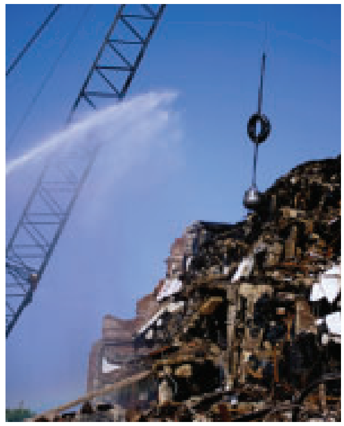
d) How far does the buoy drop from its highest point to its lowest point? Explain how you know.

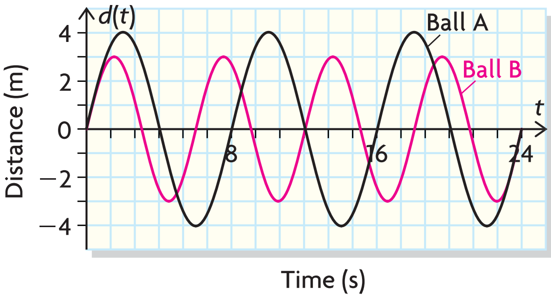
12. The average monthly temperature, $T(t)$, in degrees Celsius in Kingston, Ontario, can be modelled by the function $T(t) = 14.2 \sin(30(t - 4.2))° + 5.9$, where t represents the number of months. For $t = 1$, the month is January; for $t = 2$, the month is February; and so on.



- **a)** Using graphing technology in DEGREE mode and the WINDOW settings shown, graph $T(t)$ and sketch the graph.
- **b)** What does the period represent in this situation?
- **c)** What is the average temperature range in Kingston?
- **d)** What is the mean temperature in Kingston?
- **e)** Calculate $T(30)$, and explain what it represents in terms of the situation.

13. Two wrecking balls attached to different cranes swing back and forth. The distance the balls move to the left and the right of their resting positions in terms of time can be modelled by the graphs shown.



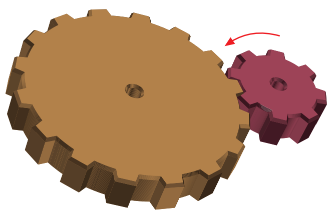


- **a)** What is the period of each function, and what does it represent in this situation?
- **b)** What is the equation of the axis of each function, and what does it represent in this situation?
- **c)** What is the amplitude of each function, and what does it represent in this situation?
- **d)** Determine the range of each function.
- **e)** Compare the motions of the two wrecking balls.

14. C How many pieces of information do you need to know to sketch a sinusoidal function. What pieces of information could they be?

Extending

15. A gear of radius 1 m turns counterclockwise and drives a larger gear of radius 4 m. Both gears have their axes along the horizontal.



- **a)** In which direction is the larger gear turning?
- **b)** If the period of the smaller gear is 2 s, what is the period of the larger gear?
- **c)** In a table, record convenient intervals for each gear, to show the vertical displacement, d, of the point where the two gears first touched. Begin the table at 0 s and end it at 24 s. Graph vertical displacement versus time.
- **d)** What is the displacement of the point on the large wheel when the drive wheel first has a displacement of -0.5 m?
- **e)** What is the displacement of the drive wheel when the large wheel first has a displacement of 2 m?
- **f)** What is the displacement of the point on the large wheel at 5 min?

6 Mid-Chapter Review

FREQUENTLY ASKED Questions

Study *Aid*

- See Lesson 6.2, Examples 1 and 2.
- Try Mid-Chapter Review Question 3.

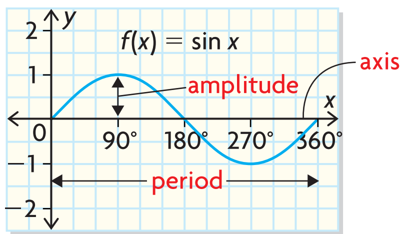
Q: What are sinusoidal functions, and what characteristics are often used to describe them?

A: Sinusoidal functions, like other periodic functions, repeat at regular intervals. Unlike other periodic functions, sinusoidal functions form smooth symmetrical waves such that any portion of a wave can be horizontally translated onto another portion of the curve. Sinusoidal functions are formed from transformations of the functions $y = \sin x$ and $y = \cos x$.

The three characteristics of a sinusoidal function, as well as any periodic function, are the period, the equation of the axis, and the amplitude.

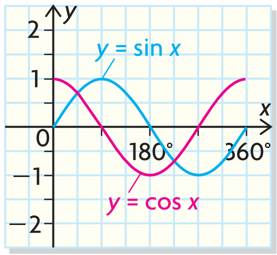
Period	Equation of the Axis	Amplitude
The period is the change in x corresponding to one cycle. (A cycle of a sinusoidal function is a portion of the graph that repeats.) One way to determine the period is to look at the change in x between two maxima.	The equation of the axis is the equation of the line halfway between the maximum and minimum values on a sinusoidal function. It can be determined with the formula $y = \dfrac{(\text{maximum value} + \text{minimum value})}{2}$	The amplitude is the vertical distance from the function's axis to the minimum or maximum value. It is always positive.

EXAMPLE



For the function $f(x) = \sin x$, the period is 360°, the equation of the axis is $y = 0$, and the amplitude is 1.

Q: How do the graphs of $y = \sin x$ and $y = \cos x$ compare?



A: Similarities

The period is 360°.

The equation of the axis is $y = 0$.

The amplitude $= 1$.

The range is $\{y \in \mathbf{R} \mid -1 \le y \le 1\}$.

Differences

A maximum for $y = \sin x$ occurs at 90° and at increments of 360° from that point.

A maximum for $y = \cos x$ occurs at 0° and at increments of 360° from that point.

A minimum for $y = \sin x$ occurs at 270° and at increments of 360° from that point.

A minimum for $y = \cos x$ occurs at 180° and at increments of 360° from that point.

The graph of the function $y = \sin x$ can be changed to a graph of the function $y = \cos x$ by applying a horizontal translation of 90° to the left.

The graph of the function $y = \cos x$ can be changed to a graph of the function $y = \sin x$ by applying a horizontal translation of 90° to the right.

Q: Why might it be useful to learn about sinusoidal functions?

A: Many real-world phenomena that have a regular repeating pattern can be modelled with sinusoidal functions. For example,

- the motion of objects in a circular orbit
- the motion of swinging objects, such as a pendulum
- the number of hours of sunlight for a particular latitude
- the phase of the Moon
- the current for an AC circuit

Study *Aid*

- See Lesson 6.3, Examples 1, 2, and 3.
- Try Mid-Chapter Review Questions 5 and 6.

PRACTICE Questions

Lesson 6.1

1. Sketch the graph of a periodic function whose period is 10 and whose range is $\{y \in \mathbf{R} \mid 4 \le y \le 10\}$.

2. The following data show the pressure (in pounds per square inch, psi) in the tank of an air compressor at different times.

Time (s)	0	1	2	3	4	5	6	7	8	9
Pressure (psi)	60	60	80	100	100	90	80	70	60	60

Time (s)	10	11	12	13	14	15	16	17	18	19
Pressure (psi)	80	100	100	90	80	70	60	60	80	100

a) Create a scatter plot of the data and the curve that best models the data.
b) How do you know that the graph is periodic?
c) Determine the period of the function.
d) Determine the equation of the axis.
e) Determine the amplitude.
f) How fast is the air pressure increasing when the compressor is on?
g) How fast is the air pressure decreasing when the equipment is in operation?
h) Is the container ever empty? Explain.

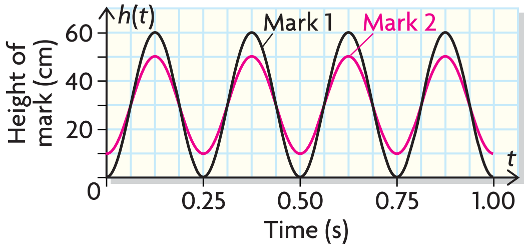
Lesson 6.2

3. a) Graph the function $g(x) = 5\cos(2x) + 7$ using a graphing calculator. Adjust the WINDOW settings so that $0° \le x \le 360°$ and $0 \le g(x) \le 15$. Determine the period, equation of the axis, amplitude, and range of the function.
b) Explain why the function is sinusoidal.
c) Calculate $g(125)$.
d) Determine the values of x, $0° \le x \le 360°$, for which $g(x) = 12$.

4. Determine the coordinates of the new point after a rotation of 64° about (0, 0) from the point (7, 0).

Lesson 6.3

5. Two white marks are made on a car tire by a parking meter inspector. One mark is made on the outer edge of the tire; the other mark is made a few centimetres from the edge. The two graphs show the relationship between the heights of the white marks above the ground in terms of time as the car moves forward.



a) What is the period of each function, and what does it represent in this situation?
b) What is the equation of the axis of each function, and what does it represent in this situation?
c) What is the amplitude of each function, and what does it represent in this situation?
d) Determine the range of each function.
e) Determine the speed of each mark, in centimetres per second.
f) If a third mark were placed on the tire but closer to the centre, how would the graph of this function compare with the other two graphs?

6. The position, $P(d)$, of the Sun at sunset, in degrees north or south of due west, depends on the latitude and the day of the year, d. For a specific latitude, the position in terms of the day of the year can be modelled by the function
$P(d) = 28\sin\left(\frac{360}{365}d - 81\right)^{\circ}$.
a) Graph the function using a graphing calculator and adjust the WINDOW settings as required.
b) What is the period of the function, and what does it represent in this situation?
c) What is the equation of the axis of the function, and what does it represent in this situation?
d) What is the amplitude of the function, and what does it represent in this situation?
e) Determine the range of the function.
f) What is the angle of sunset on February 15?

6.4 Exploring Transformations of Sinusoidal Functions

GOAL

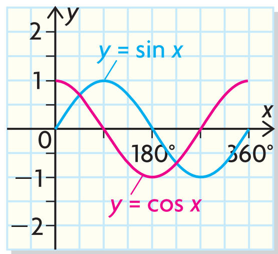
Determine how changing the values of a, c, d, and k affect the graphs of $f(x) = a\sin(k(x - d)) + c$ and $f(x) = a\cos(k(x - d)) + c$

YOU WILL NEED

- graphing calculator

EXPLORE the Math

Paula and Marcus know how various transformations affect several types of functions, such as $f(x) = x^2$, $f(x) = \frac{1}{x}$, $f(x) = \sqrt{x}$, and $f(x) = |x|$.



They want to know if these same transformations can be applied to $y = \sin x$ and $y = \cos x$, and if so, how the equations and graphs of these functions change.

? Can transformations be applied to sinusoidal functions in the same manner, and do they have the same effect on the graph and the equation?

Part 1 The graphs of $y = a\sin x$ and $y = a\cos x$

A. Predict what the graphs of $y = a\sin x$, $0° \leq x \leq 720°$, will look like for $a = 1$, 2, and 3 and for $a = \frac{1}{2}$ and $a = \frac{1}{4}$. Sketch the graphs on the same axes. Verify your sketches using a graphing calculator.

B. On a new set of axes, repeat part A for the graphs of $y = a\sin x$, $0° \leq x \leq 720°$, for $a = -1$, -2, and -3.

C. How do the graphs in part A compare with those in part B? Discuss how the zeros, amplitude, and maximum or minimum values change for each function.

D. Repeat parts A to C using $y = a\cos x$.

E. Explain how the value of a affects the graphs of $y = a\sin x$ and $y = a\cos x$.

Tech *Support*

For Parts 1 and 2, verify your sketches by graphing the parent function ($y = \sin x$ or $y = \cos x$) in Y1 and each transformed function in Y2, Y3, and so on. Use an Xscl = 90°, and graph using ZoomFit by pressing ZOOM 0.

Part 2 The graphs of $y = \sin x + c$ and $y = \cos x + c$

F. Predict what the graphs of $y = \sin x + c$, $0° \leq x \leq 720°$, will look like for $c = -2$, -1, 1, and 2. Sketch the graphs on the same axes, and then verify your predictions using a graphing calculator. Discuss which features of the graph have changed.

G. Predict what the graphs of $y = \cos x + c$, $0° \leq x \leq 720°$, will look like for $c = -2$, -1, 1, and 2. Sketch the graphs on the same axes, and then verify your predictions using a graphing calculator. Discuss which features of the graph have changed.

H. Explain how the value of c affects the graphs of $y = a\sin x + c$ and $y = a\cos x + c$.

Tech Support

For Part 3, verify your sketches by graphing the parent function in Y1 and each transformed function in Y2. Use an Xscl = 90° and graph using ZoomFit.

Part 3 The graphs of $y = \sin kx$ and $y = \cos kx$

I. Predict what the graphs of $y = \sin kx$ will look like for $k = 2, 3,$ and 4, $0° \leq x \leq 720°$. Sketch each graph, and then verify your predictions using a graphing calculator. Discuss which features of the graph have changed. Clear the previous equation, but not the base equation, from the graphing calculator before entering another equation.

J. Repeat part I for $k = \frac{1}{2}$, $k = \frac{1}{4}$, and $k = -1$. Adjust the WINDOW on the graphing calculator so that you can see one complete cycle of each graph.

K. Repeat parts I and J using $y = \cos kx$.

L. How could you determine the period of $y = \sin kx$ and $y = \cos kx$ knowing that the period of both functions is 360°?

M. Explain how the value of k affects $y = \sin kx$ and $y = \cos kx$.

i)

x	y
60°	
150°	
240°	
330°	
420°	

ii)

x	y
−120°	
−30°	
60°	
150°	
240°	

Part 4 The graphs of $y = \sin(x - d)$ and $y = \cos(x - d)$

N. **a)** Predict the effect of d on the graph of $y = \sin(x - d)$.

b) Copy and complete the tables of values at the left.

i) $y = \sin(x - 60°)$ **ii)** $y = \sin(x + 120°)$

c) Use your tables to sketch the graphs of the two sinusoidal functions from part (b) on the same coordinate system. Include the graph of $y = \sin x$. Verify your sketches using a graphing calculator, and discuss which features of the graph have changed.

i)

x	y
−45°	
45°	
135°	
225°	
315°	

ii)

x	y
120°	
210°	
300°	
390°	
480°	

O. **a)** Predict the effect of d on the graph of $y = \cos(x - d)$.

b) Copy and complete the tables of values at the left.

i) $y = \cos(x + 45°)$ **ii)** $y = \cos(x - 120°)$

c) Use your tables to sketch the graphs of the two sinusoidal functions from part (b) on the same coordinate system. Include the graph of $y = \cos x$. Verify your sketches with a graphing calculator, and discuss which features of the graph have changed.

P. Explain how the value of d affects the graphs of $y = \sin(x - d)$ and $y = \cos(x - d)$.

Tech Support

For Part 4, verify your sketches using a domain of $0° \leq x \leq 360°$ and an Xscl = 30°. Graph using ZoomFit.

Reflecting

Q. What transformation affects the period of a sinusoidal function?

R. What transformation affects the equation of the axis of a sinusoidal function?

S. What transformation affects the amplitude of a sinusoidal function?

T. What transformations affect the location of the maximum and minimum values of the sinusoidal function?

U. Summarize how the graphs of $y = a \sin(k(x - d)) + c$ and $y = a \cos(k(x - d)) + c$ compare with the graphs of $y = \sin x$ and $y = \cos x$.

In Summary

Key Ideas

- The graphs of the functions $f(x) = a\sin(k(x - d)) + c$ and $f(x) = a\cos(k(x - d)) + c$ are periodic in the same way that the graphs of $f(x) = \sin x$ and $f(x) = \cos x$ are. The differences are only in the placement of the graph and how stretched or compressed it is.
- The values a, k, c, and d in the functions $f(x) = a\sin(k(x - d)) + c$ and $f(x) = a\cos(k(x - d)) + c$ affect the graphs of $y = \sin x$ and $y = \cos x$ in the same way that they affect the graphs of $y = f(k(x - d)) + c$, where $f(x) = x^2$, $f(x) = \frac{1}{x}$, $f(x) = \sqrt{x}$, and $f(x) = |x|$.

Need to Know

- Changing the value of c results in a vertical translation and affects the equation of the axis, the maximum and minimum values, and the range of the function but has no effect on the period, amplitude, or domain.
- Changing the value of d results in a horizontal translation and slides the graph to the left or right but has no effect on the period, amplitude, equation of the axis, domain, or range unless the situation forces a change in the domain or range.
- Changing the value of a results in a vertical stretch or compression and affects the maximum and minimum values, amplitude, and range of the function but has no effect on the period or domain. If a is negative, a reflection in the x-axis also occurs.
- Changing the value of k results in a horizontal stretch or compression and affects the period, changing it to $\frac{360°}{|k|}$, but has no effect on the amplitude, equation of the axis, maximum and minimum values, domain, and range unless the situation forces a change in the domain or range. If k is negative, a reflection in the y-axis also occurs.

FURTHER Your Understanding

1. State the transformation to the graph of either $y = \sin x$ or $y = \cos x$ that has occurred to result in each sinusoidal function.

a) $y = 3\cos x$
b) $y = \sin(x - 50°)$
c) $y = -\cos x$
d) $y = \sin(5x)$
e) $y = \cos x - 6$
f) $y = \cos(x + 20°)$

2. Each sinusoidal function below has undergone one transformation that has affected either the period, amplitude, or equation of the axis. In each case, determine which characteristic has been changed and indicate its value.

a) $y = \sin x + 2$
b) $y = 4\sin x$
c) $y = \cos(8x)$
d) $y = \sin(2x + 30°)$
e) $y = 0.25\cos x$
f) $y = \sin(0.5x)$

3. Which two of these transformations do not affect the period, amplitude, or equation of the axis of a sinusoidal function?

a) reflection in the x-axis
b) vertical stretch/vertical compression
c) vertical translation
d) horizontal stretch/horizontal compression
e) horizontal translation

6.5 Using Transformations to Sketch the Graphs of Sinusoidal Functions

YOU WILL NEED

- graph paper

GOAL

Sketch the graphs of sinusoidal functions using transformations.

LEARN ABOUT the Math

Glen has been asked to graph the sinusoidal function $f(x) = 3\sin(2(x - 60°)) + 4$ without using technology.

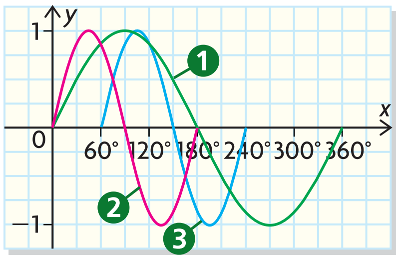
? How can you graph sinusoidal functions using transformations?

EXAMPLE 1 Using transformations to sketch the graph of a sinusoidal function

Sketch the graph of $f(x) = 3\sin(2(x - 60°)) + 4$.

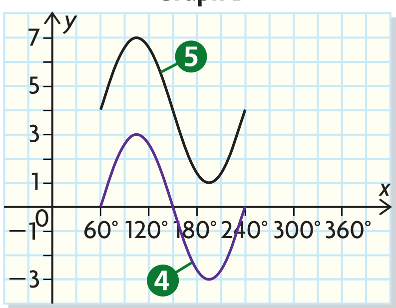
Glen's Solution

Graph A



1. I started by graphing $y = \sin x$ (in green).
2. Then I graphed $y = \sin(2x)$ (in red). It has a horizontal compression of $\frac{1}{2}$, so the period is $\frac{360°}{2} = 180°$ instead of 360° because all the x-coordinates of the points on the graph of $y = \sin x$ have been divided by 2.
3. Then I graphed $y = \sin(2(x - 60°))$ (in blue) by applying a horizontal translation of $y = \sin(2x)$ 60° to the right because 60° has been added to all the x-coordinates of the points on the previous graph.

Graph B



4. Next, I graphed $y = 3\sin(2(x - 60°))$ (in purple) by applying a vertical stretch of 3 to $y = \sin(2(x - 60°))$. The amplitude is now 3 because all the y-coordinates of the points on the previous graph have been multiplied by 3.
5. Finally, I graphed $y = 3\sin(2(x - 60°)) + 4$ (in black) by applying a vertical translation of 4 to $y = 3\sin(2(x - 60°))$. This means that the whole graph has slid up 4 units and that the equation of the axis is now $y = 4$ because 4 has been added to all the y-coordinates of the points on the previous graph.

Reflecting

A. In what order were the transformations applied to the function $y = \sin x$?

B. If the equation of the function $y = 3\sin(2(x - 60°)) + 4$ were changed to $y = 3\sin(2(x - 60°)) - 5$, how would the graph of the function change? How would it stay the same?

C. If the equation of the function $y = 3\sin(2(x - 60°)) + 4$ were changed to $y = 3\sin(9(x - 60°)) + 4$, how would the graph of the function change?

D. Which transformations affect the range of the function? How?

E. Which transformations affect the period of the function? How?

F. Could Glen graph this function faster by combining transformations? If so, which ones?

APPLY the Math

EXAMPLE 2 Connecting transformations to the graph of a sinusoidal function

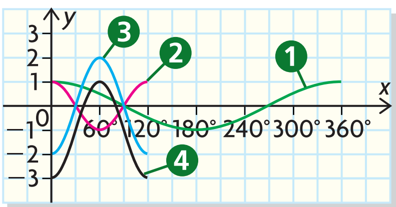
a) Graph $y = -2\cos(3x) - 1$ using transformations.

b) State the amplitude, period, equation of the axis, phase shift, and range of this sinusoidal function.

phase shift
the horizontal translation of a sinusoidal function

Steven's Solution

a)



1 I started by graphing $y = \cos x$ (in green).

2 I dealt with the horizontal compression first. I graphed $y = \cos(3x)$ (in red) using a period of $\frac{360°}{3} = 120°$ instead of 360°.

3 I dealt with the vertical stretch and the reflection in the x-axis. I graphed $y = -2\cos(3x)$ (in blue) starting at its lowest value due to the reflection, changing its amplitude to 2 due to the vertical stretch.

4 Finally, I did the vertical translation. I graphed $y = -2\cos(3x) - 1$ (in black) by sliding the previous graph down 1 unit, so the equation of the axis is $y = -1$.

b) The amplitude is 2.

The period is 120°.

The equation of the axis is $y = -1$.

The phase shift is 0.

The range is $\{y \in \mathbf{R} \mid -3 \leq y \leq 1\}$.

You can graph sinusoidal functions more efficiently if you combine and use several transformations at the same time.

EXAMPLE 3 Using a factoring strategy to determine the transformations

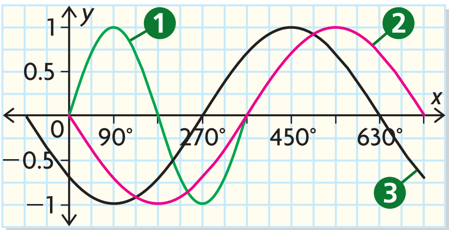
Graph $y = -\sin(0.5x + 45°)$ using transformations.

John's Solution

$y = -\sin(0.5x + 45°)$

$y = -\sin[0.5(x + 90°)]$

I factored the expression inside the brackets so that I could see all the transformations. I divided out the common factor 0.5 from 0.5x and 45.



1. I started by graphing $y = \sin x$ (in green).
2. Rather than graph this one transformation at a time, I dealt with all stretches/compressions and reflections at the same time. I graphed $y = -\sin(0.5x)$ (in red) by using a period of $\frac{360°}{0.5} = 720°$ and reflecting this across the x-axis.
3. I applied the phase shift and graphed $y = -\sin(0.5(x + 90°))$ (in black) by shifting all the points on the previous graph 90° to the left.

In Summary

Key Idea

- Functions of the form $g(x) = a\sin(k(x - d)) + c$ and $h(x) = a\cos(k(x - d)) + c$ can be graphed by applying the appropriate transformations to the graphs of $f(x) = \sin x$ and $f(x) = \cos x$, respectively, one at a time, following the order of operations (multiplication and division before addition and subtraction) for all vertical transformations and for all horizontal transformations. The horizontal and vertical transformations can be completed in either order.
- As with other functions, you can apply all stretches/compressions and reflections together followed by all translations to graph the transformed function more efficiently.

Need to Know

- To graph $g(x)$, you need to apply the transformations to the key points of $f(x) = \sin x$ or $f(x) = \cos x$ only, not to every point on $f(x)$.
  - Key points for $f(x) = \sin x$
    (0°, 0), (90°, 1), (180°, 0), (270°, −1), (360°, 0)
  - Key points for $f(x) = \cos x$
    (0°, 1), (90°, 0), (180°, −1), (270°, 0), (360°, 1)

(continued)

- By doing so, you end up with a function with
  - an amplitude of $|a|$
  - a period of $\dfrac{360°}{|k|}$
  - an equation of the axis $y = c$
- Horizontal and vertical translations of sine and cosine functions can be summarized as follows:

  Horizontal
  - Move the graph d units to the right when $d > 0$.
  - Move the graph $|d|$ units to the left when $d < 0$.

  Vertical
  - Move the graph $|c|$ units down when $c < 0$.
  - Move the graph c units up when $c > 0$.
- Horizontal and vertical stretches of sine and cosine functions can be summarized as follows:

  Horizontal
  - Compress the graph by a factor $\left|\dfrac{1}{k}\right|$ when $|k| > 1$.
  - Stretch the graph by a factor $\left|\dfrac{1}{k}\right|$ when $0 < |k| < 1$.
  - Reflect the graph in the y-axis if $k < 0$.

  Vertical
  - Stretch the graph by a factor $|a|$ when $|a| > 1$.
  - Compress the graph by a factor $|a|$ when $0 < |a| < 1$.
  - Reflect the graph in the x-axis if $a < 0$.

CHECK *Your Understanding*

1. State the transformations, in the order you would apply them, for each sinusoidal function.

   a) $f(x) = \sin(4x) + 2$

   b) $y = 0.25 \cos(x - 20°)$

   c) $g(x) = -\sin(0.5x)$

   d) $y = 12 \cos(18x) + 3$

   e) $f(x) = -20 \sin\left[\dfrac{1}{3}(x - 40°)\right]$

2. If the function $f(x) = 4 \cos 3x + 6$ starts at $x = 0$ and completes two full cycles, determine the period, amplitude, equation of the axis, domain, and range.

3. Use transformations to predict what the graph of $g(x) = 5 \sin(2(x - 30°)) + 4$ will look like. Verify with a graphing calculator.

PRACTISING

4. State the transformations in the order you would apply them for each sinusoidal function.

a) $y = -2\sin(x + 10°)$

b) $y = \cos(5x) + 7$

c) $y = 9\cos(2(x + 6°)) - 5$

d) $g(x) = \frac{1}{5}\sin(x - 15°) + 1$

e) $h(x) = -\sin\left[\frac{1}{4}(x + 37°)\right] - 2$

f) $d = -6\cos(3t) + 22$

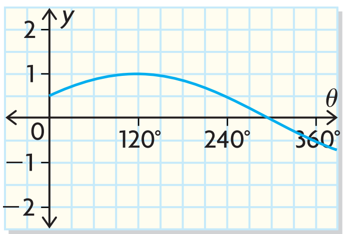
5. Match each function to its corresponding graph.

a) $y = \sin(2\theta - 90°),\ 0° \le \theta \le 360°$

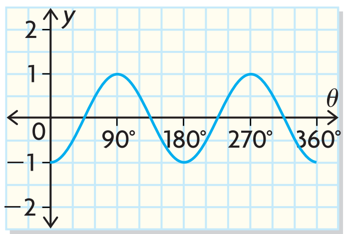
b) $y = \sin(3\theta - 90°),\ 0° \le \theta \le 360°$

c) $y = \sin\left(\frac{\theta}{2} + 30°\right),\ 0° \le \theta \le 360°$

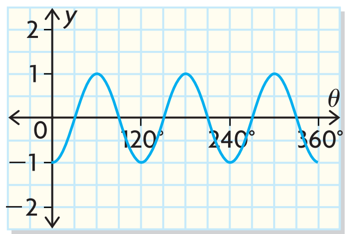
i)



ii)



iii)



6. If each function starts at $x = 0$ and finishes after three complete cycles, determine the period, amplitude, equation of the axis, domain, and range of each without graphing.

a) $y = 3\sin x + 2$

b) $g(x) = -4\cos(2x) + 7$

c) $h = -\frac{1}{2}\sin t - 5$

d) $h(x) = \cos(4(x - 12°)) - 9$

e) $d = 10\sin(180(t - 17°)) - 30$

f) $j(x) = 0.5\sin(2x - 30°)$

7. **K** Predict what the graph of each sinusoidal function will look like by describing the transformations of $y = \sin x$ or $y = \cos x$ that would result in the new graph. Sketch the graph, and then verify with a graphing calculator.

a) $y = 2\sin x + 3$

b) $y = -3\cos x + 5$

c) $y = -\sin(6x) + 4$

d) $y = 4\cos(2x) - 3$

e) $y = \frac{1}{2}\cos(3x - 120°)$

f) $y = -8\sin\left[\frac{1}{2}(x + 50°)\right] - 9$

8. Determine the appropriate WINDOW settings on your graphing calculator that enable you to see a complete cycle for each function. There is more than one acceptable answer.

a) $k(x) = -\sin(2x) + 6$

b) $j(x) = -5\sin\left(\frac{1}{2}x\right) + 20$

c) $y = 7\cos(90(x - 1°)) + 82$

d) $f(x) = \frac{1}{2}\sin(360x + 72°) - 27$

9. Each person's blood pressure is different. But there is a range of blood
A pressure values that is considered healthy. For a person at rest, the function $P(t) = -20\cos(300t)^\circ + 100$ models the blood pressure, $P(t)$, in millimetres of mercury at time t seconds.

a) What is the period of the function? What does the period represent for an individual?

b) What is the range of the function? Explain the meaning of the range in terms of a person's blood pressure.

10. **a)** Determine the equation of a sine function that would have the range
T $\{y \in \mathbf{R} \mid -1 \leq y \leq 7\}$ and a period of 720°.

b) Determine the equation of the cosine function that results in the same graph as your function in part (a).

11. Explain how you would graph the function $f(x) = -\frac{1}{2}\cos(120x) + 30$
C using transformations.

Extending

12. If the functions $y = \sin x$ and $y = \cos x$ are subjected to a horizontal compression of 0.5, what transformation would map the resulting sine curve onto the resulting cosine curve?

13. The function $D(t) = 4\sin\left[\frac{360}{365}(t - 80)\right]^\circ + 12$ is a model of the number of hours, $D(t)$, of daylight on a specific day, t, at latitude 50° north.

a) Explain why a trigonometric function is a reasonable model for predicting the number of hours of daylight.

b) How many hours of daylight do March 21 and September 21 have? What is the significance of each of these days?

c) How many hours of daylight do June 21 and December 21 have? What is the significance of each of these days?

d) Explain what the number 12 represents in the model.

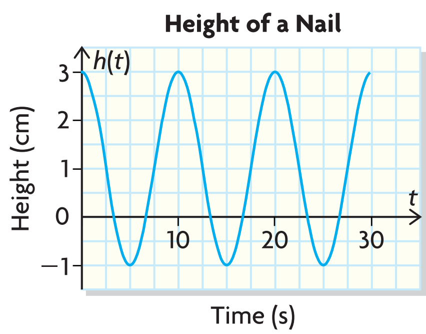
6.6 Investigating Models of Sinusoidal Functions

GOAL

Determine the equation of a sinusoidal function from a graph or a table of values.

LEARN ABOUT the Math

A nail located on the circumference of a water wheel is moving as the current pushes on the wheel. The height of the nail in terms of time can be modelled by the graph shown.



? How can you determine the equation of a sinusoidal function from its graph?

EXAMPLE 1 Representing a sinusoidal graph using the equation of a function

Determine an equation of the given graph.

Sasha's Solution

Horizontal compression factor: k

$$\text{period} = \frac{360}{|k|}$$

The period is 10 s.

$k > 0$, so $|k| = k$

I calculated the period, equation of the axis, and amplitude. Then I figured out how they are related to different transformations.

The period is 10 s since the peaks are 10 units apart. The horizontal stretch or compression factor k had to be positive because the graph was not reflected in the y-axis. I used the formula relating k to the period.

$$10 = \frac{360}{k}$$

$$k = \frac{360}{10}$$

$$k = 36$$

The graph was compressed by a factor of $\frac{1}{36}$.

Vertical translation: c

The axis is halfway between the maximum, 3, and the minimum, -1. This gave me the vertical translation and the value of c.

$$\text{equation of the axis} = \frac{\text{max} + \text{min}}{2}$$

$$= \frac{3 + (-1)}{2}$$

$$= 1 \text{ (vertical translation)}$$

$$c = 1$$

Vertical stretch: a

I calculated the amplitude by taking the maximum value, 3, and subtracting the axis, 1. Since the amplitude of $y = \cos x$ is 1, and the amplitude of this graph is 2, the vertical stretch is 2.

$a = 2$

Base graph: $y = \cos x$

The cosine curve is easier to use for my equation since the graph has its maximum on the y-axis, just as this graph does. This means that for a cosine curve, there isn't any horizontal translation, so $d = 0$. I found the equation of the function by substituting the values I calculated into $f(x) = a\cos(k(x - d)) + c$.

As a cosine curve:

$y = 2\cos(36x)^\circ + 1$

As a sine curve:

$y = 2\sin(36(x - 7.5))^\circ + 1$

I could have used the sine function instead.

A sine curve increases from a y-value of 0 at $x = 0$.

On this graph, that happens at 7.5. This means that, for a sine curve, there is a horizontal translation of 7.5, so $c = 7.5$.

For both functions, the domain is restricted to $x \geq 0$ because it represents the time elapsed.

Reflecting

A. Tanya says that another possible equation of the sinusoidal function created by Sasha is $y = 2\cos(36(x - 10))^\circ + 1$. Is she correct? Why or why not?

B. If the period on the original water wheel graph is changed from 10 to 20, what would be the new equation of the sinusoidal function?

C. If the maximum value on the original water wheel graph is changed from 3 to 5, what would be the new equation of the sinusoidal function?

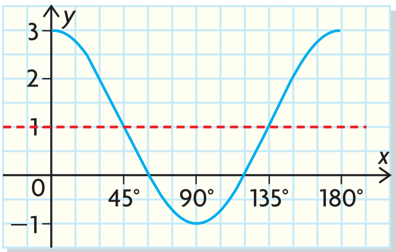
D. If the speed of the current increases so that the water wheel spins twice as fast, what would be the equation of the resulting function?

APPLY the Math

EXAMPLE 2 Connecting the equation of a sinusoidal function to its features

A sinusoidal function has an amplitude of 2 units, a period of 180°, and a maximum at (0, 3). Represent the function with an equation in two different ways.

Rajiv's Solution



The graph has a maximum at (0, 3) and a period of 180°, so the next maximum would be at (180, 3).

A minimum would be halfway between the two maximums.

Since the amplitude is 2, and $2 - 3 = -1$, the minimum would have to be at $(90°, -1)$.

Vertical translation: $c = 1$

The equation of the axis gave me the vertical translation. Since the equation is $y = 1$ instead of $y = 0$, there was a vertical translation of 1.

Vertical stretch: a

$$\text{amplitude} = 3 - 1 = 2$$
$$a = 2$$

The amplitude gives me the vertical stretch.

Horizontal compression: k

$$\text{period} = \frac{360°}{|k|}$$
$$180° = \frac{360°}{k}$$
$$k = \frac{360°}{180°}$$
$$k = 2$$

The period is 180°, so there has been a horizontal compression. Since there was no horizontal reflection, $k > 0$. To find k, I took 360° and divided it by the period.

Compression factor is $\frac{1}{2}$.

For a cosine curve:

No horizontal translation so

$d = 0$

Equation:

$y = 2\cos(2x) + 1$

Cosine curves have a maximum at $x = 0$, unless they've been translated horizontally. This curve starts at its maximum, so there would be no horizontal translation with a cosine function as a model.

I found the equation of the function by substituting the values into $f(x) = a\cos(k(x - d)) + c$.

For a sine curve:

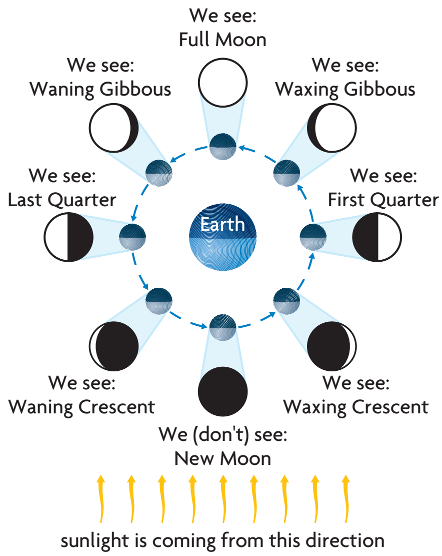
horizontal translation $= 135°$

$y = 2\sin(2(x - 135°)) + 1$

The equation of the axis of this cosine curve is $y = 1$. On this cosine curve, the point $(135°, 1)$ corresponds to the start of the cycle of the sine function. The sine curve with the same period and axis as this cosine curve has the equation $y = 2\sin(2x) + 1$, but its starting point is $(0°, 1)$. This means the function $y = 2\sin(2x) + 1$ must be translated horizontally to the right by 135°, so $d = 135°$.

EXAMPLE 3 Connecting data to the algebraic model of a sinusoidal function

The Moon is always half illuminated by the Sun. How much of the Moon we see depends on where it is in its orbit around Earth. The table shows the proportion of the Moon that was visible from Southern Ontario on days 1 to 74 in the year 2006.



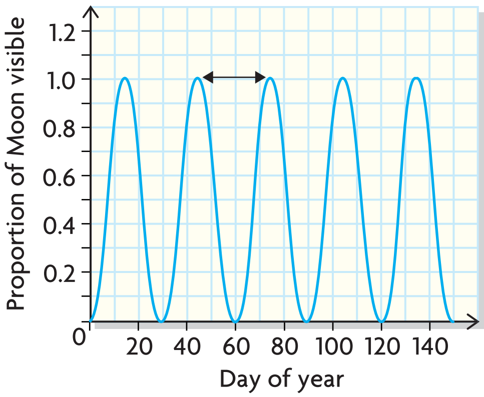
Day of Year	1	4	7	10	14	20	24	29	34
Proportion of Moon Visible	0.02	0.22	0.55	0.83	1.00	0.73	0.34	0.00	0.28

Day of Year	41	44	48	53	56	59	63	70	74
Proportion of Moon Visible	0.92	1.00	0.86	0.41	0.12	0.00	0.23	0.88	1.00

a) Determine the equation of the sinusoidal function that models the proportion of visible Moon in terms of time.
b) Determine the domain and range of the function.
c) Use the equation to determine the proportion of the Moon that is visible on day 110.

Rosalie's Solution

a) **Cycle of the Proportion of the Moon Visible**



I plotted the data. When I drew the curve, the graph looked like a sinusoidal function.

The maximum value was 1, and the minimum value 0.

The graph repeats every 30 days, so the period must be 30 days.

I figured out some of the important features of the sinusoidal function.

Vertical translation: c

Equation of the axis is $y = 0.5$.

$c = 0.5$

The axis is halfway between the maximum of 1 and the minimum of 0.

Vertical stretch: a

$$\text{amplitude} = \frac{(1 - 0)}{2}$$

The amplitude is the vertical distance between the maximum and the axis. In this case, it is 0.5, or $\frac{1}{2}$.

$$= 0.5 \quad \text{or} \quad \frac{1}{2}$$

$$a = 0.5$$

Horizontal compression: k

$$\text{period} = \frac{360}{|k|}$$

I used the period to get the compression.

$$k > 0, \text{ so } |k| = k$$

$$30 = \frac{360°}{k}$$

$$k = \frac{360°}{30}$$

$$k = 12$$

Horizontal translation: d

A sine curve or a cosine curve will work. I used the cosine curve. The horizontal translation is equal to the x-coordinate of a maximum, since $y = \cos x$ has a maximum at $x = 0$. I chose the x-coordinate of the maximum closest to the origin, $x = 14$.

Using a cosine curve:

$d = 14$

$$y = \frac{1}{2}\cos(12(x - 14)°) + 0.5$$

I put the information together to get the equation.

b) domain: $\{x \in \mathbf{R} \mid 0 \leq x \leq 365\}$

range: $\{y \in \mathbf{R} \mid 0 \leq y \leq 1\}$

The domain is only the non-negative values of x up to 365, since they are days of the year. The range is 0 to 1.

c) $y = \frac{1}{2}\cos(12(x - 14)°) + 0.5$

Since x represents the time in days, I substituted 110 for x in the equation to calculate the amount of the Moon visible at that time. Then I solved for y.

$$= \frac{1}{2}\cos(12(110 - 14)°) + 0.5$$

$$= \frac{1}{2}\cos(1152)° + 0.5$$

$$\doteq \frac{1}{2}(0.3090) + 0.5$$

$$= 0.65$$

On day 110, 65% of the Moon is exposed.

In Summary

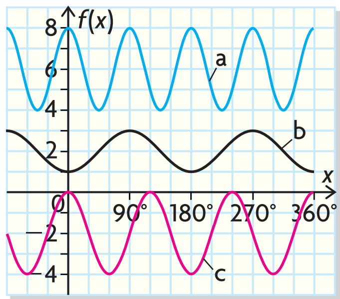
Key Idea

- If you are given a set of data and the corresponding graph is a sinusoidal function, then you can determine the equation by calculating the graph's period, amplitude, and equation of the axis. This information will help you determine the values of k, a, and c, respectively, in the equations $g(x) = a\sin(k(x - d)) + c$ and $h(x) = a\cos(k(x - d)) + c$. The value of d is determined by estimating the required horizontal shift (left or right) compared with the graph of the sine or cosine curve.

Need to Know

- If the graph begins at a maximum value, it may be easier to use the cosine function as your model.
- The domain and range of a sinusoidal model may need to be restricted for the situation you are dealing with.

CHECK Your Understanding



1. Determine an equation for each sinusoidal function at the right.

2. Determine the function that models the data in the table and does not involve a horizontal translation.

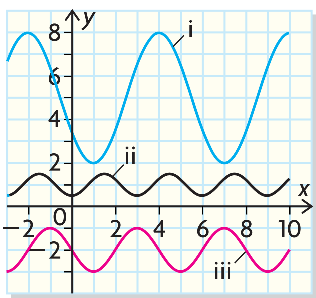
x	0°	45°	90°	135°	180°	225°	270°
y	9	7	5	7	9	7	5

3. A sinusoidal function has an amplitude of 4 units, a period of 120°, and a maximum at (0, 9). Determine the equation of the function.

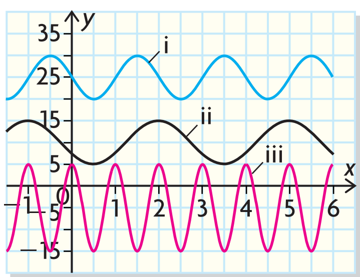
PRACTISING

4. Determine the equation for each sinusoidal function.

a)



b)



5. For each table of data, determine the equation of the function that is the simplest model.

a)

x	0°	30°	60°	90°	120°	150°	180°
y	3	2	1	2	3	2	1

b)

x	−180°	0°	180°	360°	540°	720°	900°
y	17	13	17	21	17	13	17

c)

x	−120°	−60°	0°	60°	120°	180°	240°
y	−4	−7	−4	−1	−4	−7	−4

d)

x	−20°	10°	40°	70°	100°	130°	160°
y	2	5	2	−1	2	5	2

6. **K** Determine the equation of the cosine function whose graph has each of the following features.

	Amplitude	Period	Equation of the Axis	Horizontal Translation
a)	3	360°	$y = 11$	0°
b)	4	180°	$y = 15$	30°
c)	2	40°	$y = 0$	7°
d)	0.5	720°	$y = -3$	−56°

7. A sinusoidal function has an amplitude of 6 units, a period of 45°, and a minimum at (0, 1). Determine an equation of the function.

8. **A** The table shows the average monthly high temperature for one year in Kapuskasing, Ontario.

Time (months)	J	F	M	A	M	J	J	A	S	O	N	D
Temperature (°C)	−18.6	−16.3	−9.1	0.4	8.5	13.8	17.0	15.4	10.3	4.4	−4.3	−14.8

a) Draw a scatter plot of the data and the curve of best fit. Let January be month 0.
b) What type of model describes the graph? Why did you select that model?
c) Write an equation for your model. Describe the constants and the variables in the context of this problem.
d) What is the average monthly temperature for month 20?

9. The table shows the velocity of air of Nicole's breathing while she is at rest.

Time (s)	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2	2.25	2.5	2.75	3
Velocity (L/s)	0	0.22	0.45	0.61	0.75	0.82	0.85	0.83	0.74	0.61	0.43	0.23	0

a) Explain why breathing is an example of a periodic function.
b) Graph the data, and determine an equation that models the situation.
c) Using a graphing calculator, graph the data as a scatter plot. Enter your equation and graph. Comment on the closeness of fit between the scatter plot and the graph.
d) What is the velocity of Nicole's breathing at 6 s? Justify.
e) How many seconds have passed when the velocity is 0.5 L/s?

Tech | *Support*

For help creating a scatter plot using a graphic calculator, see Technical Appendix, B-11.

10. The table shows the average monthly temperature for three cities: Athens, Lisbon, and Moscow.

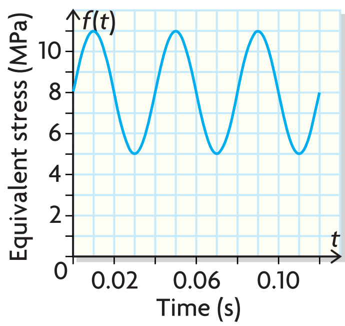
Time (month)	J	F	M	A	M	J	J	A	S	O	N	D
Athens (°C)	12	13	15	19	24	30	33	32	28	23	18	14
Lisbon (°C)	13	14	16	18	21	24	26	27	24	21	17	14
Moscow (°C)	−9	−6	0	10	19	21	23	22	16	9	1	−4

a) Graph the data to show that temperature is a function of time for each city.
b) Write the equations that model each function.
c) Explain the differences in the amplitude and the vertical translation for each city.
d) What does this tell you about the cities?

11. C The relationship between the stress on the shaft of an electric motor and time can be modelled with a sinusoidal function. (The units of stress are megapascals (MPa).)

a) Determine an equation of the function that describes the equivalent stress in terms of time.
b) What do the peaks of the function represent in this situation?
c) How much stress was the motor undergoing at 0.143 s?

Stress on a Motor Shaft



12. T Describe a procedure for writing the equation of a sinusoidal function based on a given graph.

Extending

13. The diameter of a car's tire is 60 cm. While the car is being driven, the tire picks up a nail. How high above the ground is the nail after the car has travelled 1 km?

14. Matthew is riding a Ferris wheel at a constant speed of 10 km/h. The boarding height for the wheel is 1 m, and the wheel has a radius of 7 m. What is the equation of the function that describes Matthew's height in terms of time, assuming Matthew starts at the highest point on the wheel?

6.7 Solving Problems Using Sinusoidal Models

GOAL

Solve problems related to real-world applications of sinusoidal functions.

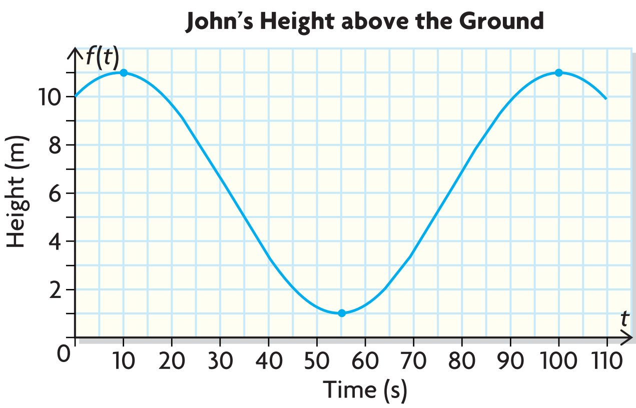
LEARN ABOUT the Math

A group of students is tracking a friend, John, who is riding a Ferris wheel. They know that John reaches the maximum height of 11 m at 10 s and then reaches the minimum height of 1 m at 55 s.

? How can you develop the equation of a sinusoidal function that models John's height above the ground to determine his height at 78 s?

EXAMPLE 1 Connecting the equation of a sinusoidal function to the situation

Justine's Solution



I plotted the two points I knew: (10, 11) and (55, 1).

Since it takes John 45 s to go from the highest point to the lowest, then it would take him 90 s to do one complete revolution and be back to a height of 11 m at 100 s.

I drew a smooth curve to connect the points to look like a wave.

Vertical translation: c

equation of the axis:

$$y = \frac{11 + 1}{2} = 6$$

$$c = 6$$

I found the equation of the axis by adding the maximum and minimum and dividing by 2. That gave me the vertical translation and the value of c.

Vertical stretch: a

amplitude $= 11 - 6 = 5$

$$a = 5$$

I found the amplitude by taking the maximum and subtracting the y-value for the equation of the axis. That gave me the vertical stretch and the value of a.

Horizontal compression: k

For the horizontal compression, I used the formula relating the period to k. The curve wasn't reflected, so k is positive.

$$\text{period} = \frac{360}{|k|}$$

$$k > 0, \text{ so the period} = \frac{360}{k}$$

$$90 = \frac{360}{k}$$

$$k = \frac{360}{90}$$

$$k = 4$$

Horizontal translation: d

If I use the cosine function, the first maximum is at $x = 0$. The first maximum of the new function is at $x = 10$. So there was a horizontal translation of 10. That gave me the value of d.

$d = 10$

$y = 5\cos(4(x - 10)^\circ) + 6$

I got the equation of the sinusoidal function by substituting the values I found into $y = a\cos(k(x - d)) + c$.

$$y = 5\cos(4(78 - 10)^\circ) + 6$$
$$= 5\cos 272^\circ + 6$$
$$y \doteq 5(0.035) + 6$$
$$\doteq 6.17 \text{ m}$$

Once I had the equation, I substituted $x = 78$, and solved for the height.

At 78 s, his height will be about 6.17 m.

The answer 6.17 m looks reasonable based on the graph.

Reflecting

A. If it took John 60 s instead of 90 s to complete one revolution, how would the sinusoidal function change? State the value and type of transformation associated with this change.

B. If the radius of the Ferris wheel remained the same but the axle of the wheel was 1 m higher, how would the sinusoidal function change? State the value and type of transformation associated with this change.

C. If both characteristics from parts A and B were changed, what would be the equation of the sinusoidal function describing John's height above the ground in terms of time?

APPLY the Math

EXAMPLE 2 Solving a problem involving a sinusoidal function

The top of a flagpole sways back and forth in high winds. The top sways 10 cm to the right $(+10 \text{ cm})$ and 10 cm to the left (-10 cm) of its resting position and moves back and forth 240 times every minute. At $t = 0$, the pole was momentarily at its resting position. Then it started moving to the right.

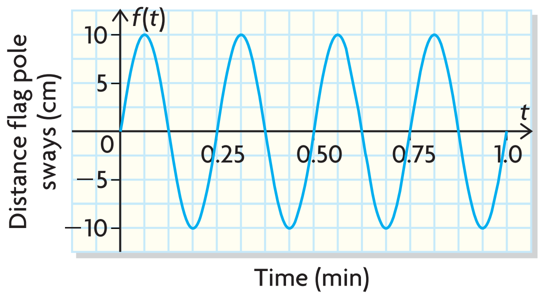
a) Determine the equation of a sinusoidal function that describes the distance the top of the pole is from its resting position in terms of time.

b) How does the situation affect the domain and range?

c) If the wind speed decreases slightly such that the sway of the top of the pole is reduced by 20%, what is the new equation of the sinusoidal function? Assume that the period remains the same.

Ryan's Solution

a)



I drew a graph where time is the independent variable, and the distance the top of the pole moves is the dependent variable.

The highest point on my graph will be 10, and the lowest will be -10.

I started at (0, 0) because the pole was at its resting position at $t = 0$.

$$\text{Number of sways each second} = \frac{240}{60} = 4$$

Since the pole sways back and forth 240 times in 60 s, the time to complete one sway must be 0.25 s. This is the period.

$$\text{period} = \frac{1}{4} = 0.25$$

Vertical translation: c

The axis is at $y = 0$. This gives the vertical translation.

equation of the axis: $y = 0$

so $c = 0$

Vertical stretch: a

I took the distance between a peak and the equation of the axis to get the amplitude.

amplitude $= 10$

so $a = 10$

Horizontal compression: k

I found the horizontal compression from the formula relating the period to the value of k.

$$\text{period} = \frac{360}{|k|}$$

$k > 0$

$$\text{period} = \frac{360}{|k|}$$

$$0.25 = \frac{360}{k}$$

$$k = \frac{360}{0.25}$$

$$k = 1440$$

The sine function:

I decided to use the sine function since this graph starts at $(0°, 0)$. Using the values of a and k, i determined the equation

$$y = 10 \sin(1440x)°$$

Horizontal translation: d

For the cosine function, the horizontal translation is equal to the x-coordinate of any maximum, since the maximum of a cosine function is at 0. I used the x-coordinate of the first maximum of the new function. That maximum is at $t = \frac{1}{16}$.

$$d = \frac{1}{16}$$

$$y = 10 \cos\left(1440\left(x - \frac{1}{16}\right)^{\circ}\right)$$

I put all these transformations together to get the equation of the function.

b) For either function, the domain is restricted to positive values because the values represent the time elapsed. The range of each function depends on its amplitudes.

c) 80% of 10

If the sway is the only thing that's changing, then the amplitude is going to change on the graph.
If the sway is reduced by 20%, it's 80% of what it used to be. The amplitude will then change from 10 to 8.
The vertical stretch is 8.

$$= 0.80 \times 10$$

$$= 8$$

$$y = 8 \cos\left(1440\left(x - \frac{1}{16}\right)^{\circ}\right)$$

or $\quad y = 8 \sin(1440x)°$

In Summary

Key Idea

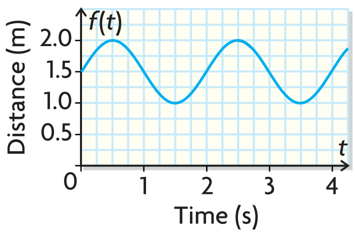
- Algebraic and graphical models of the sine and cosine functions can be used to solve a variety of real-world problems involving periodic behaviour.

Need to Know

- When you have a description of an event that can be modelled by a sinusoidal graph rather than data, it is useful to organize the information presented by drawing a rough sketch of the graph.
- You will have to determine the equation of the sinusoidal function by first calculating the period, amplitude, and equation of the axis. This information will help you determine the values of k, a, and c, respectively, in the equations $g(x) = a\sin(k(x - d)) + c$ and $h(x) = a\cos(k(x - d)) + c$.

CHECK Your Understanding

Distance Between the Tail Light and the Curb



1. The load on a trailer has shifted, causing the rear end of the trailer to swing left and right. The distance from one of the tail lights on the trailer to the curb varies sinusoidally with time. The graph models this behaviour.

a) What is the equation of the axis of the function, and what does it represent in this situation?

b) What is the amplitude of the function, and what does it represent in this situation?

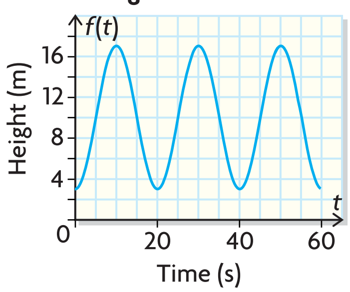
c) What is the period of the function, and what does it represent in this situation?

d) Determine the equation and the range of the sinusoidal function.

e) What are the domain and range of the function in terms of the situation?

f) How far is the tail light from the curve at $t = 3.2$ s?

Height above Ground



2. Don Quixote, a fictional character in a Spanish novel, attacked windmills because he thought they were giants. At one point, he got snagged by one of the blades and was hoisted into the air. The graph shows his height above ground in terms of time.

a) What is the equation of the axis of the function, and what does it represent in this situation?

b) What is the amplitude of the function, and what does it represent in this situation?

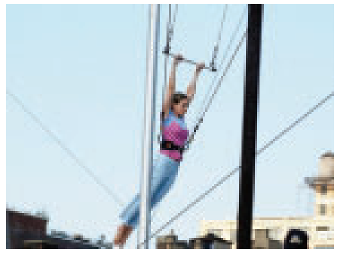
c) What is the period of the function, and what does it represent in this situation?

d) If Don Quixote remains snagged for seven complete cycles, determine the domain and range of the function.

e) Determine the equation of the sinusoidal function.

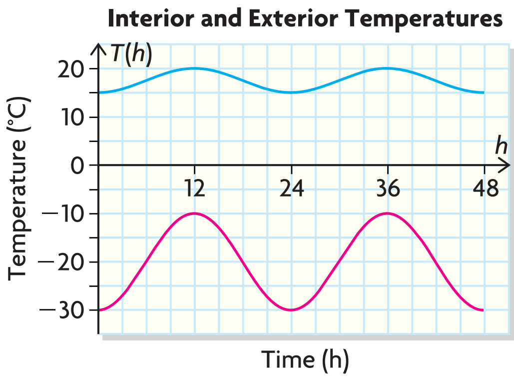
f) If the wind speed decreased, how would that affect the graph of the sinusoidal function?

3. Chantelle is swinging back and forth on a trapeze. Her distance from a vertical support beam in terms of time can be modelled by a sinusoidal function. At 1 s, she is the maximum distance from the beam, 12 m. At 3 s, she is the minimum distance from the beam, 4 m. Determine an equation of a sinusoidal function that describes Chantelle's distance from the vertical beam in relation to time.



PRACTISING

4. **K** The interior and exterior temperatures of an igloo were recorded over a 48 h period. The data were collected and plotted, and two curves were drawn through the appropriate points.



   a) How are these curves similar? Explain how each of them might be related to this situation.
   b) Describe the domain and range of each curve.
   c) Assuming that the curves can be represented by sinusoidal functions, determine the equation of each function.

5. **A** Skyscrapers sway in high-wind conditions. In one case, at $t = 2$ s, the top floor of a building swayed 30 cm to the left (-30 cm), and at $t = 12$, the top floor swayed 30 cm to the right $(+30 \text{ cm})$ of its starting position.
   a) What is the equation of a sinusoidal function that describes the motion of the building in terms of time?
   b) Dampers, in the forms of large tanks of water, are often added to the top floors of skyscrapers to reduce the severity of the sways. If a damper is added to this building, it will reduce the sway (not the period) by 70%. What is the equation of the new function that describes the motion of the building in terms of time?

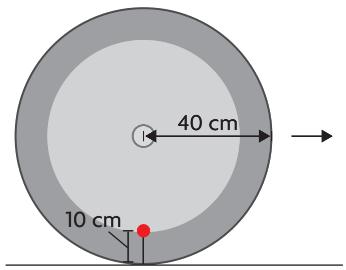
6. Milton is floating in an inner tube in a wave pool. He is 1.5 m from the bottom of the pool when he is at the trough of a wave. A stopwatch starts timing at this point. In 1.25 s, he is on the crest of the wave, 2.1 m from the bottom of the pool.
   a) Determine the equation of the function that expresses Milton's distance from the bottom of the pool in terms of time.

b) What is the amplitude of the function, and what does it represent in this situation?
c) How far above the bottom of the pool is Milton at $t = 4$ s?
d) If data are collected for only 40 s, how many complete cycles of the sinusoidal function will there be?
e) If the period of the function changes to 3 s, what is the equation of this new function?

7. An oscilloscope hooked up to an alternating current (AC) circuit shows a sine curve. The device records the current in amperes (A) on the vertical axis and the time in seconds on the horizontal axis. At $t = 0$ s, the current reads its first maximum value of 4.5 A. At $t = \frac{1}{120}$ s, the current reads its first minimum value of -4.5 A. Determine the equation of the function that expresses the current in terms of time.

8. Candice is holding onto the end of a spring that is attached to a lead ball. As she moves her hand slightly up and down, the ball moves up and down. With a little concentration, she can repeatedly get the ball to reach a maximum height of 20 cm and a minimum height of 4 cm from the top of a surface. The first maximum height occurs at 0.2 s, and the first minimum height occurs at 0.6 s.
a) Determine the equation of the sinusoidal function that represents the height of the lead ball in terms of time.
b) Determine the domain and range of the function.
c) What is the equation of the axis, and what does it represent in this situation?
d) What is the height of the lead ball at 1.3 s?

9. A paintball is shot at a wheel of radius 40 cm. The paintball leaves a circular mark 10 cm from the outer edge of the wheel. As the wheel rolls, the mark moves in a circular motion.



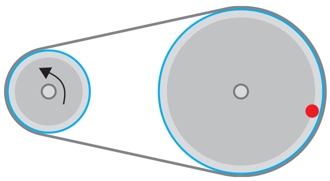
a) Assuming that the paintball mark starts at its lowest point, determine the equation of the sinusoidal function that describes the height of the mark in terms of the distance the wheel travels.
b) If the wheel completes five revolutions before it stops, determine the domain and range of the sinusoidal function.
c) What is the height of the mark when the wheel has travelled 120 cm from its initial position?

10. The population of rabbits, $R(t)$, and the population of foxes, $F(t)$, in a given
T region are modelled by the functions $R(t) = 10\,000 + 5000\cos(15t)°$ and $F(t) = 1000 + 500\sin(15t)°$, where t is the time in months. Referring to each graph, explain how the number of rabbits and the number of foxes are related.

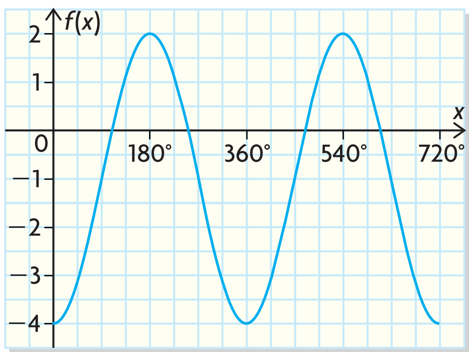
11. What information would you need to determine an algebraic or graphical
C model of a situation that could be modelled with a sinusoidal function?

Extending

12. Two pulleys are connected by a belt. Pulley A has a radius of 3 cm, and Pulley B has a radius of 6 cm. As Pulley A rotates, a drop of paint on the circumference of Pulley B rotates around the axle of Pulley B. Initially, the paint drop is 7 cm above the ground. Determine the equation of a sinusoidal function that describes the height of the drop of paint above the ground in terms of the rotation of Pulley A.



13. Examine the graph of the function $f(x)$.



a) Determine the equation of the function.
b) Evaluate $f(20)$.
c) If $f(x) = 2$, then which of the following is true for x?
  i) $180° + 360°k,\ k \in \mathbf{I}$ **iii)** $90° + 180°k,\ k \in \mathbf{I}$
  ii) $360° + 180°k,\ k \in \mathbf{I}$ **iv)** $270° + 360°k,\ k \in \mathbf{I}$
d) If $f(x) = -1$, then which of the following is true for x?
  i) $180° + 360°k,\ k \in \mathbf{I}$ **iii)** $90° + 360°k,\ k \in \mathbf{I}$
  ii) $360° + 90°k,\ k \in \mathbf{I}$ **iv)** $90° + 180°k,\ k \in \mathbf{I}$

14. Using graphing technology, determine x when $f(x) = 7$ for the function $f(x) = 4\cos(2x) + 3$ in the domain $\{x \in \mathbf{R} \mid 0° \le x \le 360°\}$.

Curious Math

Music

YOU WILL NEED

- graphing calculator

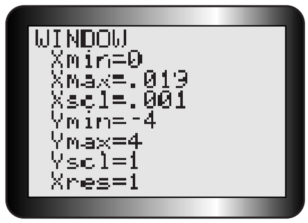
Pressing certain piano keys at the same time produces consonance, or pleasant sounds. Some combinations of keys produce dissonance, or unpleasant sounds.

When you strike a key, a string vibrates, causing the air to vibrate. This vibration of air produces a sound wave that your ear detects. The sound waves caused by striking various notes can be described by the functions in the table, where x is time in seconds and $f(x)$ is the displacement (or movement) of air molecules in micrometres $(1 \times 10^{-6}\text{ m})$.

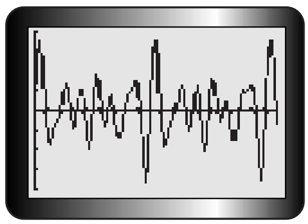
Equations for Notes (n.o. means next octave)

Note	Equation	Note	Equation	Note	Equation
A	$f(x) = \sin(158\,400x)°$	D	$f(x) = \sin(211\,427x)°$	G	$f(x) = \sin(282\,239x)°$
A#	$f(x) = \sin(167\,831x)°$	D#	$f(x) = \sin(224\,026x)°$	G#	$f(x) = \sin(299\,015x)°$
B	$f(x) = \sin(177\,806x)°$	E	$f(x) = \sin(237\,348x)°$	A n.o.	$f(x) = \sin(316\,800x)°$
C	$f(x) = \sin(188\,389x)°$	F	$f(x) = \sin(251\,465x)°$	B n.o.	$f(x) = \sin(355\,612x)°$
C#	$f(x) = \sin(199\,584x)°$	F#	$f(x) = \sin(266\,402x)°$	C n.o.	$f(x) = \sin(376\,777x)°$

One combination of notes is the A major chord, which is made up of A, C#, E, and A in the next octave. The sound can be modelled by graphing the sum of the equations for each note in Y1 using the WINDOW settings shown.



A major



1. Is the function for the A major chord periodic, sinusoidal, or both?
2. The C major chord is made up of C, E, G, and C in the next octave. Graph this function using your graphing calculator. Sketch the graph in your notebook. Compare the C major graph with the A major graph.
3. If you strike the keys A, B, C#, and F, the sound will be dissonance rather than consonance. Graph the function for this series of notes using your graphing calculator. Sketch the resulting curve. Compare with the C major and the A major graphs.
4. Graph and sketch each combination of notes below using your graphing calculator and the WINDOW settings shown above. Which combinations display consonance and which display dissonance?
   a) CC (C in first octave, C in next octave)
   b) CF
   c) CD
   d) CB (B in next octave)

6 Chapter Review

FREQUENTLY ASKED Questions

Q: How do you use transformations to determine the domain and range of a sinusoidal function?

A: The domain of a sinusoidal function is $\{x \in \mathbf{R}\}$. A restriction in the domain can occur when you consider the real-world situation you are trying to model.

To determine the range, you must determine the equation of the axis, based on the vertical translation. You then determine the amplitude, based on the vertical stretch or compression. Determine the equation of the axis, and then go above and below that value an amount equivalent to the amplitude. For example, if the equation of the axis is $y = 7$ and the amplitude is 3, then the range would be $\{y \in \mathbf{R} \mid 4 \leq y \leq 10\}$.

Study *Aid*

- See Lesson 6.5, Example 2.
- Try Chapter Review Question 11.

Q: How do you determine the equation of a sinusoidal function from its graph?

A:
1. Use the formula

$$y = \frac{\text{maximum} + \text{minimum}}{2}$$

to determine the equation of the axis, which is equivalent to the vertical translation and the value of c.

2. Use the formula amplitude = maximum − axis to determine the amplitude of the function, which is equivalent to the vertical stretch or compression and the value of a. If the graph is reflected in the x-axis, then a is negative.

3. Use the formula

$$\text{period} = \frac{360°}{|k|}$$

to determine the horizontal stretch or compression, $\frac{1}{|k|}$.

4. Determine the horizontal translation. It is often easier to transform the function $y = \cos x$ than to transform $y = \sin x$ because, in many questions, it is easier to identify the coordinates of the peak of the function rather than points on the axis. If you are transforming $y = \cos x$, the horizontal translation is equivalent to the x-coordinate of any maximum. Determining this gives you the value of d.

5. Incorporate all the transformations into the equation $y = a\cos(k(x - d)) + c$ or $y = a\sin(k(x - d)) + c$.

Study *Aid*

- See Lesson 6.6, Example 1.
- Try Chapter Review Question 12.

PRACTICE Questions

Lesson 6.1

1. The automatic dishwasher in a school cafeteria runs constantly through lunch. The table shows the amount of water in the dishwasher at different times.

Time (min)	0	1	2	3	4	5	6	7
Volume (L)	0	16	16	16	16	16	0	16

Time (min)	8	9	10	11	12	13	14	15
Volume (L)	16	16	0	16	16	16	16	16

Time (min)	16	17	18	19	20
Volume (L)	0	16	16	16	0

   a) Plot the data, and draw the resulting graph.
   b) Is the graph periodic?
   c) What is the period of the function, and what does it represent in this situation?
   d) Determine the equation of the axis.
   e) Determine the amplitude.
   f) What is the range of this function?

2. Sketch a graph of a periodic function whose period is 20 and whose range is $\{y \in \mathbf{R} | 3 \leq y \leq 8\}$.

Lesson 6.2

3. Sketch the graph of a sinusoidal function that has a period of 6, an amplitude of 4, and whose equation of the axis is $y = -2$.

4. Colin is on a unique Ferris wheel: it is situated on the top of a building. Colin's height above the ground at various times is recorded in the table.

Time (s)	0	10	20	30	40	50
Height (m)	25	22.4	16	9.7	7	9.7

Time (s)	60	70	80	90	100	110
Height (m)	16	22.4	25	22.4	16	9.7

Time (s)	120	130	140	150	160
Height (m)	7	9.7	16	22.4	25

   a) What is the period of the function, and what does it represent in this situation?
   b) What is the equation of the axis, and what does it represent in this situation?
   c) What is the amplitude of the function, and what does it represent in this situation?
   d) Was the Ferris wheel already in motion when the data were recorded? Explain.
   e) How fast is Colin travelling around the wheel, in metres per second?
   f) What is the range of the function?
   g) If the building is 6 m tall, what was Colin's boarding height in terms of the building?

Lesson 6.3

5. a) Graph the function $h(x) = 4\cos(3x) + 9$ using a graphing calculator in DEGREE mode for $0° \leq x \leq 360°$. Use Xscl $= 90°$. Determine the period, equation of the axis, amplitude, and the range of the function.
   b) Is the function sinusoidal?
   c) Calculate $h(45)$.
   d) Determine the values of x, $0° \leq x \leq 360°$, for which $h(x) = 5$.

6. A ship is docked in port and rises and falls with the waves. The function $d(t) = 2\sin(30t)° + 5$ models the depth of the propeller, $d(t)$, in metres at t seconds. Graph the function using a graphing calculator, and answer the following questions.
   a) What is the period of the function, and what does it represent in this situation?
   b) If there were no waves, what would be the depth of the propeller?
   c) What is the depth of the propeller at $t = 5.5$ s?
   d) What is the range of the function?
   e) Within the first 10 s, at what times is the propeller at a depth of 3 m?

7. Determine the coordinates of the image point after a rotation of 25° about (0, 0) from the point (4, 0).

Lesson 6.4

8. Each sinusoidal function has undergone one transformation that may have affected the period, amplitude, or equation of the axis of the function. In each case, determine which characteristic has been changed. If one has, indicate its new value.

a) $y = \sin x - 3$
b) $y = \sin(4x)$
c) $y = 7 \cos x$
d) $y = \cos(x - 70°)$

Lesson 6.5

9. Use transformations to graph each function for $0° \leq x \leq 360°$.
a) $y = 5 \cos(2x) + 7$
b) $y = -0.5 \sin(x - 30°) - 4$

10. Determine the range of each sinusoidal function without graphing.
a) $y = -3 \sin(4x) + 2$
b) $y = 0.5 \cos(3(x - 40°))$

Lesson 6.6

11. The average daily maximum temperature in Kenora, Ontario, is shown for each month.

Time (months)	J	F	M	A
Temperature (°C)	−13.1	−9.0	−1.1	8.5

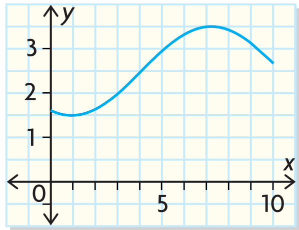
Time (months)	M	J	J	A
Temperature (°C)	16.8	21.6	24.7	22.9

Time (months)	S	O	N	D
Temperature (°C)	16.3	9.3	−1.2	−10.2

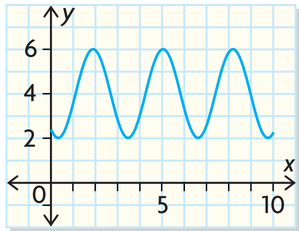
a) Prepare a scatter plot of the data. Let January represent month 0.
b) Draw a curve of good fit. Explain why this type of data can be expressed as a periodic function.
c) State the maximum and minimum values.
d) What is the period of the curve? Explain why this period is appropriate within the context of the question.
e) Write an equation for the axis of the curve.
f) What is the phase shift if the cosine function acts as the base curve?
g) Use the cosine function to write an equation that models the data.
h) Use the equation to predict the temperature for month 38. How can the table be used to confirm this prediction?

12. Determine the sine function $y = a \sin k(\theta - d) + c$ for each graph.

a)



b)

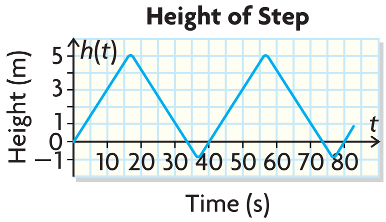


Lesson 6.7

13. Meagan is sitting in a rocking chair. The distance, $d(t)$, between the wall and the rear of the chair varies sinusoidally with time t. At $t = 1$ s, the chair is closest to the wall and $d(1) = 18$ cm. At $t = 1.75$ s, the chair is farthest from the wall and $d(1.75) = 34$ cm.
a) What is the period of the function, and what does it represent in this situation?
b) How far is the chair from the wall when no one is rocking in it?
c) If Meagan rocks back and forth 40 times only, what is the domain of the function?
d) What is the range of the function in part (c)?
e) What is the amplitude of the function, and what does it represent in this situation?
f) What is the equation of the sinusoidal function?
g) What is the distance between the wall and the chair at $t = 8$ s?

14. Summarize how you can determine the equation of a sinusoidal function that represents real phenomena from data, a graph, or a description of the situation. In your summary, explain how each part of the equation relates to the characteristics of the graph.

Chapter Self-Test



1. Steven is monitoring the height of one particular step on an escalator that takes passengers from the ground level to the second floor. The height of the step in terms of time can be modelled by the graph shown.

a) What is the period of the function, and what does it represent in this situation?

b) Determine the equation of the axis for this periodic function.

c) What do the peaks of the periodic function represent in this situation?

d) State the range of the function.

e) If the escalator completes only 10 cycles before being shut down, what is the domain of the periodic function?

f) Steven states that the stair will be at ground level at $t = 300$ s. Is he correct? Justify your answer.

2. Sketch a sinusoidal function that passes through $(0, -4)$ and has a period of 20, an amplitude of 3, and an equation of the axis $y = -1$.

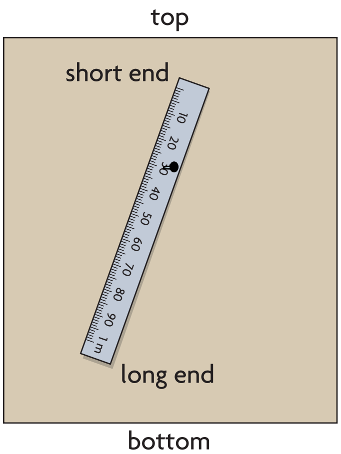
3. Determine the coordinates of the point after a rotation of 65° about (0, 0) from the point (7, 0).

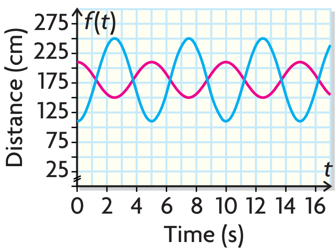
4. a) Graph $f(x) = -4\cos(0.5(x + 90°)) - 6$ using transformations of $f(x) = \cos x$.

b) State the amplitude, period, and equation of the axis.

c) Calculate $f(135°)$.

d) Determine the range of $f(x)$.





5. Keri has drilled a hole at the 30 cm mark in a metre stick. She then nails the metre stick onto a piece of plywood, through the hole. If she rotates the stick at a constant rate, then the distance from its long end to the top of the plywood can be modelled by the function in blue in the graph shown. If she rotates the stick at the same constant rate, then the distance from its short end to the top of the plywood can be modelled by the function in red.

a) What do the troughs of the sinusoidal functions represent in this situation?

b) How do the periods of the sinusoidal functions compare? Why is this so?

c) How far is the nail from the top of the plywood?

d) What is the amplitude of each sinusoidal function, and what does it represent in this situation?

e) What is the range of each sinusoidal function?

f) If Keri rotates the metre stick five complete revolutions, what is the domain of the sinusoidal function?

g) Determine the equation of each sinusoidal function.

h) What is the distance between the short end of the metre stick and the top of the plywood at $t = 19$ s?

6 Chapter Task

Cylinders and Sinusoidal Functions

? Can sinusoidal functions be obtained from cylinders?

As you follow the instructions, complete the table.

Cylinder	Radius of the Cylinder	Circumference of the Cylinder	Height of Point D	Height of Point E	Equation of the Resulting Sinusoidal Function
1					
2					
3					

A. Select one of the cylindrical objects. Determine its circumference, and record the measurement in the table. Take a sheet of paper, wrap it around the cylinder, and tape it in place. Make sure that the paper is narrow enough that the top portion of the cylinder is still exposed.

B. Mark a point D along the seam of the paper, somewhere near the top of the paper. Record the position in the table.

C. Mark a point E on the opposite side of the cylinder at least 4 cm below the height of point D. Record the position in the table.

D. Draw a line around the cylinder connecting points D and E and continue back to D.

E. Remove the cylindrically shaped object, leaving the tube of paper. Using scissors, cut along the line you drew.

F. Remove the tape and unroll the paper.

G. Determine an equation that models the resulting curve. Record the equation in the table.

H. Repeat this procedure two more times using the other cylindrical objects, marking the points D and E in different locations.

a) What is the relationship between the circumference of the cylinder and the resulting sinusoidal function?

b) What effect does changing the locations of points D and E have on the resulting sinusoidal function?

c) If you wanted to see three complete cycles on the paper, what would have to be included in the instructions?

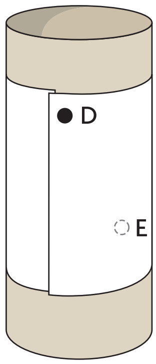
d) How could you do a similar activity and create a function that was periodic but not sinusoidal?

e) If the period of the resulting sinusoidal function was 69.12 cm, calculate the radius of the cylinder.

f) Another cylinder has a radius of 7 cm, point D at 12 cm high, and point E at 8 cm high. Determine the equation of the resulting sinusoidal function.

YOU WILL NEED

- three cylindrically shaped objects, for example, a pop can, a wooden dowel, and an empty paper towel roll
- 216 × 279 mm (letter-size) paper
- tape
- scissors



Task *Checklist*

- ✔ Did you show and explain the steps you used to determine the equations?
- ✔ Did you support your choice of data used to determine each equation?
- ✔ Did you explain your thinking clearly when answering the questions asked in part H?

Chapters 4–6 Cumulative Review

Multiple Choice

1. Which of the following expressions has a value of -7?

a) $25^{\frac{1}{2}} + 16^{\frac{3}{4}}$

b) $8^{\frac{2}{3}} - 81^{\frac{3}{4}} + 4^2$

c) $8^{-\frac{3}{4}} - 81^{-\frac{3}{4}} + 8^{-3}$

d) $81^{-\frac{3}{4}} + 16^{-\frac{3}{4}} - 16^{-\frac{1}{2}}$

2. Identify the expressions that are true when $x = 2$.

a) $3^{2x-1} = 27$

b) $6^{2x-3} = \sqrt{6}$

c) $5^{3x+2} = \frac{1}{5}$

d) $(2^{2x})(2^{x-1}) = 32$

3. Identify the expression that simplifies to 1.

a) $(a^{10+2p})(a^{-p-8})$

b) $(2x^2)^{3-2m}\left(\frac{1}{x}\right)^{2m}$

c) $[(c)^{2n-3m}](c^3)^m \div (c^2)^n$

d) $\left[(x^{4n-m})\left(\frac{1}{x}\right)\right]^6$

4. The population of a town is growing at an average rate of 5% per year. In 2000, its population was 15 000. What is the best estimate of the population in 2020 if the town continues to grow at this rate?

a) 40 000 **c)** 35 000

b) 30 000 **d)** 45 000

5. Point $P(-7, 24)$ is on the terminal arm of an angle in standard position. What is the measure of the related acute angle and the principal angle to the nearest degree?

a) 74° and 106° **c)** 16° and 164°

b) 16° and 344° **d)** 74° and 286°

6. What is the exact value of csc 300°?

a) $\frac{\sqrt{3}}{2}$ **b)** $\frac{2}{\sqrt{3}}$ **c)** $-\frac{2\sqrt{3}}{3}$ **d)** $\frac{1}{2}$

7. Which equation is not an identity?

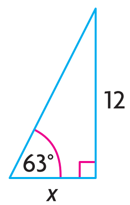
a) $(1 - \tan^2\theta)(1 - \cos^2\theta) = \dfrac{\sin^2\theta - 4\sin^4\theta}{1 - \sin^2\theta}$

b) $\dfrac{\tan x \sin x}{\tan x + \sin x} = \dfrac{\tan x \sin x}{\tan x \sin x}$

c) $\dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin\theta\cos\theta} = 1 - \tan\theta$

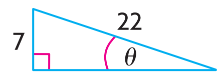
d) $\dfrac{\dfrac{1 - \cos x}{\cos x}}{\tan x} = \dfrac{1 - \cos x}{\sin x}$

8. What is the measure of x to the nearest unit?



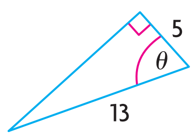
a) 4 **b)** 5 **c)** 6 **d)** 7

9. What is the measure of θ to the nearest degree?



a) 19° **b)** 22° **c)** 15° **d)** 27°

10. Which is the correct ratio for csc θ?

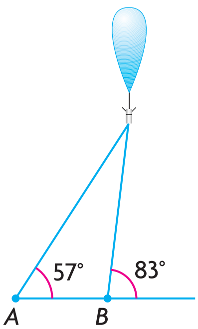


a) $\frac{5}{13}$ **b)** $\frac{13}{5}$ **c)** $\frac{13}{12}$ **d)** $\frac{12}{5}$

11. If $\tan\theta = \frac{4}{3}$ and θ lies in the third quadrant, which is the correct ratio for $\cos\theta$?

a) $\frac{4}{5}$ **b)** $-\frac{3}{5}$ **c)** $-\frac{4}{5}$ **d)** $\frac{3}{5}$

12. A weather balloon is spotted from two angles of elevation, 57° and 83°, from two different tracking stations. The tracking stations are 15 km apart. Determine the altitude of the balloon if the tracking stations and the point directly below the ballon lie along the same straight line.



a) 28.5 km **c)** 984 km
b) 32 km **d)** 23.7 km

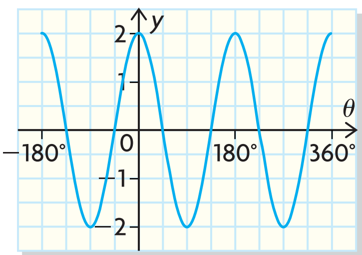
13. At a concert, a spotlight is placed at a height of 12.0 m. The spotlight beam shines down at an angle of depression of 35°. How far is the spotlight from the stage?

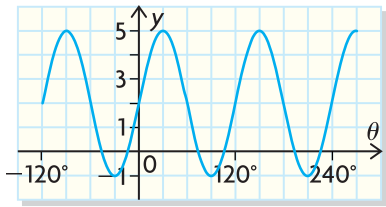
a) 20.9 m **c)** 25 m
b) 12.1 m **d)** 9.6 m

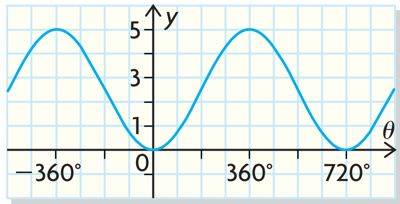
14. In $\triangle ABC$, $\angle A = 32°$, $\angle C = 81°$, and $a = 24.1$. Solve the triangle, and identify the correct solution.

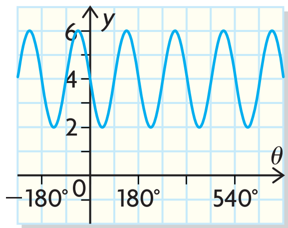
a) $\angle B = 125°$, $AC = 14.2$, $AB = 44.9$
b) $\angle B = 52°$, $AC = 41.9$, $AB = 44.9$
c) $\angle B = 107°$, $AC = 29.4$, $AB = 44.9$
d) $\angle B = 67°$, $AC = 41.9$, $AB = 44.9$

15. Which is the graph of $y = 2\cos 2(\theta + 45°) + 4$?

a)


b)


c)


d)


16. Refer to the graphs in question 15. Which is the graph of $y = 2\cos 2\theta$?

a) graph a) **c)** graph c)
b) graph b) **d)** graph d)

17. A sine function has an amplitude of 5, a period of 720°, and range $\{y \in \mathbf{R} \mid 2 \leq y \leq 12\}$. Identify the correct equation of this function.

a) $y = 5\sin 2\theta + 7$
b) $y = 5\sin 2\theta - 7$
c) $y = 5\sin 0.5\theta + 7$
d) $y = 5\sin 0.5\theta - 7$

18. Identify which of the following statements is true regarding sinusoidal functions of the form $y = a \sin(k(x - d)) + c$.

a) Changing the value of a affects the maximum and minimum values, the amplitude, and the range.

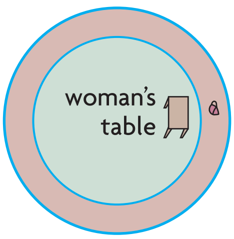
b) Changing the value of k affects the amplitude, the equation of the axis, and the domain and range.

c) Changing the value of c affects the period, the amplitude, or the domain.

d) Changing the value of d affects the period, the amplitude, and the equation of the axis.

19. A circular dining room at the top of a skyscraper rotates in a counterclockwise direction so that diners can see the entire city. A woman sits next to the window ledge and places her purse on the ledge as shown. Eighteen minutes later she realizes that her table has moved, but her purse is on the ledge where she left it. The coordinates of her position are $(x, y) = (20 \cos (7.5t)°, 20 \sin (7.5t)°)$, where t is the time in minutes and x and y are in metres. What is the shortest distance she has to walk to retrieve her purse?

a) 54.1 m **c)** 114.0 m
b) 37.0 m **d)** 62.9 m



20. Which of the following statements is not true about the graph of $y = \sin x$?

a) The period is 360°.

b) The amplitude is 1.

c) The equation of the axis is $y = 0$.

d) The range is $\{y \in \mathbf{R} \mid 0 < y < 1\}$.

21. A regular octagon is inscribed inside a circle with a radius of 14 cm. The perimeter is

a) 32.9 cm **c)** 85.7 cm
b) 56.0 cm **d)** 42.9 cm

22. In $\triangle ABC$, $\angle A = 85°$, $c = 10$ cm, and $b = 15$ cm. A possible height of $\triangle ABC$ is

a) 10.0 cm **c)** 13.8 cm
b) 8.6 cm **d)** 12.5 cm

23. The exact value of $\cos(-420°)$ is

a) $\frac{1}{2}$ **c)** $\frac{\sqrt{3}}{2}$
b) $-\frac{\sqrt{3}}{2}$ **d)** 1

24. Using the definitions $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, and $\tan \theta = \frac{y}{x}$, the simplified form of the expression $\dfrac{\sin^2 \theta + \cos^2 \theta}{\frac{\cos \theta}{\sin \theta}}$ is

a) $\frac{x}{y}$ **b)** $\frac{y}{x}$ **c)** $\frac{x}{r}$ **d)** $\frac{y}{r}$

25. The simplified form of the expression $\dfrac{\sin x \sin x}{(1 - \sin x)(1 + \sin x)}$ is

a) $\frac{\sin^2 x}{\cos x}$ **c)** $\tan^2 x$
b) $\frac{\sin^2 x}{\sin x}$ **d)** $\frac{\sin^2 x}{1 + \sin^2 x}$

26. The period of the function $y = \sin 4\theta$ in degrees is

a) 360° **b)** 180° **c)** 90° **d)** 1440°

27. $\left(\left(\frac{1}{a}\right)\left(\frac{1}{b^{-1}}\right)\right)^{-1}$ is equivalent to

a) $\frac{a}{b}$ **b)** $\frac{b}{a}$ **c)** $\frac{-a}{b}$ **d)** $\frac{-b}{a}$

28. If $3x^{\frac{1}{2}} = 12$, then x is equal to

a) 576 **b)** 64 **c)** 16 **d)** $\frac{1}{64}$

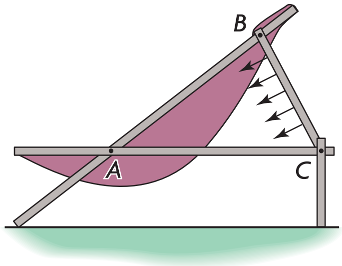
Investigations

29. The Paper Folding Problem

The Paper Folding Problem was a well-known challenge to fold paper in half more than seven or eight times, using paper of any size or shape. The task was commonly known to be impossible until April 2005, when Britney Gallivan solved it.

A sheet of letter paper is about 0.1 mm thick. On the third fold it is about as thick as your fingernail. On the 7th fold it is about as thick as a notebook. If it was possible to keep folding indefinitely, how many folds would be required to end up with a thickness that surpasses the height of the CN Tower, which is 553 m?

30. Lawn Chairs



The manufacturer of a reclining lawn chair would like to have the chair positioned at the following angles: 105°, 125°, 145°, 165°, and 175°. In the figure, AC is 75 cm and AB is 55 cm. Determine the positions for the notches on BC that will produce the required angles. Give a complete solution.

31. Dock Dilemma

The Arps recently bought a cottage on a small, sheltered inlet on Prince Edward Island. They wish to build a dock on an outcropping of level rocks. To determine the tide's effect at this position, they measured the depth of the water every hour over a 24 h period.

Time	1:00	2:00	3:00	4:00	5:00	6:00
Depth (m)	3.81	5.05	5.94	6.25	5.89	4.95

Time	7:00	8:00	9:00	10:00	11:00	12:00
Depth (m)	3.69	2.45	1.56	1.25	1.62	2.55

Time	13:00	14:00	15:00	16:00	17:00	18:00
Depth (m)	3.81	5.05	5.94	6.25	5.89	4.95

Time	19:00	20:00	21:00	22:00	23:00	24:00
Depth (m)	3.69	2.45	1.56	1.25	1.62	2.55

a) Graph the data, and determine an equation that models this situation over a 24 h period.
b) What is the maximum depth of the water at this location?
c) The hull of their boat must have a clearance of at least 1 m at all times. Is this location suitable for their dock? Explain.


1st generation
(1 ancestor)
2nd generation
(1 ancestor)
3rd generation
(2 ancestors)
4th generation
(3 ancestors)
5th generation
(5 ancestors)
6th generation
(8 ancestors)
drone
queen

Chapter 7

Discrete Functions: Sequences and Series

GOALS

You will be able to

- Identify and classify sequences
- Create functions for describing sequences and use the sequences to make predictions
- Investigate efficient ways to add the terms of a sequence
- Model real-life problems using sequences

? Unlike humans, honeybees don't always have two biological parents. Male drones have only one parent (a female queen), while a queen has two parents (a drone and a queen). How can you determine the number of ancestors a bee has over a given number of generations?

7 Getting Started

SKILLS AND CONCEPTS You Need

1. State the equation of each line.

a) slope $= -\frac{2}{5}$ and y-intercept $= 8$

b) slope $= -9$ and passing through $(5, 4)$

c) x-intercept $= 5$ and y-intercept $= -7$

d) passing through $(-12, 17)$ and $(5, -17)$

Study *Aid*

- For help with Question 2, see Essential Skills Appendix, A-7.

2. Evaluate.

a) $g(-2)$, for $g(x) = 3x^2 + x - 4$

b) $f\left(\frac{3}{4}\right)$, for $f(x) = \frac{4}{5}x + \frac{7}{10}$

c) $g(\sqrt{6})$, for $g(x) = 4x^2 - 24$

d) $h\left(\frac{1}{3}\right)$, for $h(x) = 64^x$

3. Calculate the 1st and 2nd differences to determine whether each relation is linear, quadratic, or neither.

a)

x	$f(x)$
0	18
1	23
2	28
3	33
4	38

b)

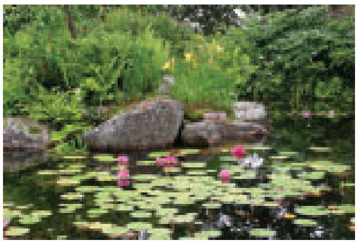
x	$f(x)$
0	6
6	12
12	24
18	48
24	96

c)

x	$f(x)$
0	5
1	7
2	13
3	23
4	37

Study *Aid*

- For help with Questions 5 and 6, see Lesson 4.7, Examples 2 and 3.



4. Solve each equation.

a) $2x - 3 = 7$

b) $5x + 8 = 2x - 7$

c) $5(3x - 2) + 7x - 4 = 2(4x + 8) - 2x + 3$

d) $-8x + \frac{3}{4} = \frac{1}{3}x - 12$

5. A radioactive material has a **half-life** of 100 years. If 2.3 kg of the substance is placed in a special disposal container, how much of the radioactive material will remain after 1000 years?

6. 0.1% of a pond is covered by lily pads. Each week the number of lily pads doubles. What percent of the pond will be covered after nine weeks?

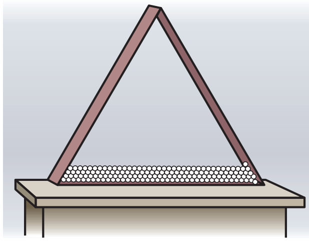
7. Complete the chart to show what you know about exponential functions.

Definition:	Rules/Method:
Exponential Functions	
Examples:	Non-examples:

APPLYING What You Know

Stacking Golf Balls

George's Golf Garage is having a grand-opening celebration. A display involves constructing a stack of golf balls within a large equilateral triangle frame on one of the walls. The base of the triangle will contain 40 golf balls, with each stacked row using one less ball than the previous row.



YOU WILL NEED

- counters or coins
- graphing calculator
- spreadsheet software (optional)

? How many golf balls are needed to construct the triangle?

A. Use counters or coins to construct a series of equilateral triangles with side lengths 1, 2, 3, 4, 5, 6, and 7, respectively. Record the total number of counters used to make each triangle in a table.

Side Length	Diagram	Number of Counters Used
1	•	1
2	• • •	3
3	• • • • • •	6
4		

B. Create a scatter plot of number of counters versus side length. Then determine the 1st differences between the numbers of counters used. What does this tell you about the type of function that models the number of balls needed to create an equilateral triangle?

C. How is the triangle with side length
- 4 related to the triangle with side length 2?
- 6 related to the triangle with side length 3?
- $2n$ related to the triangle with side length n?

D. Repeat part C for triangles with side lengths of
- 5 and 2
- 7 and 3
- $2n + 1$ and n

E. Use your rules from parts C and D to determine the number of golf balls in a triangle with side length 40.

7.1 Arithmetic Sequences

YOU WILL NEED

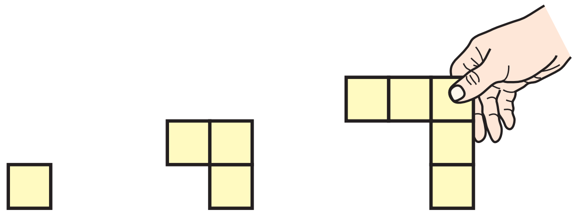
- linking cubes
- graphing calculator or graph paper
- spreadsheet software

GOAL

Recognize the characteristics of arithmetic sequences, and express the general terms in a variety of ways.

INVESTIGATE the Math

Chris used linking cubes to create different shapes. The first three shapes are shown. He wrote the sequence that represents the number of cubes in each shape.



? How many linking cubes are there in the 100th figure?

A. Create the next three terms of Chris's arithmetic sequence.

B. How is each term of this recursive sequence related to the previous term?

C. Construct a graph of term (number of cubes) versus figure number. What type of relation is this?

D. Determine a formula for the general term of the sequence.

E. Use the general term to calculate the 100th term.

Reflecting

F. Chris's sequence is an arithmetic sequence. Another arithmetic sequence is 635, 630, 625, 620, 615, ... How are the two sequences similar? Different?

G. How does the definition of an arithmetic sequence help you predict the shape of the graph of the sequence?

H. A recursive formula for Chris's sequence is $t_1 = 1$, $t_n = t_{n-1} + 2$, where $n \in \mathbf{N}$ and $n > 1$. How is this recursive formula related to the characteristics of Chris's arithmetic sequence?

sequence
an ordered list of numbers

term
a number in a sequence. Subscripts are usually used to identify the positions of the terms.

arithmetic sequence
a sequence that has the same difference, the **common difference**, between any pair of consecutive terms

recursive sequence
a sequence for which one term (or more) is given and each successive term is determined from the previous term(s)

general term
a formula, labelled t_n, that expresses each term of a sequence as a function of its position. For example, if the general term is $t_n = 2n$, then to calculate the 12th term (t_{12}), substitute $n = 12$.

$$\begin{aligned} t_{12} &= 2(12) \\ &= 24 \end{aligned}$$

recursive formula
a formula relating the general term of a sequence to the previous term(s)

APPLY the Math

EXAMPLE 1 Representing the general term of an arithmetic sequence

a) Determine a formula that defines the arithmetic sequence 3, 12, 21, 30, ...
b) State a formula that defines each term of any arithmetic sequence.

Wanda's Solution: Using Differences

a) $12 - 3 = 9$

I knew that the sequence is arithmetic, so the terms increase by the same amount. I subtracted t_1 from t_2 to determine the common difference.

$$\begin{aligned} t_n &= 3 + (n - 1)(9) \\ &= 3 + 9n - 9 \\ &= 9n - 6 \end{aligned}$$

I wrote this sequence as 3, 3 + 9, 3 + 2(9), 3 + 3(9), ... Each multiple of 9 that I added was one less than the position number. So for the nth term, I needed to add $(n - 1)$ 9s.

The general term is $t_n = 9n - 6$.

b) $a, a + d, (a + d) + d, (a + 2d) + d, \ldots$
$= a, a + d, a + 2d, a + 3d, \ldots$

I wrote an arithmetic sequence using a general first term, a, and a common difference, d. I simplified by collecting like terms.

General term:

$t_n = a + (n - 1)d$

Each multiple of d that I added was one less than the position number. So I knew that I had a formula for the general term.

Nathan's Solution: Using Multiples of 9

a) $12 = 3 + 9$

Since the sequence is arithmetic, to get each new term, I added 9 to the previous term.

$21 = 12 + 9$

$30 = 21 + 9$

$t_n = 9n$

Since I added 9 each time, I thought about the sequence of multiples of 9 because each term of that sequence goes up by 9s.

9, 18, 27, 36, ...

3, 12, 21, 30, ...

Each term of my sequence is 6 less than the term in the same position in the sequence of multiples of 9.

The general term is $t_n = 9n - 6$.

The general term of the sequence of multiples of 9 is $9n$, so I subtracted 6 to get the general term of my sequence.

b) $t_n = nd$

$d, 2d, 3d, 4d, \ldots$

Since I added the common difference *d* each time, I thought about the sequence of multiples of *d*.

$d + (a - d), 2d + (a - d), 3d + (a - d), 4d + (a - d), \ldots$

But the first term of my sequence was *a*, not *d*. So to get my sequence, I had to add *a* to, and subtract *d* from, each term of the sequence of multiples of *d*.

General term:

$$\begin{aligned} t_n &= nd + (a - d) \\ &= a + nd - d \\ &= a + (n - 1)d \end{aligned}$$

I simplified to get the general term.

Tina's Solution: Using a Recursive Formula

a) $12 = 3 + 9$

$21 = 12 + 9$

$30 = 21 + 9$

Since the sequence is arithmetic, to get each new term, I added 9 to the previous term.

The recursive formula is

$t_1 = 3, t_n = t_{n-1} + 9$, where $n \in \mathbf{N}$ and $n > 1$.

Since I added 9 each time, I expressed the general term of the sequence using a recursive formula.

b) $a, a + d, a + 2d, a + 3d, \ldots$

To get the terms of any arithmetic sequence, I would add *d* to the previous term each time, where *a* is the first term.

Recursive formula:

$t_1 = a, t_n = t_{n-1} + d$, where $n \in \mathbf{N}$ and $n > 1$

Once you know the general term of an arithmetic sequence, you can use it to determine *any* term in the sequence.

EXAMPLE 2 Connecting a specific term to the general term of an arithmetic sequence

What is the 33rd term of the sequence $18, 11, 4, -3, \ldots$?

David's Solution: Using Differences and the General Term

$11 - 18 = -7$

$4 - 11 = -7$

$-3 - 4 = -7$

I subtracted consecutive terms and found that each term is 7 less than the previous term. So the sequence is arithmetic.

$a = 18, d = -7$

$t_n = a + (n - 1)d$

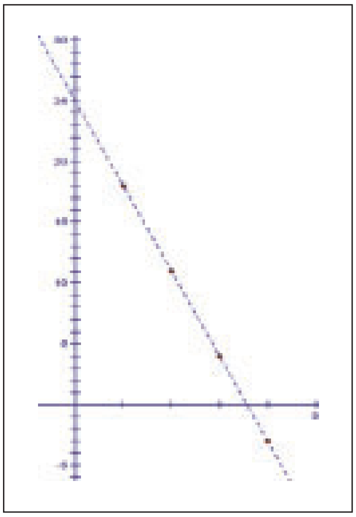
The first term of the sequence is 18. Since the terms are decreasing, the common difference is -7.

$$t_{33} = 18 + (33 - 1)(-7)$$
$$= -206$$

I substituted these numbers into the formula for the general term of an arithmetic sequence. To get the 33rd term, I let $n = 33$.

The 33rd term is -206.

Leila's Solution: Using a Graph and Function Notation



I represented the sequence as a function using ordered pairs with the term number (n) as the x-coordinate and the term (t_n) as the y-coordinate.

n	t_n	1st Differences
1	18	
		-7
2	11	
		-7
3	4	
		-7
4	-3	

The 1st differences are constant so these points lie on a line.

Since $n \in \mathbf{N}$ and the terms lie on this line, I used a dashed line to connect the points.

$f(x) = -7x + 25$

The slope of the line is $m = -7$ and the y-intercept is $b = 25$. I used this information to write the function that describes the line.

The y-intercept corresponds to the term t_0 but it is *not* a term of the sequence since $x \in \mathbf{N}$.

$$f(33) = -7(33) + 25$$
$$= -206$$

To get the 33rd term of the sequence, I substituted $x = 33$ into the equation $f(x) = -7x + 25$.

The 33rd term is -206.

Communication *Tip*

A dashed line on a graph indicates that the x-coordinates of the points on the line are natural numbers.

Arithmetic sequences can be used to model problems that involve increases or decreases that occur at a constant rate.

EXAMPLE 3 Representing an arithmetic sequence

Terry invests $300 in a GIC (guaranteed investment certificate) that pays 6% simple interest per year. When will his investment be worth $732?

Philip's Solution: Using a Spreadsheet

6% of $300 = $18

Terry earns 6% of $300, which is $18 interest per year.

	A	B
1	Year	Terry's $
2	1	$300.00
3	2	= B2+18
4		

I set up a spreadsheet. In one column I entered the year number, and in the other column, I entered the amount. I set up a formula to increase the amount by $18 per year.

	A	B
1	Year	Terry's $
2	1	$300.00
3	2	$318.00
4	3	$336.00
5	4	$354.00
6	5	$372.00
7	6	$390.00
8	7	$408.00
9	8	$426.00
10	9	$444.00
11	10	$462.00
12	11	$480.00
13	12	$498.00
14	13	$516.00
15	14	$534.00
16	15	$552.00
17	16	$570.00
18	17	$588.00
19	18	$606.00
20	19	$624.00
21	20	$642.00
22	21	$660.00
23	22	$678.00
24	23	$696.00
25	24	$714.00
26	25	$732.00

I used the spreadsheet to continue the pattern until the amount reached $732.

From the spreadsheet, Terry's investment will be worth $732 at the beginning of the 25th year.

Tech | *Support*

For help using a spreadsheet to enter values and formulas, fill down, and fill right, see Technical Appendix, B-21.

Jamie's Solution: Using the General Term

$t_n = a + (n - 1)d$

$t_n = 300 + (n - 1)(18)$

Terry earns 6% of $300, or $18 interest, per year. So his investment increases by $18/year. This is an arithmetic sequence, where $a = 300$ and $d = 18$.

$732 = 300 + (n - 1)(18)$

I needed to determine when $t_n = 732$.

$$732 = 300 + 18n - 18$$

I solved for n.

$$732 - 300 + 18 = 18n$$
$$450 = 18n$$
$$25 = n$$

Terry's investment will be worth $732 in the 25th year.

Suzie's Solution: Using Reasoning

$a = 300, d = 18$

Terry earns 6% of $300 = $18 interest per year. So the amount at the start of each year will form an arithmetic sequence.

$732 - 300 = 432$

I calculated the difference between the starting and ending values to know how much interest was earned.

$432 \div 18 = 24$

I divided by the amount of interest paid per year to determine how many interest payments were made.

The investment will be worth $732 at the beginning of the 25th year.

Since interest was paid every year except the first year, $732 must occur in the 25th year.

If you know two terms of an arithmetic sequence, you can determine *any* term in the sequence.

EXAMPLE 4 Solving an arithmetic sequence problem

The 7th term of an arithmetic sequence is 53 and the 11th term is 97. Determine the 100th term.

Tanya's Solution: Using Reasoning

$$\begin{aligned} t_{11} - t_7 &= 97 - 53 \\ &= 44 \end{aligned}$$

I knew that the sequence is arithmetic, so the terms increase by the same amount each time.

$$\begin{aligned} 4d &= 44 \\ d &= 11 \end{aligned}$$

There are four differences to go from t_7 to t_{11}. So I divided 44 by 4 to get the common difference.

$$\begin{aligned} t_{100} &= 97 + 89 \times 11 \\ &= 97 + 979 \\ &= 1076 \end{aligned}$$

Since the common difference is 11, I knew that to get the 100th term, I would have to add it to t_{11} 89 times.

The 100th term is 1076.

Deepak's Solution: Using Algebra

$$t_n = a + (n - 1)d$$

I knew that the sequence is arithmetic, so I wrote the formula for the general term.

t_7

$$\begin{aligned} 53 &= a + (7 - 1)d \\ 53 &= a + 6d \end{aligned}$$

t_{11}

$$\begin{aligned} 97 &= a + (11 - 1)d \\ 97 &= a + 10d \end{aligned}$$

For the 7th term, I substituted $t_7 = 53$ and $n = 7$ into the general term. For the 11th term, I substituted $t_{11} = 97$ and $n = 11$. Since both equations describe terms of the same arithmetic sequence, a and d are the same in both equations.

$$\begin{aligned} 97 &= a + 10d \\ -53 &= -(a + 6d) \\ \hline 44 &= 4d \\ 11 &= d \end{aligned}$$

The equations for t_7 and t_{11} represent a linear system. To solve for d, I subtracted the equation for t_7 from the equation for t_{11}.

$$53 = a + 6(11)$$
$$53 = a + 66$$
$$-13 = a$$

To solve for a, I substituted $d = 11$ into the equation for t_7.

$$t_n = a + (n - 1)d$$
$$t_{100} = -13 + (100 - 1)(11)$$
$$= 1076$$

To get the 100th term, I substituted $a = -13$, $d = 11$, and $n = 100$ into the formula for the general term.

The 100th term is 1076.

In Summary

Key Ideas

- Every sequence is a discrete function. Since each term is identified by its position in the list (1st, 2nd, and so on), the domain is the set of natural numbers, **N** = {1, 2, 3, ...}. The range is the set of all the terms of the sequence.
  For example, 4, 12, 20, 28, ...
  1st term ↗ 4, 2nd term ↖ 12

  Domain: {1, 2, 3, 4, ...}
  Range: {4, 12, 20, 28, ...}

- An arithmetic sequence is a recursive sequence in which new terms are created by adding the same value (the common difference) each time.
  For example, 2, 6, 10, 14, ... is increasing with a common difference of 4,
  +4 +4 +4

  $$t_2 - t_1 = 6 - 2 = 4$$
  $$t_3 - t_2 = 10 - 6 = 4$$
  $$t_4 - t_3 = 14 - 10 = 4$$
  $$\vdots$$

  and 9, 6, 3, 0, ... is decreasing with a common difference of -3.
  −3 −3 −3

  $$t_2 - t_1 = 6 - 9 = -3$$
  $$t_3 - t_2 = 3 - 6 = -3$$
  $$t_4 - t_3 = 0 - 3 = -3$$
  $$\vdots$$

Need to Know

- An arithmetic sequence can be defined
  - by the general term $t_n = a + (n - 1)d$,
  - recursively by $t_1 = a$, $t_n = t_{n-1} + d$, where $n > 1$, or
  - by a discrete linear function $f(n) = dn + b$, where $b = t_0 = a - d$.

  In all cases, $n \in$ **N**, a is the first term, and d is the common difference.

CHECK Your Understanding

1. Determine which sequences are arithmetic. For those that are, state the common difference.

a) 1, 5, 9, 13, 17, ...
b) 3, 7, 13, 17, 23, 27, ...
c) 3, 6, 12, 24, ...
d) 59, 48, 37, 26, 15, ...

2. State the general term and the recursive formula for each arithmetic sequence.

a) 28, 42, 56, ... **b)** 53, 49, 45, ... **c)** $-1, -111, -221, \ldots$

3. The 10th term of an arithmetic sequence is 29 and the 11th term is 41. What is the 12th term?

4. What is the 15th term of the arithmetic sequence 85, 102, 119, ...?

PRACTISING

5. i) Determine whether each sequence is arithmetic.
ii) If a sequence is arithmetic, state the general term and the recursive formula.

a) 8, 11, 14, 17, ...
b) 15, 16, 18, 19, ...
c) 13, 31, 13, 31, ...
d) 3, 6, 12, 24, ...
e) 23, 34, 45, 56, ...
f) $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots$

6. Determine the recursive formula and the general term for the arithmetic sequence in which

a) the first term is 19 and consecutive terms increase by 8
b) $t_1 = 4$ and consecutive terms decrease by 5
c) the first term is 21 and the second term is 26
d) $t_4 = 35$ and consecutive terms decrease by 12

7. i) Determine whether each recursive formula defines an arithmetic sequence, where $n \in \mathbf{N}$ and $n > 1$.
ii) If the sequence is arithmetic, state the first five terms and the common difference.

a) $t_1 = 13, t_n = 14 + t_{n-1}$
b) $t_1 = 5, t_n = 3t_{n-1}$
c) $t_1 = 4, t_n = t_{n-1} + n - 1$
d) $t_1 = 1, t_n = 2t_{n-1} - n + 2$

8. For each arithmetic sequence, determine

K **i)** the general term **ii)** the recursive formula **iii)** t_{11}

a) 35, 40, 45, ...
b) 31, 20, 9, ...
c) $-29, -41, -53, \ldots$
d) 11, 11, 11, ...
e) $1, \frac{6}{5}, \frac{7}{5}, \ldots$
f) 0.4, 0.57, 0.74, ...

9. **i)** Determine whether each general term defines an arithmetic sequence.
ii) If the sequence is arithmetic, state the first five terms and the common difference.

a) $t_n = 8 - 2n$

b) $t_n = n^2 - 3n + 7$

c) $f(n) = \frac{1}{4}n + \frac{1}{2}$

d) $f(n) = \frac{2n + 5}{7 - 3n}$

10. **A** An opera house has 27 seats in the first row, 34 seats in the second row, 41 seats in the third row, and so on. The last row has 181 seats.
a) How many seats are in the 10th row?
b) How many rows of seats are in the opera house?

11. Janice gets a job and starts out earning \$9.25/h. Her boss promises her a raise of \$0.15/h after each month of work. When will Janice start earning at least twice her starting wage?

12. Phil invests \$5000 in a high-interest savings account and earns 3.5% simple interest per year. How long will he have to leave his money in the account if he wants to have \$7800?

13. Determine the number of terms in each arithmetic sequence.
a) 7, 9, 11, 13, ... , 63
b) $-20, -25, -30, -35, \ldots, -205$
c) $31, 27, 23, 19, \ldots, -25$
d) 9, 16, 23, 30, ... , 100
e) $-33, -26, -19, -12, \ldots, 86$
f) $28, 19, 10, 1, \ldots, -44$

14. **T** You are given the 4th and 8th terms of a sequence. Describe how to determine the 100th term *without* finding the general term.

15. The 50th term of an arithmetic sequence is 238 and the 93rd term is 539. State the general term.

16. **C** Two terms of an arithmetic sequence are 20 and 50.
a) Create three different arithmetic sequences given these terms. Each of the three sequences should have a different first term and a different common difference.
b) How are the common differences related to the terms 20 and 50?

Extending

17. The first term of an arithmetic sequence is 13. Two other terms of the sequence are 37 and 73. The common difference between consecutive terms is an integer. Determine all possible values for the 100th term.

18. Create an arithmetic sequence that has $t_1 > 0$ and in which each term is greater than the previous term. Create a new sequence by picking, from the original sequence, the terms described by the sequence. (For example, for the sequence 3, 7, 11, 15, ... , you would choose the 3rd, 7th, 11th, 15th, ... terms of the original sequence as $t_1, t_2, t_3, t_4, \ldots$ of your new sequence.) Is this new sequence always arithmetic?

7.2 Geometric Sequences

YOU WILL NEED

- graphing calculator
- graph paper

GOAL

Recognize the characteristics of geometric sequences and express the general terms in a variety of ways.

INVESTIGATE the Math

A local conservation group set up a challenge to get trees planted in a community. The challenge involves each person planting a tree and signing up seven other people to each do the same. Denise and Lise both initially accepted the challenge.

? If the pattern continues, how many trees will be planted at the 10th stage?

geometric sequence

a sequence that has the same ratio, the **common ratio**, between any pair of consecutive terms

A. Create the first five terms of the geometric sequence that represents the number of trees planted at each stage.

B. How is each term of this recursive sequence related to the previous term?

C. Use a graphing calculator to graph the term (number of trees planted) versus stage number. What type of relation is this?

D. Determine a formula for the general term of the sequence.

E. Use the general term to calculate the 10th term.

Reflecting

F. The tree-planting sequence is a geometric sequence. Another geometric sequence is 1 000 000, 500 000, 250 000, 125 000, ... How are the two sequences similar? Different?

G. How is the general term of a geometric sequence related to the equation of its graph?

H. A recursive formula for the tree-planting sequence is $t_1 = 2, t_n = 7t_{n-1}$, where $n \in \mathbf{N}$ and $n > 1$. How is this recursive formula related to the characteristics of this geometric sequence?

APPLY the Math

EXAMPLE 1 Connecting a specific term to the general term of a geometric sequence

a) Determine the 13th term of a geometric sequence if the first term is 9 and the common ratio is 2.

b) State a formula that defines each term of any geometric sequence.

Leo's Solution: Using a Recursive Formula

a)

n	1	2	3	4	5	6	7
t_n	9	18	36	72	144	288	576

n	8	9	10	11	12	13
t_n	1152	2304	4608	9216	18 432	36 864

I knew that the sequence is geometric so the terms increase by the same multiple each time. I made a table starting with the first term, and I multiplied each term by 2 to get the next term until I got the 13th term.

The 13th term is 36 864.

b) $a, ar, (ar)r, (ar^2)r, \ldots$

To get the terms of any geometric sequence, I would multiply the previous term by r each time, where a is the first term.

Recursive formula:

$t_1 = a, t_n = rt_{n-1}$, where $n \in \mathbf{N}$ and $n > 1$

Tamara's Solution: Using Powers of *r*

a) $a = 9$

$r = 2$

I knew that the sequence is geometric with first term 9 and common ratio 2.

$t_{13} = 9 \times 2^{12}$

$= 36\ 864$

To get the 13th term, I started with the first term. Then I multiplied the common ratio 12 times.

The 13th term is 36 864.

b) $a, ar, (ar)r, (ar^2)r, \ldots$

$= a, ar, ar^2, ar^3, \ldots$

I wrote a geometric sequence using a general first term, a, and a common ratio, r. I simplified the terms.

General term:

$t_n = ar^{n-1}$ or $f(n) = ar^{n-1}$

Each time I multiplied by r, the result was one less than the position number. I recognized this as an exponential function, so I knew that I had a formula for the general term.

Geometric sequences can be used to model problems that involve increases or decreases that change exponentially.

EXAMPLE 2 Solving a problem by using a geometric sequence

A company has 3 kg of radioactive material that must be stored until it becomes safe to the environment. After one year, 95% of the radioactive material remains. How much radioactive material will be left after 100 years?

Jacob's Solution

$$3,\ 3 \times 0.95,\ (3 \times 0.95) \times 0.95,\ (3 \times 0.95^2) \times 0.95,\ \ldots$$
$$= 3,\ 3 \times 0.95,\ 3 \times 0.95^2,\ 3 \times 0.95^3,\ \ldots$$

Every year, 95% of the radioactive material remains. I represented the amount of radioactive material as a sequence. The terms show the amounts in each year.

$$a = 3$$
$$r = 0.95$$

The sequence is geometric with first term 3 and common ratio 0.95.

$$f(n) = ar^{n-1}$$

I wrote the formula for the general term.

$$f(100) = 3 \times 0.95^{100-1}$$
$$= 3 \times 0.95^{99}$$
$$\doteq 0.019$$

I needed to determine the value of $f(n)$ when $n = 100$. So I substituted $a = 3$, $r = 0.95$, and $n = 100$ into the formula.

After the 100th year, there will be about 19 g of radioactive material left.

EXAMPLE 3 Selecting a strategy to determine the number of terms in a geometric sequence

How many terms are in the geometric sequence 52 612 659, 17 537 553, ... , 11?

Suzie's Solution

$$a = 52\,612\,659$$
$$r = \frac{17\,537\,553}{52\,612\,659} = \frac{1}{3}$$

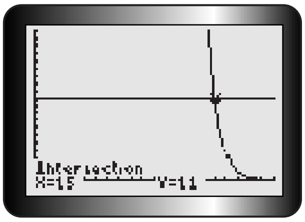
I knew that the sequence is geometric with first term 52 612 659. I calculated the common ratio by dividing t_2 by t_1.

$$f(n) = ar^{n-1}$$

I wrote the formula for the general term of a geometric sequence.

$$11 = 52\,612\,659 \times \left(\frac{1}{3}\right)^{n-1}$$

The last term of the sequence is 11, so its position number will be equal to the number of terms in the sequence. I determined the value of n when $f(n) = 11$ by substituting $a = 52\,612\,659$, $r = \frac{1}{3}$, and $f(n) = 11$ into the formula.



Instead of using guess and check to determine n, I graphed the functions Y1 = 52 612 659(1/3)^(X−1) and Y2 = 11 using my graphing calculator. Then I found the point of intersection. The x-coordinate represents the number of terms in the sequence.

There are 15 terms in the geometric sequence.

Tech Support

For help using a graphing calculator to determine the point of intersection of two functions, see Technical Appendix, B-12.

In Summary

Key Idea

- A geometric sequence is a recursive sequence in which new terms are created by multiplying the previous term by the same value (the common ratio) each time. For example, 2, 6, 18, 54, ... is increasing with a common ratio of 3,

  $\times 3 \quad \times 3 \quad \times 3$

$$\frac{t_2}{t_1} = \frac{6}{2} = 3$$
$$\frac{t_3}{t_2} = \frac{18}{6} = 3$$
$$\frac{t_4}{t_3} = \frac{54}{18} = 3$$
$$\vdots$$

  and 144, 72, 36, 18, ... is decreasing with a common ratio of $\frac{1}{2}$.

  $\times \frac{1}{2} \quad \times \frac{1}{2} \quad \times \frac{1}{2}$

$$\frac{t_2}{t_1} = \frac{72}{144} = \frac{1}{2}$$
$$\frac{t_3}{t_2} = \frac{36}{72} = \frac{1}{2}$$
$$\frac{t_4}{t_3} = \frac{18}{36} = \frac{1}{2}$$
$$\vdots$$

  If the common ratio is negative, the sequence has terms that alternate from positive to negative. For example, 5, −20, 80, −320, ... has a common ratio of −4.

  $\times (-4) \quad \times (-4) \quad \times (-4)$

Need to Know

- A geometric sequence can be defined
  - by the general term $t_n = ar^{n-1}$,
  - recursively by $t_1 = a$, $t_n = rt_{n-1}$, where $n > 1$, or
  - by a discrete exponential function $f(n) = ar^{n-1}$.

  In all cases, $n \in \mathbf{N}$, a is the first term, and r is the common ratio.

CHECK Your Understanding

1. Determine which sequences are geometric. For those that are, state the common ratio.
   a) 15, 26, 37, 48, ...
   b) 5, 15, 45, 135, ...
   c) 3, 9, 81, 6561, ...
   d) 6000, 3000, 1500, 750, 375, ...

2. State the general term and the recursive formula for each geometric sequence.
   a) 9, 36, 144, ...
   b) 625, 1250, 2500, ...
   c) 10 125, 6750, 4500, ...

3. The 31st term of a geometric sequence is 123 and the 32nd term is 1107. What is the 33rd term?

4. What is the 10th term of the geometric sequence 1 813 985 280, 302 330 880, 50 388 480, ...?

PRACTISING

5. i) Determine whether each sequence is geometric.
   ii) If a sequence is geometric, state the general term and the recursive formula.
   a) 12, 24, 48, 96, ...
   b) 1, 3, 7, 15, ...
   c) 3, 6, 9, 12, ...
   d) 5, -15, 45, -135, ...
   e) 6, 7, 14, 15, ...
   f) 125, 50, 20, 8, ...

6. For each geometric sequence, determine
   K i) the general term ii) the recursive formula iii) t_6
   a) 4, 20, 100, ...
   b) -11, -22, -44, ...
   c) 15, -60, 240, ...
   d) 896, 448, 224, ...
   e) 6, 2, $\frac{2}{3}$, ...
   f) 1, 0.2, 0.04, ...

7. i) Determine whether each sequence is arithmetic, geometric, or neither.
   ii) If a sequence is arithmetic or geometric, state the general term.
   a) 9, 13, 17, 21, ...
   b) 7, -21, 63, -189, ...
   c) 18, -18, 18, -18, ...
   d) 31, 32, 34, 37, ...
   e) 29, 19, 9, -1, ...
   f) 128, 96, 72, 54, ...

8. Determine the recursive formula and the general term for the geometric sequence in which
   a) the first term is 19 and the common ratio is 5
   b) $t_1 = -9$ and $r = -4$
   c) the first term is 144 and the second term is 36
   d) $t_1 = 900$ and $r = \frac{1}{6}$

9. **i)** Determine whether each recursive formula defines a geometric sequence, where $n \in \mathbf{N}$.

ii) If the sequence is geometric, state the first five terms and the common ratio.

a) $t_1 = 18$, $t_n = \left(\frac{2}{3}\right)^{n-1} t_{n-1}$, where $n > 1$

b) $t_1 = -8$, $t_n = -3t_{n-1}$, where $n > 1$

c) $t_1 = 123$, $t_n = \frac{t_{n-1}}{3}$, where $n > 1$

d) $t_1 = 10$, $t_2 = 20$, $t_n = 4t_{n-2}$, where $n > 2$

10. **i)** Determine whether each general term defines a geometric sequence, where $n \in \mathbf{N}$.

ii) If the sequence is geometric, state the first five terms and the common ratio.

a) $t_n = 4^n$

b) $t_n = 3^n + 5$

c) $f(n) = n^2 - 13n + 8$

d) $t_n = 7 \times (-5)^{n-4}$

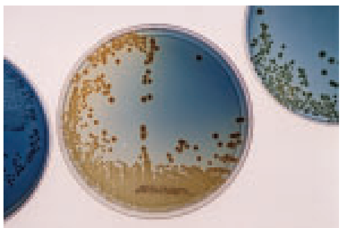
e) $f(n) = \frac{2}{3n + 1}$

f) $f(n) = \frac{11}{13^n}$

11. The 5th term of a geometric sequence is 45 and the 8th term is 360. Determine the 20th term.

12. A A doctor makes observations of a bacterial culture at fixed time intervals. The table below shows his first four observations. If the pattern continues, how many bacteria will be present at the 9th observation?

Observation	Number of Bacteria
1	5 120
2	7 680
3	11 520
4	17 280



13. Sam invested \$5000 in a GIC earning 8% compound interest per year. The interest gets added to the amount invested, so the next year Sam gets interest on the interest already earned, as well as on the original amount. How much will Sam's investment be worth at the end of 10 years?

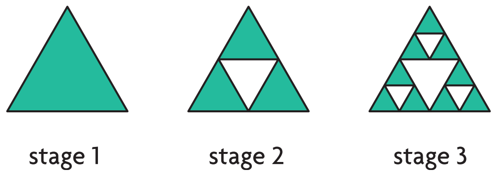
14. A certain antibiotic reduces the number of bacteria in your body by 10% each dose.

a) If four doses of the antibiotic are taken, what percent of the original bacterial population is left?

b) Biologists have determined that when a person has a bacterial infection, if the bacterial level can be reduced to 5% of its initial population, the person can fight off the infection. How many doses must be administered to reduce the bacterial population to the desired level?

15. You are given the 5th and 7th terms of a geometric sequence. Is it possible to determine the 29th term *without* finding the general term? If so, describe how you would do it.

16. **T** The Sierpinski gasket is a fractal created from an equilateral triangle. At each stage, the "middle" is cut out of each remaining equilateral triangle. The first three stages are shown.



a) If the process continues indefinitely, the stages get closer to the Sierpinski gasket. How many shaded triangles would be present in the sixth stage?

b) If the triangle in the first stage has an area of 80 cm^2, what is the area of the shaded portion of the 20th stage?

17. **C** In what ways are arithmetic and geometric sequences similar? Different?

Extending

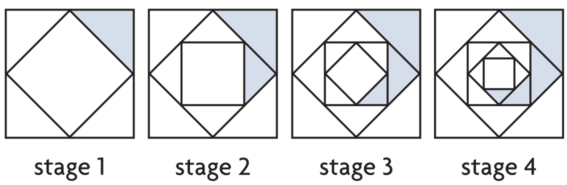
18. Given the geometric sequence with $t_1 = 1$ and $r = \frac{1}{2}$, calculate the sum of the first 1, 2, 3, and 4 terms. What would happen to the sum if you added more and more terms?

19. Determine the 10th term of the sequence 3, 10, 28, 72, 176, ... State the general term.

20. Is it possible for the first three terms of an arithmetic sequence to be equal to the first three terms of a geometric sequence? If so, provide an example.

21. Create an arithmetic sequence such that some of its terms form a geometric sequence. How is the geometric sequence related to the arithmetic sequence?

22. A square has a side length of 12 cm. The midpoints of the square are joined creating a smaller square and four triangles. If you continue this process, what will be the total area of the shaded region in stage 6?



7.3 Creating Rules to Define Sequences

GOAL

Create rules for generating sequences that are neither arithmetic nor geometric.

YOU WILL NEED

- circles (or squares) of paper of increasing size

LEARN ABOUT the Math

The Tower of Hanoi is a game played with three pegs and 10 discs of increasing size. At the start of the game, all of the discs are arranged in order of size on one of the pegs, with the smallest on top. The object of the game is to stack all the discs on a different peg in the same order of size as you started with in the fewest number of moves. The rules for moving discs are:

- You may move only one disc at a time.
- You may move only a disc that is alone on a peg, or one that is on top of a pile.
- You may place a disc only on an open peg, or on top of another disc that is larger than it.

start

end

? What is the minimum number of moves required to complete the game?

EXAMPLE 1 Using a pattern to represent the moves

Determine the minimum number of moves needed to move 10 discs to another peg.

Mario's Solution

Number of Discs	Number of Moves
1	1
2	3
3	7

I started with a simpler problem by counting the moves needed with 1, 2, and 3 discs, respectively. I noticed that for 3 discs, I first had to move the top two discs to another peg, then move the third disc to the open peg, and finally move the top two discs on top of the third disc.

$t_1 = 1, t_n = 2t_{n-1} + 1$

$t_2 = 3, t_2 = 2 \times 1 + 1$

$t_3 = 7, t_3 = 2 \times 3 + 1$

I noticed that each term was double the previous term plus 1. I wrote my pattern rule as a recursive formula, and it worked for the first three cases.

$t_4 = 2 \times 7 + 1$

$= 15$

I assumed that this pattern was correct. Then I used my formula to calculate the number of moves needed for 4 discs.

$t_5 = 2 \times 15 + 1 = 31$

$t_6 = 2 \times 31 + 1 = 63$

$t_7 = 2 \times 63 + 1 = 127$

$t_8 = 2 \times 127 + 1 = 255$

$t_9 = 2 \times 255 + 1 = 511$

$t_{10} = 2 \times 511 + 1 = 1023$

I then used the formula to calculate the number of moves needed for 10 discs.

To move 10 discs to a new peg requires 1023 moves.

Reflecting

A. How is Mario's recursive formula useful for understanding this sequence?

B. Why would it be difficult to use a recursive formula to figure out the number of moves if there were 1000 discs?

C. Add 1 to each term in the sequence. Use these new numbers to help you determine the general term of the sequence.

APPLY the Math

If a sequence is neither arithmetic nor geometric, identify the type of pattern (if one exists) that relates the terms to each other to get the general term.

EXAMPLE 2 Using reasoning to determine the next terms of a sequence

Given the sequence 1, 8, 16, 26, 39, 56, 78, ... , determine the next three terms. Explain your reasoning.

Tina's Solution

Term	1st Difference
1	
	7
8	
	8
16	
	10
26	
	13
39	

I calculated the 1st differences of the first five terms to determine whether the sequence is arithmetic. The sequence is not arithmetic since the 1st differences were not constant.

$$\frac{t_2}{t_1} = \frac{8}{1} = 8$$

$$\frac{t_3}{t_2} = \frac{16}{8} = 2$$

I checked to see if the sequence is geometric. There was no common ratio so the sequence is not geometric.

Term	1st Difference	2nd Difference
1		
	7	
8		
	8	1
16		2
	10	
26		3
	13	
39		

I calculated the 2nd differences. The 2nd differences go up by 1. If this pattern continues, I could determine the next terms of the sequence. I first checked whether this pattern was valid by calculating the next two terms.

Term	1st Difference	2nd Difference
1		
	7	
8		1
	8	
16		2
	10	
26		3
	13	
39		$3 + 1 = \mathbf{4}$
	$13 + 4 = \mathbf{17}$	
$39 + 17 = \mathbf{56}$		$4 + 1 = \mathbf{5}$
	$17 + 5 = \mathbf{22}$	
$56 + 22 = \mathbf{78}$		

I calculated the next two 2nd differences and worked backward to get the next two 1st differences. I worked backward again to calculate the next two terms. My values of t_6 and t_7 matched those in the given sequence, so the pattern rule seemed to be valid.

Term	1st Difference	2nd Difference
1	7	
8	8	1
16	10	2
26	13	3
39	**17**	**4**
56	**22**	**5**
78	22 + 6 = **28**	5 + 1 = **6**
78 + 28 = **106**	28 + 7 = **35**	6 + 1 = **7**
106 + 35 = **141**	35 + 8 = **43**	7 + 1 = **8**
141 + 43 = **184**		

I applied the rule to determine the next three terms.

The next three terms of the sequence are 106, 141, and 184.

Sometimes the pattern between terms in a sequence that is neither arithmetic nor geometric can be best described using a recursive formula.

EXAMPLE 3 Using reasoning to determine the recursive formula of a sequence

Given the sequence 5, 14, 41, 122, 365, 1094, 3281, ... , determine the recursive formula. Explain your reasoning.

Ali's Solution

$$t_2 - t_1 = 14 - 5 = 9$$
$$t_3 - t_2 = 41 - 14 = 27$$

I calculated some 1st differences and found they were not the same. The sequence is not arithmetic.

$$\frac{t_2}{t_1} = \frac{14}{5} = 2.8$$

I calculated a few ratios and found they were not the same. The sequence is not geometric.

$$\frac{t_3}{t_2} = \frac{41}{14} \doteq 2.93$$

$$\frac{t_4}{t_3} = \frac{122}{41} \doteq 2.98$$

$$\frac{t_5}{t_4} = \frac{365}{122} \doteq 2.99$$

The ratios seemed to be getting closer to 3.

n	t_n	$3t_{n-1}$
1	5	—
2	14	15
3	41	42
4	122	123
5	365	366
6	1094	1095
7	3281	3282

Since the ratios were almost 3, I pretended that they were actually 3. I created a table to compare each term t_n in the given sequence with 3 times the previous term, $3t_{n-1}$. I noticed that the value of t_n was one less than the value of $3t_{n-1}$.

$t_1 = 5$, $t_n = 3t_{n-1} - 1$, where $n \in \mathbf{N}$ and $n > 1$

I wrote a recursive formula for the sequence based on the pattern in my table.

Assuming that the pattern continues, the recursive formula for the sequence 5, 14, 41, 122, 365, 1094, 3281, ... is $t_1 = 5$, $t_n = 3t_{n-1} - 1$, where $n \in \mathbf{N}$ and $n > 1$.

If the terms of a sequence are rational numbers, you can sometimes find a pattern between terms if you look at the numerators and the denominators on their own.

EXAMPLE 4 Using reasoning to determine the general term of a sequence

Given the sequence $\frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \frac{11}{36}, \frac{13}{49}, \frac{15}{64}, \ldots$, determine the general term. Explain your reasoning.

Monica's Solution

3, 5, 7, 9, 11, 13, 15, ...

I looked at just the numerators to see if they formed a sequence. There was a common difference of 2 between terms, so the numerators formed an arithmetic sequence.

$$\begin{aligned} N_n &= a + (n - 1)d \\ &= 3 + (n - 1)(2) \\ &= 2n + 1 \end{aligned}$$

I wrote the numerator in terms of n by substituting $a = 3$ and $d = 2$ into the general formula for an arithmetic sequence.

$4, 9, 16, 25, 36, 49, 64, \ldots$
$2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, \ldots$

Next, I looked at the denominators to see if they formed a sequence. The terms were perfect squares but of the next position's value, not the position value for the current term.

$$D_n = (n + 1)^2$$

I wrote the denominator in terms of n by using a general expression for perfect squares.

$$t_n = \frac{N_n}{D_n}$$

The general term of the given sequence is $t_n = \frac{N_n}{D_n}$.

$$= \frac{2n + 1}{(n + 1)^2}$$

I substituted the expressions for N_n and D_n into t_n.

Assuming that the pattern continues, the general term of the given sequence is $t_n = \frac{2n + 1}{(n + 1)^2}$, where $n \in \mathbf{N}$.

In Summary

Key Idea

- A sequence is an ordered list of numbers that may or may not follow a predictable pattern. For example, the sequence of primes, 2, 3, 5, 7, 11, … , is well understood, but no function or recursive formula has ever been discovered to generate them.

Need to Know

- A sequence has a general term if an algebraic rule using the term number, n, can be found to generate each term.
- If a sequence is arithmetic or geometric, a general term can always be found because arithmetic and geometric sequences follow a predictable pattern. For any other type of sequence, it is not always possible to find a general term.

CHECK Your Understanding

1. Sam wrote a solution to determine the 10th term of the sequence 1, 5, 4, -1, -5, -4, ...

Sam's Solution

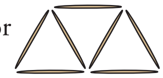
$$t_n = t_{n-1} - t_{n-2}$$
$$\therefore t_{10} = -1$$

Do you think Sam is right? Explain.

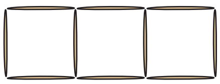
2. Determine a rule for calculating the terms of the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$, ... Explain your reasoning.

PRACTISING

3. Leila used toothpicks to make a row of triangles.



   a) Determine a rule for calculating t_n, the number of toothpicks needed for n triangles. Explain your reasoning.
   b) How will your rule change if the row of triangles is replaced with a row of squares? Explain your reasoning.



   c) Determine a rule for calculating t_n, the number of toothpicks needed to create an $n \times n$ grid of squares. Explain your reasoning.

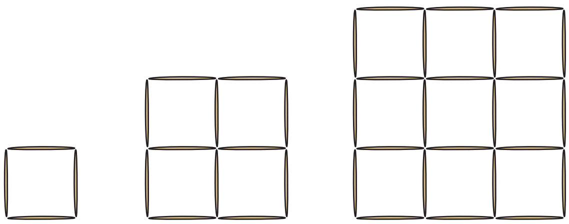


figure 1 figure 2 figure 3

   d) Show that both your rules work for $n = 4$.

4. You are given the sequence 0, 1, -1, 2, -2, 3, -3, ...
   a) Determine a rule for calculating the general term. Explain your reasoning.
   b) Compare your rule with that of a classmate. Did you come up with the same rule? Which rule is "better"? Why?
   c) Determine t_{12345}. Did you have to modify your rule to do this? If so, what is your new rule?

5. Determine an expression for the general term of the sequence $x + \frac{1}{y}$, $2x + \frac{1}{y^2}$, $3x + \frac{1}{y^3}$, ...

6. Determine a rule for calculating the terms of the sequence $\frac{3}{5}, \frac{21}{55}, \frac{147}{555}, \frac{1029}{5555}, \frac{7203}{55\,555}, \frac{50\,421}{555\,555}, \ldots$. Explain your reasoning.

7. Determine the next three terms of each sequence. Explain your reasoning.
K
a) 4, 9, 19, 39, 79, ...
b) 100, 99, 97, 94, 90, ...
c) 1, 1, 2, 3, 5, 8, 13, 21, ...
d) 3, 5, 10, 12, 24, 26, 52, ...
e) 1, −8, 27, −64, 125, ...
f) 6, 13, 27, 55, ...

8. In computer science, a bubble sort is an algorithm used to sort numbers from lowest to highest.
A
- The algorithm compares the first two numbers in a list to see if they are in the correct order. If they are not, the numbers switch places. Otherwise, they are left alone.
- The process continues with the 2nd and 3rd numbers, and then the 3rd and 4th, all the way through the list until the last two numbers.
- The algorithm starts at the beginning and repeats the whole process.
- The algorithm stops after it goes through the complete list and makes no switches. For example, a bubble sort of the numbers 3, 1, 5, 4, 2 would look like this:

Compare **3, 1**, 5, 4, 2. ⇒ Switch to give 1, 3, 5, 4, 2.
Compare 1, **3, 5**, 4, 2. ⇒ Leave as is.
Compare 1, 3, **5, 4**, 2. ⇒ Switch to give 1, 3, 4, 5, 2.
Compare 1, 3, 4, **5, 2**. ⇒ Switch to give 1, 3, 4, 2, 5.
Compare **1, 3**, 4, 2, 5. ⇒ Leave as is.
Compare 1, **3, 4**, 2, 5. ⇒ Leave as is.
Compare 1, 3, **4, 2**, 5. ⇒ Switch to give 1, 3, 2, 4, 5.
Compare 1, 3, 2, **4, 5**. ⇒ Leave as is.
Compare **1, 3**, 2, 4, 5. ⇒ Leave as is.
Compare 1, **3, 2**, 4, 5. ⇒ Switch to give 1, 2, 3, 4, 5.

The algorithm would then make 6 more comparisons, with no changes, and stop.

Suppose you had the numbers 100, 99, 98, 97, ... , 3, 2, 1. How many comparisons would the algorithm have to make to arrange these numbers from lowest to highest?

9. Determine the next three terms of the sequence 2, 11, 54, 271, 1354, 6771, 33 854, ... Explain your reasoning.
T

10. A sequence is defined by

$$t_1 = 1$$
$$t_n = \begin{cases} \frac{1}{2}t_{n-1}, & \text{if } t_{n-1} \text{ is even} \\ \frac{5}{2}(t_{n-1} + 1), & \text{if } t_{n-1} \text{ is odd} \end{cases}$$

Determine t_{1000}. Explain your reasoning.

11. Create your own sequence that is neither arithmetic nor geometric. State a rule for generating the sequence.
C

7.4 Exploring Recursive Sequences

GOAL

Explore patterns in sequences in which a term is related to the previous two terms.

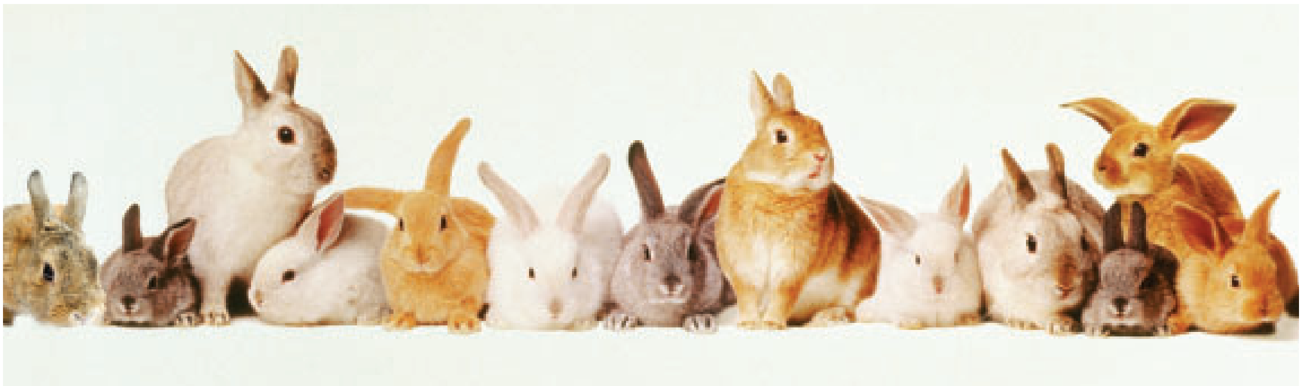
YOU WILL NEED

- graph paper

EXPLORE the Math

In his book *Liber Abaci* (*The Book of Calculation*), Italian mathematician Leonardo Pisano (1170–1250), nicknamed Fibonacci, described a situation like this:

A man put a pair of newborn rabbits (one male and one female) in an area surrounded on all sides by a wall. When the rabbits are in their second month of life, they produce a new pair of rabbits every month (one male and one female), which eventually mate. If the cycle continues, how many pairs of rabbits are there every month?



The sequence that represents the number of pairs of rabbits each month is called the Fibonacci sequence in Pisano's honour.

? What relationships can you determine in the Fibonacci sequence?

A. The first five terms of the Fibonacci sequence are 1, 1, 2, 3, and 5. Explain how these terms are related and generate the next five terms. Determine an expression for generating any term, F_n, in the sequence.

B. French mathematician Edouard Lucas (1842–91) named the sequence in the rabbit problem "the Fibonacci sequence." He studied the related sequence 1, 3, 4, ... , whose terms are generated in the same way as the Fibonacci sequence. Generate the next five terms of the Lucas sequence.

C. Starting with the Fibonacci sequence, create a new sequence by adding terms that are two apart. The first four terms are shown.

Fibonacci	1	1	2	3	5	8
New		1 + 2	1 + 3	2 + 5	3 + 8	

Repeat this process with the Lucas sequence. How are these new sequences related to the Fibonacci and Lucas sequences?

D. Determine the ratios of consecutive terms in the Fibonacci sequence. The first three ratios are shown.

$$\frac{F_2}{F_1} = \frac{1}{1} = 1, \qquad \frac{F_3}{F_2} = \frac{2}{1} = 2, \qquad \frac{F_4}{F_3} = \frac{3}{2} = 1.5$$

What happens to the ratios if you continue the process? What happens if you repeat this process with the Lucas sequence? Based on your answers, how are the Fibonacci and Lucas sequences related to a geometric sequence?

E. Starting with the Fibonacci sequence, create two new sequences as shown.

Fibonacci	1	1	2	3	
New 1	1 x 1	1 x 1	2 x 2	3 x 3	
New 2	1 x 2	1 x 3	2 x 5	3 x 8	

The first new sequence is the squares of the Fibonacci terms. The second is the products of Fibonacci terms that are two apart. How are these two sequences related? What happens if you repeat this process with the Lucas sequence?

F. Create a new sequence by multiplying a Fibonacci number by a Lucas number from the same position. How is this new sequence related to the Fibonacci sequence?

Reflecting

G. How are the Fibonacci and Lucas sequences similar? different?

H. Although the Fibonacci and Lucas sequences have different starting values, they share the same relationship between consecutive terms, and they have many similar properties. What properties do you think different sequences with the same relationship between consecutive terms have? How would you check your conjecture?

I. From part D, the Fibonacci and Lucas sequences are closely related to a geometric sequence. How are these sequences similar? different?

In Summary

Key Ideas

- The Fibonacci sequence is defined by the recursive formula $t_1 = 1$, $t_2 = 1$, $t_n = t_{n-1} + t_{n-2}$, where $n \in \mathbf{N}$ and $n > 2$. This sequence models the number of petals on many kinds of flowers, the number of spirals on a pineapple, and the number of spirals of seeds on a sunflower head, among other naturally occurring phenomena.
- The Lucas sequence is defined by the recursive formula $t_1 = 1$, $t_2 = 3$, $t_n = t_{n-1} + t_{n-2}$, where $n \in \mathbf{N}$ and $n > 2$, and has many of the properties of the Fibonacci sequence.

Need to Know

- In a recursive sequence, the terms depend on one or more of the previous terms.
- Two different sequences with the same relationship between consecutive terms have similar properties.

FURTHER Your Understanding

1. Pick any two numbers and use the same relationship between consecutive terms as the Fibonacci and Lucas sequences to generate a new sequence. What properties does this new sequence share with the Fibonacci and Lucas sequences?

2. Since the ratios of consecutive terms of the Fibonacci and Lucas sequences are *almost* constant, these sequences are similar to a geometric sequence. Substitute the general term for a geometric sequence, $t_n = ar^{n-1}$, into the recursive formulas for the Fibonacci and Lucas sequences, and solve for r. How does this value of r relate to what you found in part D?

3. A sequence is defined by the recursive formula $t_1 = 1$, $t_2 = 5$, $t_n = t_{n-1} + 2t_{n-2}$, where $n \in \mathbf{N}$ and $n > 2$.
   a) Generate the first 10 terms.
   b) Calculate the ratios of consecutive terms. What happens to the ratios?
   c) Develop a formula for the general term.

Tech | *Support*

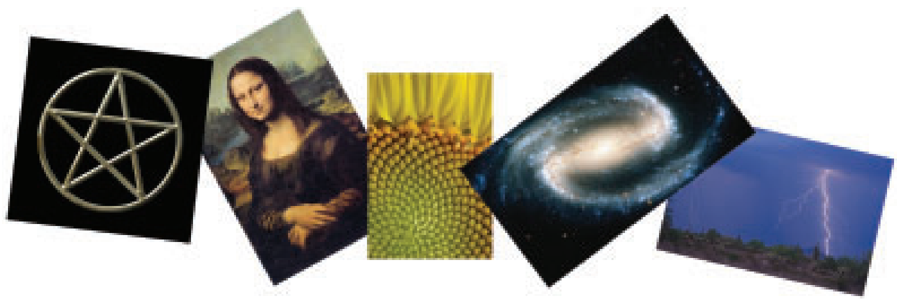
For help using a graphing calculator to generate sequences using recursive formulas, see Technical Appendix, B-16.

Curious | *Math*

The Golden Ratio

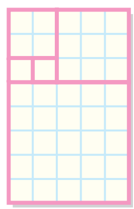
The golden ratio (symbolized by ϕ, Greek letter phi) was known to the ancient Greeks. Euclid defined the golden ratio by a point C on a line segment AB such that $\phi = \frac{AC}{CB} = \frac{AB}{CB}$.

The golden ratio, like the Fibonacci sequence, seems to pop up in unexpected places. The ancient Greeks thought that it defined the most pleasing ratio to the eye, so they used it in their architecture. Artists have been known to incorporate the golden ratio into their works. It has even received some exposure in an episode of the crime series NUMB3RS, as well as in the movie and book *The Da Vinci Code*.



Human works aren't the only places where the golden ratio occurs. The ratio of certain proportions in the human body are close to the golden ratio, and spirals in seed heads of flowers can be expressed using the golden ratio.

- On a piece of graph paper, trace a 1×1 square.
- Draw another 1×1 square touching the left side of the first square.
- On top of these two squares, draw a 2×2 square.
- On the right side of your picture, draw a 3×3 square touching one of the 1×1 squares and the 2×2 square.
- Below your picture, draw a 5×5 square touching both 1×1 squares and the 3×3 square.
- Repeat this process of adding squares around the picture, alternating directions left, up, right, down, and so on. The start of the spiral is shown at the right.



1. How is this spiral related to the Fibonacci sequence and the golden ratio?

7 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How do you know if a sequence is arithmetic?

A: A sequence is arithmetic if consecutive terms differ by a constant called the common difference, d.

$$t_2 - t_1 = d, \quad t_3 - t_2 = d, \quad t_4 - t_3 = d$$
$$\vdots$$

The recursive formula of an arithmetic sequence is based on adding the same value to the previous term. Its general term is defined by a discrete linear function since the graph of term versus position number gives a straight line.

Study *Aid*

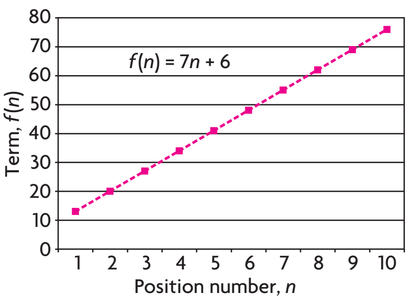
- See Lesson 7.1, Examples 1 to 4
- Try Mid-Chapter Review Questions 1 to 5.

EXAMPLE

If the sequence 13, 20, 27, 34, 41, ... is arithmetic, state the recursive formula and the general term.

Solution

Each term is 7 more than the previous term. So the recursive formula is $t_1 = 13$, $t_n = t_{n-1} + 7$, where $n \in \mathbf{N}$ and $n > 1$. The general term is $t_n = 13 + (n - 1)(7) = 7n + 6$ and its graph is a discrete linear function.



Q: How do you know if a sequence is geometric?

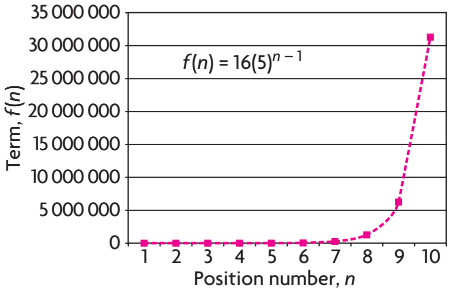
A: A sequence is geometric if the ratio of consecutive terms is a constant called the common ratio, r.

$$\frac{t_2}{t_1} = r, \quad \frac{t_3}{t_2} = r, \quad \frac{t_4}{t_3} = r$$
$$\vdots$$

The recursive formula of a geometric sequence is based on multiplying the previous term by the same value. Its general term is defined by a discrete exponential function since the graph of term versus position number gives an exponential curve.

Study *Aid*

- See Lesson 7.2, Examples 1, 2, and 3.
- Try Mid-Chapter Review Questions 4 to 6.



EXAMPLE

If the sequence 16, 80, 400, 2000, 10 000, ... is geometric, state the recursive formula and the general term.

Solution

Each term is 5 times the previous term. So the recursive formula is $t_1 = 16$, $t_n = 5t_{n-1}$, where $n \in \mathbf{N}$ and $n > 1$. The general term is $t_n = 16 \times 5^{n-1}$, and its graph is a discrete exponential function.

Study *Aid*

- See Lesson 7.3, Examples 1 to 4.
- Try Mid-Chapter Review Questions 7, 8, and 9.

Q: How do you determine terms of a sequence that is neither arithmetic nor geometric?

A: Look for a pattern among the terms. It is also useful to look at the 1st, 2nd, 3rd, and possibly higher, differences. Once you find a pattern, you can use it to generate terms of the sequence.

EXAMPLE

Determine the next three terms of the sequence 1, 6, 7, 6, 5, 6, 11, ...

Solution

Start by looking at the 1st, 2nd, and 3rd differences.

Term	1st Difference	2nd Difference	3rd Difference
1	5	−4	2
6	1	−2	2
7	−1	0	2
6	−1	2	2
5	1	4	**2**
6	5	**6**	**2**
11	**11**	**8**	**2**
22	**19**	**10**	
41	**29**		
70			

The 1st differences are not constant. Since the 2nd differences seem to be going up by a constant, the 3rd differences are the same. To determine the next three terms, work backward using the 3rd, 2nd, and then the 1st differences. Assuming that the pattern continues, the next three terms are 22, 41, and 70.

PRACTICE Questions

Lesson 7.1

1. For each arithmetic sequence, determine
i) the recursive formula
ii) the general term
iii) t_{10}

a) 29, 21, 13, ...
b) $-8, -16, -24, \ldots$
c) $-17, -9, -1, \ldots$
d) 3.25, 9.5, 15.75, ...
e) $\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \ldots$
f) $x, 3x + 3y, 5x + 6y, \ldots$

2. Determine the recursive formula and the general term for the arithmetic sequence in which
a) the first term is 17 and the common difference is 11
b) $t_1 = 38$ and $d = -7$
c) the first term is 55 and the second term is 73
d) $t_3 = -34$ and $d = -38$
e) the fifth term is 91 and the seventh term is 57

3. The number of seats in the rows of a stadium form an arithmetic sequence. Two employees of the stadium determine that the 13th row has 189 seats and the 25th row has 225 seats. How many seats are in the 55th row?

Lesson 7.2

4. i) Determine whether each sequence is arithmetic or geometric.
ii) Determine the general term, the recursive formula, and t_6.

a) 15, 30, 45, ...
b) 640, 320, 160, ...
c) $23, -46, 92, \ldots$
d) 3000, 900, 270, ...
e) 3.8, 5, 6.2, ...
f) $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \ldots$

5. i) Determine the type of each sequence (arithmetic, geometric, or neither), where $n \in \mathbf{N}$.
ii) State the first five terms.

a) $t_n = 5n$
b) $t_n = \dfrac{3}{4^n + 3}$
c) $t_1 = 5, t_n = t_{n-1} - 12$, where $n > 1$
d) $t_1 = -2, \dfrac{t_n}{t_{n-1}} = -2$, where $n > 1$
e) $t_1 = 8, t_2 = 11, t_n = 2t_{n-1} - t_{n-2}$, where $n > 2$

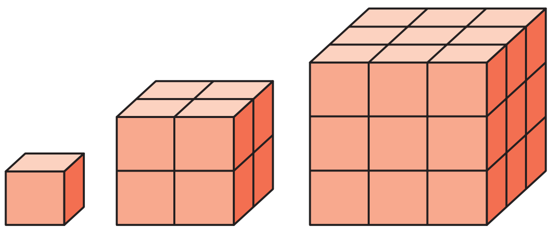
6. A work of art is priced at $10 000. After one week, if the art isn't sold, its price is reduced by 10%. Each week after that, if it hasn't sold, its price is reduced by another 10%. Your mother really likes the art and you would like to purchase it for her, but you have only $100. If the art is not sold, how many weeks will you have to wait before being able to afford it?

Lesson 7.3

7. An IQ test has the question "Determine the next three numbers in the sequence 1, 9, 29, 67, 129, 221, ___, ___, ___." What are the next three terms? Explain your reasoning.

8. Determine the general term of the sequence $x + y, x^2 + 2y, x^3 + 3y, \ldots$. Explain your reasoning.

9. Sarah built a sequence of large cubes using unit cubes.
a) State the sequence of the number of unit cubes in each larger cube.
b) Determine the next three terms of the sequence.
c) State the general term of the sequence.
d) How many unit cubes does Sarah need to build the 15th cube?



Lesson 7.4

10. a) Determine the 15th term of the sequence 3, 2, 5, 7, 12, ... Explain your reasoning.
b) Write the recursive formula for the sequence in part (a).

7.5 Arithmetic Series

YOU WILL NEED
- linking cubes

GOAL

Calculate the sum of the terms of an arithmetic sequence.

INVESTIGATE the Math

Marian goes to a party where there are 23 people present, including her. Each person shakes hands with every other person once and only once.

? How can Marian determine the total number of handshakes that take place?

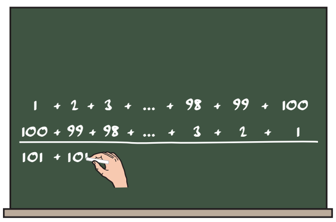
series
the sum of the terms of a sequence

arithmetic series
the sum of the terms of an arithmetic sequence

A. Suppose the people join the party one at a time. When they enter, they shake hands with the host and everyone who is already there. Create a sequence representing the number of handshakes each person will make. What type of sequence is this?

B. Write your sequence from part A, but include plus signs between terms. This expression is a **series** and represents the total number of handshakes.

C. When German mathematician Karl Friedrich Gauss (1777–1855) was a child, his teacher asked him to calculate the sum of the numbers from 1 to 100. Gauss wrote the list of numbers twice, once forward and once backward. He then paired terms from the two lists to solve the problem. Use this method to determine the sum of your **arithmetic series.**



D. Solve the handshake problem without using Gauss's method.

Reflecting

E. Suppose the partial sums of an arithmetic series are the terms of an arithmetic sequence. What would you notice about the 1st and 2nd differences?

partial sum

the sum, S_n, of the first n terms of a sequence

F. Why is Gauss's method for determining the sum of an arithmetic series efficient?

G. Consider the arithmetic series $1 + 6 + 11 + 16 + 21 + 26 + 31 + 36$. Use Gauss's method to determine the sum of this series. Do you think this method will work for *any* arithmetic series? Justify your answer.

APPLY the Math

EXAMPLE 1 Representing the sum of an arithmetic series

Determine the sum of the first n terms of the arithmetic series
$a + (a + d) + (a + 2d) + (a + 3d) + \ldots$.

Barbara's Solution

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + t_n$$

The series is arithmetic. To find S_n, I added all terms up to t_n. The nth term of the series corresponds to the general term of an arithmetic sequence, $t_n = a + (n - 1)d$.

$$\begin{array}{rcccccccc} S_n &=& a &+& (a + d) &+ \ldots +& [a + (n - 2)d] &+& [a + (n - 1)d] \\ + S_n &=& [a + (n - 1)d] &+& [a + (n - 2)d] &+ \ldots +& (a + d) &+& a \\ \hline 2S_n &=& [2a + (n - 1)d] &+& [2a + (n - 1)d] &+ \ldots +& [2a + (n - 1)d] &+& [2a + (n - 1)d] \end{array}$$

$$2S_n = n \times [2a + (n - 1)d]$$

$$S_n = \frac{n[2a + (n - 1)d]}{2}$$

Using Gauss's method, I wrote the sum out twice, first forward and then backward. Next, I added each column. Since the terms in the top row go *up* by d and the terms in the bottom row go *down* by d, each pair of terms has the same sum.

There are n pairs that add up to $2a + (n - 1)d$, but that represents $2S_n$, so I divided by 2.

The sum of the first n terms of an arithmetic series is

$$S_n = \frac{n[2a + (n - 1)d]}{2}$$

$$= \frac{n[a + a + (n - 1)d]}{2}$$

$$= \frac{n[a + (a + (n - 1)d)]}{2}$$

$$= \frac{n(t_1 + t_n)}{2}$$

I knew that $2a = a + a$, so I wrote this formula another way. I regrouped the terms in the numerator. I noticed that a is the first term of the series and $a + (n - 1)d$ is the nth term.

If a problem involves adding the terms of an arithmetic sequence, you can use the formula for the sum of an arithmetic series.

EXAMPLE 2 Solving a problem by using an arithmetic series

In an amphitheatre, seats are arranged in 50 semicircular rows facing a domed stage. The first row contains 23 seats, and each row contains 4 more seats than the previous row. How many seats are in the amphitheatre?

Kew's Solution

$a = 23, d = 4$

Since each row has 4 more seats than the previous row, the number of seats in each row forms an arithmetic sequence.

$23 + 27 + 31 + ... + t_{50}$

I wrote an arithmetic series to represent the total number of seats in the amphitheatre.

$$S_n = \frac{n[2a + (n - 1)d]}{2}$$

$$S_{50} = \frac{(50)[2(23) + (50 - 1)(4)]}{2}$$

$$= 6050$$

Since I knew the first term and the common difference, I used the formula for the sum of an arithmetic series in terms of *a* and *d*. I substituted $n = 50$ since there are 50 rows of seats.

There are 6050 seats in the amphitheatre.

In order to determine the sum of any arithmetic series, you need to know the number of terms in the series.

EXAMPLE 3 Selecting a strategy to calculate the sum of a series when the number of terms is unknown

Determine the sum of $-31 - 35 - 39 - ... -403$.

Jasmine's Solution

$$t_2 - t_1 = -35 - (-31) = -4$$
$$t_3 - t_2 = -39 - (-35) = -4$$

I checked to see if the series was arithmetic. So I calculated a few 1st differences. The differences were the same, so the series is arithmetic.

$$
\begin{aligned}
t_n &= a + (n-1)d \\
-403 &= -31 + (n-1)(-4) \\
-403 + 31 &= (n-1)(-4) \\
-372 &= (n-1)(-4) \\
\frac{-372}{-4} &= \frac{(n-1)(-4)}{-4} \\
93 &= n - 1 \\
93 + 1 &= n \\
94 &= n
\end{aligned}
$$

I needed to determine the value of n when $t_n = -403$. So I substituted $a = 31$, $d = 4$, and t_n into the formula for the general term of an arithmetic sequence and solved for n.

There are 94 terms in this sequence.

$$
\begin{aligned}
S_n &= \frac{n(t_1 + t_n)}{2} \\
S_{94} &= \frac{94[-31 + (-403)]}{2} \\
&= -20\,398
\end{aligned}
$$

Since I knew the first and last terms of the series, I used the formula for the sum of an arithmetic series in terms of t_1 and t_n. I substituted $n = 94$, $t_1 = -31$, and $t_{94} = -403$.

The sum of the series $-31 - 35 - 39 - \ldots -403$ is $-20\,398$.

In Summary

Key Idea

- An arithmetic series is created by adding the terms of an arithmetic sequence. For the sequence 2, 10, 18, 26, ... , the related arithmetic series is $2 + 10 + 18 + 26 + \ldots$.
- The partial sum, S_n, of a series is the sum of a finite number of terms from the series, $S_n = t_1 + t_2 + t_3 + \ldots + t_n$.
  For example, for the sequence 2, 10, 18, 26, ... ,

$$
\begin{aligned}
S_4 &= t_1 + t_2 + t_3 + t_4 \\
&= 2 + 10 + 18 + 26 \\
&= 56
\end{aligned}
$$

Need to Know

- The sum of the first n terms of an arithmetic sequence can be calculated using
  - $S_n = \dfrac{n[2a + (n-1)d]}{2}$ or
  - $S_n = \dfrac{n(t_1 + t_n)}{2}$.

  In both cases, $n \in \mathbf{N}$, a is the first term, and d is the common difference.
- You can use either formula, but you need to know the number of terms in the series and the first term. If you know the last term, use the formula in terms of t_1 and t_n. If you can calculate the common difference, use the formula in terms of a and d.

CHECK Your Understanding

1. Calculate the sum of the first 10 terms of each arithmetic series.
   a) $59 + 64 + 69 + \dots$
   b) $31 + 23 + 15 + \dots$
   c) $-103 - 110 - 117 - \dots$
   d) $-78 - 56 - 34 - \dots$

2. Calculate the sum of the first 20 terms of an arithmetic sequence with first term 18 and common difference 11.

3. Bricks are stacked in 20 rows such that each row has a fixed number of bricks more than the row above it. The top row has 5 bricks and the bottom row 62 bricks. How many bricks are in the stack?

PRACTISING

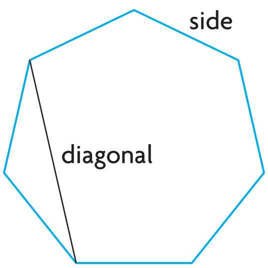
4. i) Determine whether each series is arithmetic.
   ii) If the series is arithmetic, calculate the sum of the first 25 terms.

   a) $-5 + 1 + 7 + 13 + \dots$
   b) $2 + 10 + 50 + 250 + \dots$
   c) $1 + 1 + 2 + 3 + \dots$
   d) $18 + 22 + 26 + 30 + \dots$
   e) $31 + 22 + 13 + 4 + \dots$
   f) $1 - 3 + 5 - 7 + \dots$

5. For each series, calculate t_{12} and S_{12}.
   K
   a) $37 + 41 + 45 + 49 + \dots$
   b) $-13 - 24 - 35 - 46 - \dots$
   c) $-18 - 12 - 6 + 0 + \dots$
   d) $\frac{1}{5} + \frac{7}{10} + \frac{6}{5} + \frac{17}{10} + \dots$
   e) $3.19 + 4.31 + 5.43 + 6.55 + \dots$
   f) $p + (2p + 2q) + (3p + 4q) + (4p + 6q) + \dots$

6. Determine the sum of the first 20 terms of the arithmetic series in which
   a) the first term is 8 and the common difference is 5
   b) $t_1 = 31$ and $t_{20} = 109$
   c) $t_1 = 53$ and $t_2 = 37$
   d) the 4th term is 18 and the terms increase by 17
   e) the 15th term is 107 and the terms decrease by 3
   f) the 7th term is 43 and the 13th term is 109

7. Calculate the sums of these arithmetic series.
   a) $1 + 6 + 11 + \dots + 96$
   b) $24 + 37 + 50 + \dots + 349$
   c) $85 + 77 + 69 + \dots - 99$
   d) $5 + 8 + 11 + \dots + 2135$
   e) $-31 - 38 - 45 - \dots - 136$
   f) $-63 - 57 - 51 - \dots + 63$



8. A diagonal in a regular polygon is a line segment joining two nonadjacent vertices.
   A
   a) Develop a formula for the number of diagonals for a regular polygon with n sides.
   b) Show that your formula works for a regular heptagon (a seven-sided polygon).

9. Joe invests $1000 at the start of each year for five years and earns 6.3% simple interest on his investments. How much will all his investments be worth at the start of the fifth year?

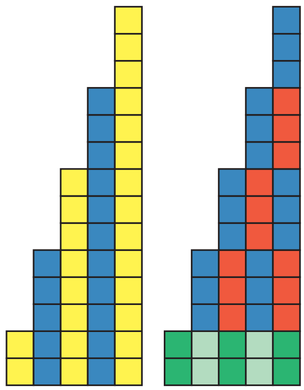
10. During a skydiving lesson, Chandra jumps out of a plane and falls 4.9 m during the first second. For each second afterward, she continues to fall 9.8 m more than the previous second. After 15 s, she opens her parachute. How far did Chandra fall before she opened her parachute?

11. Jamal got a job working on an assembly line in a toy factory. On the 20th day of work, he assembled 137 toys. He noticed that since he started, every day he assembled 3 more toys than the day before. How many toys did Jamal assemble altogether during his first 20 days?

12. In the video game "Geometric Constructors," a number of shapes have to be arranged into a predefined form. In level 1, you are given 3 min 20 s to complete the task. At each level afterward, a fixed number of seconds are removed from the time until, at level 20, 1 min 45 s are given. What would be the total amount of time given if you were to complete the first 20 levels?

13. **T** Sara is training to run a marathon. The first week she runs 5 km each day. The next week, she runs 7 km each day. During each successive week, each day she runs 2 km farther than she ran the days of the previous week. If she runs for five days each week, what total distance will Sara run in a 10 week training session?

14. **C** Joan is helping a friend understand the formulas for an arithmetic series. She uses linking cubes to represent the sum of the series $2 + 5 + 8 + 11 + 14$ two ways. These representations are shown at the right. Explain how the linking-cube representations can be used to explain the formulas for an arithmetic series.



Extending

15. The 10th term of an arithmetic series is 34, and the sum of the first 20 terms is 710. Determine the 25th term.

16. The arithmetic series $1 + 4 + 7 + \ldots + t_n$ has a sum of 1001. How many terms does the series have?

7.6 Geometric Series

YOU WILL NEED

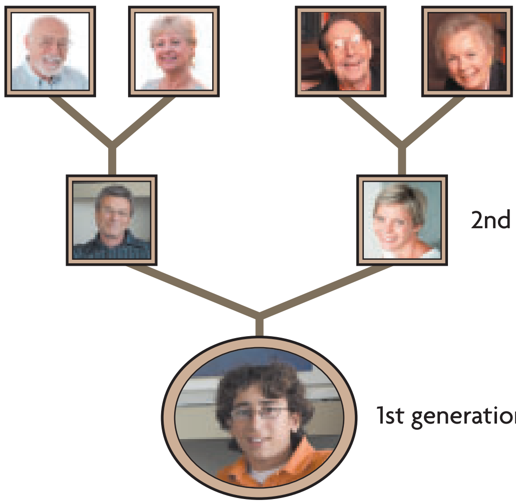
- spreadsheet software

GOAL

Calculate the sum of the terms of a geometric sequence.

INVESTIGATE the Math

An ancestor tree is a family tree that shows only the parents in each generation. John started to draw his ancestor tree, starting with his own parents. His complete ancestor tree includes 13 generations.



? How many people are in John's ancestor tree?

A. Create a sequence to represent the number of people in each generation for the first six generations. How do you know that this sequence is geometric?

B. Based on your sequence, create a geometric series to represent the total number of people in John's ancestor tree.

geometric series
the sum of the terms of a geometric sequence

C. Write the series again, but this time multiply each term by the common ratio. Write both series, rS_n and S_n, so that equal terms are aligned one above the other. Subtract S_n from rS_n.

D. Based on your calculation in part C, determine how many people are in John's ancestor tree.

Reflecting

E. How is the sum of a geometric series related to an exponential function?

F. Why did lining up equal terms make the subtraction easier?

APPLY the Math

EXAMPLE 1 Representing the sum of a geometric series

Determine the sum of the first n terms of the geometric series.

Tara's Solution

$$t_n = ar^{n-1}$$

The series is geometric. To find S_n, I added all terms up to t_n. The nth term of the series corresponds to the general term of a geometric sequence.

$$\begin{array}{rrl} rS_n = & & ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \\ -S_n = & -(& a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}) \\ \hline (r-1)S_n = & & -a + 0 + 0 + 0 + \ldots + 0 + 0 + ar^n \\ (r-1)S_n = & & -a + ar^n \end{array}$$

I wrote the sum out. If I multiplied every term by the common ratio, most of the terms would be repeated. I wrote this new series above the original series and lined up equal terms, so I would get zero for most of the terms when I subtracted. Only the first term of S_n and the last term of rS_n would remain.

$$(r-1)S_n = a(r^n - 1)$$

$$\left(\frac{\cancel{r-1}^{\,1}}{\cancel{r-1}_{\,1}}\right)S_n = \frac{a(r^n-1)}{r-1}$$

$$S_n = \frac{a(r^n-1)}{r-1}$$

I solved for S_n by dividing both sides by $r - 1$.

The sum of the first n terms of a geometric series is

$$S_n = \frac{a(r^n-1)}{r-1}, \text{ where } r \neq 1.$$

$$S_n = \frac{ar^n - a}{r-1}$$

$$S_n = \frac{t_{n+1} - t_1}{r-1}, \text{ where } r \neq 1.$$

I wrote this formula another way by expanding the numerator. I noticed that a is the first term in the series and ar^n is the $(n + 1)$th term.

If a problem involves adding together the terms of a geometric sequence, you can use the formula for the sum of geometric series.

EXAMPLE 2 — Solving a problem by using a geometric series

At a fish hatchery, fish hatch at different times even though the eggs were all fertilized at the same time. The number of fish that hatched on each of the first four days after fertilization was 2, 10, 50, and 250, respectively. If the pattern continues, calculate the total number of fish hatched during the first 10 days.

Joel's Solution

$$\frac{t_2}{t_1} = \frac{10}{2} = 5 \qquad \frac{t_3}{t_2} = \frac{50}{10} = 5 \qquad \frac{t_4}{t_3} = \frac{250}{50} = 5$$

I checked to see if the sequence 2, 10, 50, 250, ... is geometric. So I calculated the ratio of consecutive terms. Since all the ratios are the same, the sequence is geometric.

$$\therefore r = 5$$
$$a = 2$$
$$n = 10$$
$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_{10} = \frac{2(5^{10} - 1)}{5 - 1}$$
$$= 4\,882\,812$$

The first term is 2 and there are 10 terms. Since I knew the first term, the common ratio, and the number of terms, I used the formula for the sum of a geometric series in terms of a, r, and n. I substituted $a = 2$, $r = 5$, and $n = 10$.

A total of 4 882 812 fish hatched during the first 10 days.

EXAMPLE 3 Selecting a strategy to calculate the sum of a geometric series when the number of terms is unknown

Calculate the sum of the geometric series
$7\,971\,615 + 5\,314\,410 + 3\,542\,940 + \ldots + 92\,160$.

Jasmine's Solution: Using a Spreadsheet

$$\frac{t_2}{t_1} = \frac{5\,314\,410}{7\,971\,615}$$

$$= \frac{2}{3}$$

$$\therefore r = \frac{2}{3}$$

I knew that the series is geometric. So I calculated the common ratio.

	A	B
1	n	tn
2	1	7971615
3	2	5314410
4	3	3542940
5	4	2361960
6	5	1574640
7	6	1049760
8	7	699840
9	8	466560
10	9	311040
11	10	207360
12	11	138240
13	12	92160
14	13	61440

I needed to determine the number of terms, n, to get to $t_n = 92\,160$. So I set up a spreadsheet to generate the terms of the series. I saw that the 12th term is 92 160.

$$S_n = \frac{t_{n+1} - t_1}{r - 1}$$

$$S_{12} = \frac{61\,440 - 7\,971\,615}{\frac{2}{3} - 1}$$

$$= 23\,730\,525$$

From the spreadsheet, 61 440 corresponds to the $(n + 1)$th term. Since I knew the first term and the $(n + 1)$th term, I used the formula for the sum of a geometric series in terms of t_1 and t_{n+1}.

The sum of the series $7\,971\,615 + 5\,314\,410 + 3\,542\,940 + \ldots + 92\,160$ is $23\,730\,525$.

Mario's Solution: Using a Graphing Calculator

$$\frac{t_2}{t_1} = \frac{5\ 314\ 410}{7\ 971\ 615}$$
$$= \frac{2}{3}$$
$$\therefore\ r = \frac{2}{3}$$

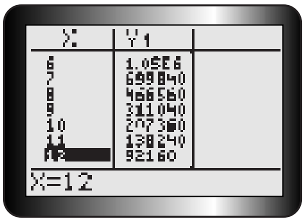
I knew that the series is geometric. So I calculated the common ratio.

$$t_n = ar^{n-1}$$
$$= 7\ 971\ 615 \times \left(\frac{2}{3}\right)^{n-1}$$

I wrote the formula for the general term of a geometric sequence. I substituted $a = 7\ 971\ 615$ and $r = \frac{2}{3}$ into the formula.

$$92\ 160 = 7\ 971\ 615 \times \left(\frac{2}{3}\right)^{n-1}$$

I needed to determine the number of terms, n, to get to $t_n = 92\ 160$.



I entered the function Y1 = 7 971 615(2/3)^(X−1) into my graphing calculator. Then I used the table function to determine the term number whose value was 92 160. It was $n = 12$.

Tech Support

For help using the table on a graphing calculator, see Technical Appendix, B-6.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Since I knew that $a = 7\ 971\ 615$, $r = \frac{2}{3}$, and $n = 12$, I substituted these values into the formula for the sum of a geometric series in terms of a, r, and n. I used my calculator to evaluate.

The sum of the series 7 971 615 + 5 314 410 + 3 542 940 + ... + 92 160 is 23 730 525.

In Summary

Key Idea

- A geometric series is created by adding the terms of a geometric sequence. For the sequence 3, 6, 12, 24, ... , the related geometric series is $3 + 6 + 12 + 24 + ...$

Need to Know

- The sum of the first n terms of a geometric sequence can be calculated using
  - $S_n = \dfrac{a(r^n - 1)}{r - 1}$, where $r \neq 1$ or
  - $S_n = \dfrac{t_{n+1} - t_1}{r - 1}$, where $r \neq 1$.

  In both cases, $n \in \mathbf{N}$, a is the first term, and r is the common ratio.
- You can use either formula, but you need to know the common ratio and the first term. If you know the $(n + 1)$th term, use the formula in terms of t_1 and t_{n+1}. If you can calculate the number of terms, use the formula in terms of a, r, and n.

CHECK Your Understanding

1. Calculate the sum of the first seven terms of each geometric series.

a) $6 + 18 + 54 + ...$
b) $100 + 50 + 25 + ...$
c) $8 - 24 + 72 - ...$
d) $\dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{12} + ...$

2. Calculate the sum of the first six terms of a geometric sequence with first term 11 and common ratio 4.

PRACTISING

3. For each geometric series, calculate t_6 and S_6.

a) $6 + 30 + 150 + ...$
b) $-11 - 33 - 99 - ...$
c) $21\,000\,000 + 4\,200\,000 + 840\,000 + ...$
d) $\dfrac{4}{5} + \dfrac{8}{15} + \dfrac{16}{45} + ...$
e) $3.4 - 7.14 + 14.994 - ...$
f) $1 + 3x^2 + 9x^4 + ...$

4. **i)** Determine whether each series is arithmetic, geometric, or neither.
K **ii)** If the series is geometric, calculate the sum of the first eight terms.

a) $5 + 10 + 15 + 20 + ...$
b) $7 + 21 + 63 + 189 + ...$
c) $2048 - 512 + 128 - 32 + ...$
d) $10 - 20 + 30 - 40 + ...$
e) $1.1 + 1.21 + 1.331 + 1.4641 + ...$
f) $81 + 63 + 45 + 27 + ...$

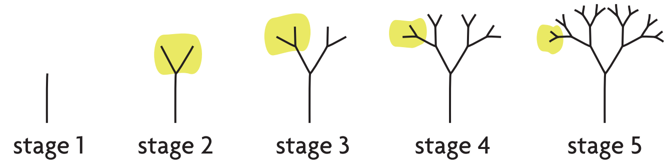
5. Determine the sum of the first seven terms of the geometric series in which
   a) $t_1 = 13$ and $r = 5$
   b) the first term is 11 and the seventh term is 704
   c) $t_1 = 120$ and $t_2 = 30$
   d) the third term is 18 and the terms increase by a factor of 3
   e) $t_8 = 1024$ and the terms decrease by a factor of $\frac{2}{3}$
   f) $t_5 = 5$ and $t_8 = -40$

6. Calculate the sum of each geometric series.
   a) $1 + 6 + 36 + \ldots + 279\,936$
   b) $960 + 480 + 240 + \ldots + 15$
   c) $17 - 51 + 153 - \ldots - 334\,611$
   d) $24\,000 + 3600 + 540 + \ldots + 1.8225$
   e) $-6 + 24 - 96 + \ldots + 98\,304$
   f) $4 + 2 + 1 + \ldots + \frac{1}{1024}$

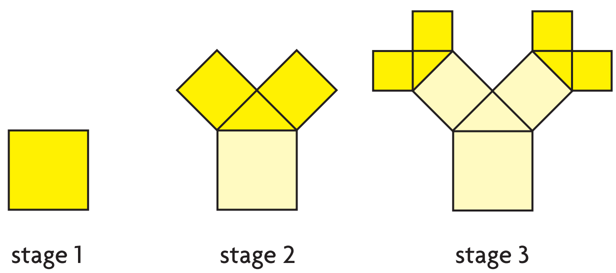
7. A ball is dropped from a height of 3 m and bounces on the ground. At the top of each bounce, the ball reaches 60% of its previous height. Calculate the total distance travelled by the ball when it hits the ground for the fifth time.

8. The formula for the sum of a geometric series is $S_n = \frac{a(r^n - 1)}{r - 1}$ or $S_n = \frac{t_{n+1} - t_1}{r - 1}$, each of which is valid only if $r \neq 1$. Explain how you would determine the sum of a geometric series if $r = 1$.

9. **A** A simple fractal tree grows in stages. At each new stage, two new line segments branch out from each segment at the top of the tree. The first five stages are shown. How many line segments need to be drawn to create stage 20?



10. A Pythagorean fractal tree starts at stage 1 with a square of side length 1 m. At every consecutive stage, an isosceles right triangle and two squares are attached to the last square(s) drawn. The first three stages are shown. Calculate the area of the tree at the 10th stage.



11. A large company has a phone tree to contact its employees in case of an emergency factory shutdown. Each of the five senior managers calls three employees, who each call three other employees, and so on. If the tree consists of seven levels, how many employees does the company have?

12. John wants to calculate the sum of a geometric series with 10 terms, where the 10th term is 5 859 375 and the common ratio is $\frac{5}{3}$. John solved the problem by considering another geometric series with common ratio $\frac{3}{5}$. Use John's method to calculate the sum. Justify your reasoning.

13. **T** A cereal company attempts to promote its product by placing certificates for a cash prize in selected boxes. The company wants to come up with a number of prizes that satisfy all of these conditions:

a) The total of the prizes is at most \$2 000 000.

b) Each prize is in whole dollars (no cents).

c) When the prizes are arranged from least to greatest, each prize is a constant integral multiple of the next smaller prize and is

- more than double the next smaller prize
- less than 10 times the next smaller prize

Determine a set of prizes that satisfies these conditions.

14. **C** Describe several methods for calculating the partial sums of an arithmetic and a geometric series. How are the methods similar? different?

Extending

15. In a geometric series, $t_1 = 12$ and $S_3 = 372$. What is the greatest possible value for t_5? Justify your answer.

16. In a geometric series, $t_1 = 23$, $t_3 = 92$, and the sum of all of the terms of the series is 62 813. How many terms are in the series?

17. Factor $x^{15} - 1$.

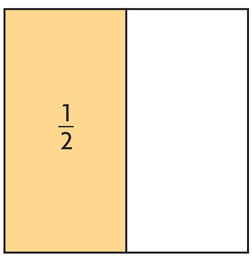
18. Suppose you want to calculate the sum of the *infinite* geometric series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots.$$

a) The diagram shown illustrates the first term of this series. Represent the next three terms on the diagram.

b) How can the formula for the sum of a geometric series be used in this case?

c) Does it make sense to talk about adding together an infinite number of terms? Justify your reasoning.



7.7

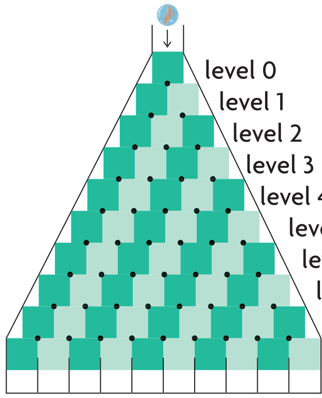
Pascal's Triangle and Binomial Expansions

GOAL

Investigate patterns in Pascal's triangle, and use one of these patterns to expand binomials efficiently.

INVESTIGATE the Math

A child's toy called "Rockin' Rollers" involves dropping a marble into its top. When the marble hits a pin, it has the same chance of going either left or right. A version of the toy with nine levels is shown at the right.



? How many paths are there to each of the bins at the bottom of this version of "Rockin' Rollers"?

A. Consider a "Rockin' Rollers" toy that has only one level. Calculate the number of paths to each bin at the bottom. Repeat the calculation with a toy having two and three levels.

B. How is the number of paths for a toy with three levels related to the number of paths for a toy with two levels? Why is this so?

C. Use the pattern to predict how many paths lead to each bin in a toy with four levels. Check your prediction by counting the number of paths.

D. Use your pattern to calculate the number of paths to each bin in a toy with nine levels.

Blaise Pascal

Reflecting

E. How is the number of paths for each bin in a given level related to the number of paths in the level above it?

F. The triangular pattern of numbers in the "Rockin' Rollers" toy is known as Pascal's triangle, named after French mathematician Blaise Pascal (1623–62), who explored many of its properties. What other pattern(s) can you find in Pascal's triangle?

APPLY the Math

EXAMPLE 1 Connecting Pascal's triangle to the expansion of a binomial power

Expand $(x + y)^6$.

Pedro's Solution

$$(x + y)^1 = 1x + 1y$$

The binomial to the 1st power is the same as the binomial itself.

$$(x + y)^2 = 1x^2 + 2xy + 1y^2$$

I expanded the square of a binomial.

$$\begin{aligned}(x + y)^3 &= (x + y)(x + y)^2\\ &= (x + y)(x^2 + 2xy + y^2)\\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3\\ &= 1x^3 + 3x^2y + 3xy^2 + 1y^3\end{aligned}$$

I expanded the binomial to the 3rd power. I noticed that the coefficients in each of the expansions so far are numbers in each row of Pascal's triangle.

1
1 1
1 2 1
1 3 3 1

Each term in the expansion is in terms of a product of an exponent of x and an exponent of y. The exponents of x start from 3 (the exponent of the binomial), and go down to zero, while the exponents of y start at zero and go up to 3. In each term, the sum of the x and y exponents is always 3.

1	0th row
1 1	1st row
1 2 1	2nd row
1 3 3 1	3rd row
1 4 6 4 1	4th row
1 5 10 10 5 1	5th row

I wrote out 6 rows of Pascal's triangle. The one at the top must correspond to $(x + y)^0 = 1$.

$$\begin{aligned}(x + y)^4 &= (x + y)(x + y)^3\\ &= (x + y)(x^3 + 3x^2y + 3xy^2 + y^3)\\ &= x^4 + 3x^3y + 3x^2y^2 + xy^3 + 3x^2y^2 + 3xy^3 + y^4\\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

I tried one more expansion to check that these patterns continue.

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^3 + 6xy^5 + y^6$$

To expand $(x + y)^6$, I used my patterns and the 6th row of the triangle.

Any binomial can be expanded by using Pascal's triangle to help determine the coefficients of each term.

EXAMPLE 2 Selecting a strategy to expand a binomial power involving a variable in one term

Expand and simplify $(x - 2)^5$.

Tanya's Solution

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Since the exponent of the binomial is 5, I used the 5th row of Pascal's triangle to determine the coefficients.

$$(x - 2)^5 = 1(x)^5 + 5(x)^4(-2)^1 + 10(x)^3(-2)^2 + 10(x)^2(-2)^3 + 5(x)^1(-2)^4 + 1(-2)^5$$

I used the terms x and -2 and applied the pattern for expanding a binomial. The exponents in each term always add up to 5. As the x exponents decrease by 1 each time, the exponents of -2 increase by 1.

$$= x^5 + 5(x^4)(-2) + 10(x^3)(4) + 10(x^2)(-8) + 5(x)(16) + (-32)$$

I simplified each term.

$$= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$$

EXAMPLE 3 Selecting a strategy to expand a binomial power involving a variable in each term

Expand and simplify $(5x + 2y)^3$.

Jason's Solution

			1			
		1		1		
	1		2		1	
1		3		3		1

Since the exponent of the binomial is 3, I wrote out the 3rd row of Pascal's triangle.

$$(5x + 2y)^3$$
$$= 1(5x)^3 + 3(5x)^2(2y)^1 + 3(5x)^1(2y)^2 + 1(2y)^3$$

I used the terms $5x$ and $2y$ and applied the pattern for expanding a binomial.

$$= 1(125x^3) + 3(25x^2)(2y) + 3(5x)(4y^2) + 1(8y^3)$$

I simplified each term.

$$= 125x^3 + 150x^2y + 60xy^2 + 8y^3$$

In Summary

Key Ideas

- The arrangement of numbers shown is called Pascal's triangle. Each row is generated by calculating the sum of pairs of consecutive terms in the previous row.

1	0th row
1 1	1st row
1 2 1	2nd row
1 3 3 1	3rd row
1 4 6 4 1	4th row
1 5 10 10 5 1	5th row

- The numbers in Pascal's triangle correspond to the coefficients in the expansion of binomials raised to whole-number exponents.

Need to Know

- Pascal's triangle has many interesting relationships among its numbers. Some of these relationships are recursive.
  - For example, down the sides are constant sequences: 1, 1, 1, ...
  - The diagonal beside that is the counting numbers, 1, 2, 3, ... , which form an arithmetic sequence.
  - The next diagonal is the triangular numbers, 1, 3, 6, 10, ... , which can be defined by the recursive formula

$$t_1 = 1,\ t_n = t_{n-1} + n, \quad \text{where } n \in \mathbf{N} \text{ and } n > 1.$$

- There are patterns in the expansions of a binomial $(a + b)^n$:
  - Each term in the expansion is the product of a number from Pascal's triangle, a power of a, and a power of b.
  - The coefficients in the expansion correspond to the numbers in the nth row in Pascal's triangle.
  - In the expansion, the exponents of a start at n and decrease by 1 down to zero, while the exponents of b start at zero and increase by 1 up to n.
  - In each term, the sum of the exponents of a and b is always n.

CHECK Your Understanding

1. The first four entries of the 12th row of Pascal's triangle are 1, 12, 66, and 220. Determine the first four entries of the 13th row of the triangle.

2. Expand and simplify each binomial power.
   a) $(x + 2)^5$ b) $(x - 1)^6$ c) $(2x - 3)^3$

3. Expand and simplify the first three terms of each binomial power.
   a) $(x + 5)^{10}$ b) $(x - 2)^8$ c) $(2x - 7)^9$

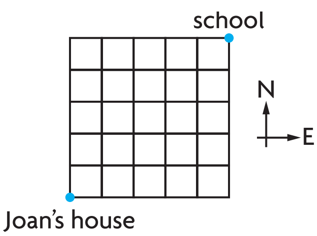
PRACTISING

4. Expand and simplify each binomial power.
   K a) $(k + 3)^4$ b) $(y - 5)^6$ c) $(3q - 4)^4$ d) $(2x + 7y)^3$ e) $\left(\sqrt{2}x + \sqrt{3}\right)^6$ f) $(2z^3 - 3y^2)^5$

5. Expand and simplify the first three terms of each binomial power.
   a) $(x - 2)^{13}$ b) $(3y + 5)^9$ c) $(z^5 - z^3)^{11}$ d) $\left(\sqrt{a} + \sqrt{5}\right)^{10}$ e) $\left(3b^2 - \frac{2}{b}\right)^{14}$ f) $(5x^3 + 3y^2)^8$

6. Using the pattern for expanding a binomial, expand each binomial power to describe a pattern in Pascal's triangle.
   a) $2^n = (1 + 1)^n$
   b) $0 = (1 - 1)^n$, where $n \geq 1$

7. Using the pattern for expanding a binomial, expand and simplify the expression $\frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right]$, where $n = 1, 2, 3$, and 4. How are the terms related?



8. **A** Using the diagram at the left, determine the number of different ways that Joan could walk to school from her house if she always travels either north or east.

9. **T** Explain, without calculating, how you can use the pattern for expanding a binomial to expand $(x + y + z)^{10}$.

10. Expand and simplify $(3x - 5y)^6$.

11. **C** Summarize the methods of expanding a binomial power and determining a term in an expansion.

Extending

12. If a relation is linear, the 1st differences are constant. If the 2nd differences are also constant, the relation is quadratic. Use the pattern for expanding a binomial to demonstrate that if a relation is cubic, the third differences are constant. (*Hint:* You may want to look at x^3 and $(x + 1)^3$.)

13. When a fair coin is tossed, the probability of getting heads or tails is $\frac{1}{2}$. Expand and simplify the first three terms in the expression $\left(\frac{1}{2} + \frac{1}{2}\right)^{10}$. How are the terms related to tossing the coin 10 times?

7 Chapter Review

FREQUENTLY ASKED Questions

Q: What strategies can you use to determine the sum of an arithmetic sequence?

A1: Write the series out twice, one above the other, once forward and once backward. When the terms of the two series are paired together, they have the same sum. This method works for calculating the sum of any arithmetic series.

A2: You can use either of the formulas $S_n = \frac{n[2a + (n - 1)d]}{2}$ or $S_n = \frac{n(t_1 + t_n)}{2}$. In both cases, $n \in \mathbf{N}$, a is the first term, and d is the common difference. For either formula, you need to know the number of terms in the series and the first term. If you know the last term, use the formula in terms of t_1 and t_n. If you can calculate the common difference, use the formula in terms of a and d.

Study Aid

- See Lesson 7.5, Examples 1, 2, and 3.
- Try Chapter Review Questions 14 to 17.

Q: How do you determine the sum of a geometric series?

A: You can use either of the formulas $S_n = \frac{a(r^n - 1)}{r - 1}$ or $S_n = \frac{t_{n+1} - t_1}{r - 1}$, where $r \neq 1$. In both cases, $n \in \mathbf{N}$, a is the first term, and r is the common ratio. For either formula, you need to know the common ratio and the first term. If you know the $(n + 1)$th term, use the formula in terms of t_1 and t_{n+1}. If you can calculate the number of terms, use the formula in terms of a, r, and n.

Study Aid

- See Lesson 7.6, Examples 1, 2, and 3.
- Try Chapter Review Questions 18 to 22.

Q: How do you expand a binomial power?

A: Use the pattern for expanding a binomial. Suppose you have the binomial $(a + b)^n$, where n is a whole number. Choose the nth row of Pascal's triangle for the coefficients. Each term in the expansion is a product of a number from Pascal's triangle, a power of a, and a power of b. The exponents of a start at n and decrease by 1 down to zero, while the exponents of b start at zero and increase by 1 up to n. In each term of the expansion, the sum of the exponents of a and b is always n.

Study Aid

- See Lesson 7.7, Examples 1, 2, and 3.
- Try Chapter Review Question 23.

PRACTICE Questions

Lesson 7.1

1. Represent the sequence 2, 8, 14, 20, ...
- **a)** in words
- **b)** algebraically
- **c)** graphically

2. How can you determine whether a sequence is arithmetic?

3. For each arithmetic sequence, state
i) the general term **ii)** the recursive formula
- **a)** 58, 73, 88, ...
- **b)** $-49, -40, -31, \ldots$
- **c)** 81, 75, 69, ...

4. Determine the 100th term of the arithmetic sequence with $t_7 = 465$ and $t_{13} = 219$.

5. A student plants a seed. After the seed sprouts, the student monitors the growth of the plant by measuring its height every week. The height after each of the first three weeks was 7 mm, 20 mm, and 33 mm, respectively. If this pattern of growth continues, in what week will the plant be more than 100 mm tall?

Lesson 7.2

6. How can you determine whether a sequence is geometric?

7. **i)** Determine whether each sequence is arithmetic, geometric, or neither.
ii) If a sequence is arithmetic or geometric, determine t_6.
- **a)** 5, 15, 45, ...
- **b)** 0, 3, 8, ...
- **c)** 288, 14.4, 0.72, ...
- **d)** 10, 50, 90, ...
- **e)** 19, 10, 1, ...
- **f)** 512, 384, 288, ...

8. For each geometric sequence, determine
i) the recursive formula **ii)** the general term
iii) the first five terms
- **a)** the first term is 7 and the common ratio is -3
- **b)** $a = 12$ and $r = \frac{1}{2}$
- **c)** the second term is 36 and the third term is 144

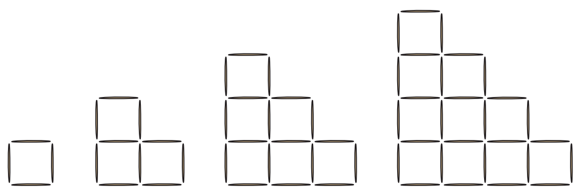
9. **i)** Determine the type of each sequence (arithmetic, geometric, or neither), where $n \in \mathbf{N}$.
ii) State the first five terms.
- **a)** $t_n = 4n + 5$
- **b)** $t_n = \dfrac{1}{7n - 3}$
- **c)** $t_n = n^2 - 1$
- **d)** $t_1 = -17, t_n = t_{n-1} + n - 1$, where $n > 1$

10. In a laboratory experiment, the count of a certain bacteria doubles every hour.
- **a)** At 1 p.m., there were 23 000 bacteria present. How many bacteria will be present at midnight?
- **b)** Can this model be used to determine the bacterial population at any time? Explain.

11. Guy purchased a rare stamp for $820 in 2001. If the value of the stamp increases by 10% per year, how much will the stamp be worth in 2010?

Lesson 7.3

12. Toothpicks are used to make a sequence of stacked squares as shown. Determine a rule for calculating t_n, the number of toothpicks needed for a stack of squares n high. Explain your reasoning.



13. Determine the 100th term of the sequence $\frac{1}{2}, \frac{2}{5}, \frac{3}{8}, \frac{4}{11}, \ldots$. Explain your reasoning.

Lesson 7.5

14. For each arithmetic series, calculate the sum of the first 50 terms.
- **a)** $1 + 9 + 17 + \ldots$
- **b)** $21 + 17 + 13 + \ldots$
- **c)** $31 + 52 + 73 + \ldots$
- **d)** $-9 - 14 - 19 - \ldots$
- **e)** $17.5 + 18.9 + 20.3 + \ldots$
- **f)** $-39 - 31 - 23 - \ldots$

15. Determine the sum of the first 25 terms of an arithmetic series in which

a) the first term is 24 and the common difference is 11

b) $t_1 = 91$ and $t_{25} = 374$

c) $t_1 = 84$ and $t_2 = 57$

d) the third term is 42 and the terms decrease by 11

e) the 12th term is 19 and the terms decrease by 4

f) $t_5 = 142$ and $t_{15} = 12$

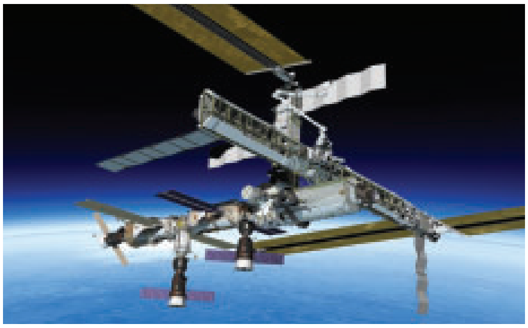
16. Calculate the sum of each series.

a) $1 + 13 + 25 + ... + 145$

b) $9 + 42 + 75 + ... + 4068$

c) $123 + 118 + 113 + ... - 122$

17. A spacecraft leaves an orbiting space station to descend to the planet below. The spacecraft descends 64 m during the first second and then engages its reverse thrusters to slow down its descent. It travels 7 m less during each second afterward. If the spacecraft lands after 10 s, how far did it descend?



Lesson 7.6

18. For each geometric series, calculate t_6 and S_6.

a) $11 + 33 + 99 + ...$

b) $0.111\,111 + 1.111\,11 + 11.1111 + ...$

c) $6 - 12 + 24 - ...$

d) $32\,805 + 21\,870 + 14\,580 + ...$

e) $17 - 25.5 + 38.25 - ...$

f) $\frac{1}{2} + \frac{3}{10} + \frac{9}{50} + ...$

19. Determine the sum of the first eight terms of the geometric series in which

a) the first term is -6 and the common ratio is 4

b) $t_1 = 42$ and $t_9 = 2112$

c) the first term is 320 and the second term is 80

d) the third term is 35 and the terms increase by a factor of 5

20. A catering company has 15 customer orders during its first month. For each month afterward, the company has double the number of orders than the previous month. How many orders in total did the company fill at the end of its first year?

21. The 1st, 5th, and 13th terms of an arithmetic sequence are the first three terms of a geometric sequence with common ratio 2. If the 21st term of the arithmetic sequence is 72, calculate the sum of the first 10 terms of the geometric sequence.

22. Calculate the sum of each series.

a) $7 + 14 + 28 + ... + 3584$

b) $-3 - 6 - 12 - 24 - ... - 768$

c) $1 + \frac{5}{2} + \frac{25}{4} + ... + \frac{15\,625}{64}$

d) $96\,000 - 48\,000 + 24\,000 - ... + 375$

e) $1000 + 1000(1.06) + 1000(1.06)^2 + ... + 1000(1.06)^{12}$

Lesson 7.7

23. Expand and simplify.

a) $(a + 6)^4$

b) $(b - 3)^5$

c) $(2c + 5)^3$

d) $(4 - 3d)^6$

e) $(5e - 2f)^4$

f) $\left(3f^2 - \frac{2}{f}\right)^4$

7 Chapter Self-Test

1. **i)** Determine the first five terms of each sequence, where $n \in \mathbf{N}$.
   ii) Determine whether each sequence is arithmetic, geometric, or neither.
   a) $t_n = 5 \times 3^{n+1}$
   b) $t_n = \dfrac{3n + 2}{2n + 1}$
   c) $t_n = 5n$
   d) $t_1 = 5, t_n = 7t_{n-1}$, where $n > 1$
   e) $t_1 = 19, t_n = 1 - t_{n-1}$, where $n > 1$
   f) $t_1 = 7, t_2 = 13, t_n = 2t_{n-1} - t_{n-2}$, where $n > 2$

2. For each sequence, determine
   i) the general term
   ii) the recursive formula
   a) a geometric sequence with $a = -9$ and $r = -11$
   b) an arithmetic sequence with second term 123 and third term -456

3. Determine the number of terms in each sequence.
   a) 18, 25, 32, ... , 193
   b) 2, -10, 50, ... , $-156\ 250$

4. Expand and simplify each binomial power.
   a) $(x - 5)^4$
   b) $(2x + 3y)^3$

5. Calculate the sum of each series.
   a) $19 + 33 + 47 + \ldots + 439$
   b) the first 10 terms of the series $10\ 000 + 12\ 000 + 14\ 400 + \ldots$

6. A sequence is defined by the recursive formula $t_1 = 4$, $t_2 = 5$, $t_n = \dfrac{t_{n-1} + 1}{t_{n-2}}$, where $n \in \mathbf{N}$ and $n > 2$. Determine t_{123}. Explain your reasoning.

7. Your grandparents put aside \$100 for you on your first birthday. Every following year, they put away \$75 more than they did the previous year. How much money will have been put aside by the time you are 21?

8. Determine the next three terms of each sequence.
   a) 1, 7, 8, 15, 23, 38, ...
   b) $p^2 + 2q, p^3 - 3q, p^4 + 4q, p^5 - 5q, \ldots$
   c) $\dfrac{25}{3}, \dfrac{19}{6}, \dfrac{13}{9}, \dfrac{7}{12}, \dfrac{1}{15}, \ldots$

7 Chapter Task

Allergy Medicine

It is estimated that 1 in 7 Canadians suffers from seasonal allergies such as hay fever. A typical treatment for hay fever is over-the-counter antihistamines. Tom decides to try a certain brand of antihistamine. The label says:

- The half-life of the antihistamine in the body is 16 h.
- For his size, maximum relief is felt when there are 150 mg to 180 mg in the body. Side effects (sleepiness, headaches, nausea) can occur when more than 180 mg are in the body.
- Each pill contains 30 mg of the active ingredient. It is unhealthy to ingest more than 180 mg within a 24 h period.

? How many hay fever pills should Tom take, and how often should he take them?

A. What are some conditions that would be reasonable when taking medication? For example, think about dosages, as well as times of the day when you would take the medication.

B. Determine three different schedules for taking the pills considering the appropriate amounts of the medication to ingest and your conditions in part A.

C. For each of your schedules, determine the amount of the medication present in Tom's body for the first few days.

D. Based on your calculations in part C, which schedule is best for Tom? Is another schedule more appropriate?

Task | *Checklist*

- ✔ Did you justify your "reasonable" conditions?
- ✔ Did you show your work?
- ✔ Did you support your choice of medication schedule?
- ✔ Did you explain your thinking clearly?

Chapter 8

Discrete Functions: Financial Applications

▶ GOALS

You will be able to

- Determine how interest is earned and charged
- Use the difference between future value and present value to solve problems
- Solve problems about money invested at regular intervals over a period of time
- Calculate payments that must be made when a purchase is financed over a period of time

? On your first birthday, your parents deposit $1000 into a bank account that pays 3% interest per year on the balance. How can you determine which amount will be closest to what will be in the account when you are ready to go to college or university, $1100, $1500, $2000, or $2500?

8 Getting Started

Study Aid

For help, see the following lessons in Chapters 4 and 7.

Question	Lesson
1, 2, 3	7.1 and 7.2
4	7.5 and 7.6
5, 6	4.7

SKILLS AND CONCEPTS You Need

1. For each sequence, determine
i) the next two terms **ii)** the general term **iii)** the recursive formula

a) 7, 11, 15, 19, ...
b) 58, 31, 4, −23, ...
c) 5, 20, 80, 320, ...
d) 1000, −500, 250, ...

2. The fourth term of an arithmetic sequence is 46 and the sixth term is 248. Determine
a) the 5th term
b) the common difference
c) the 1st term
d) the 100th term

3. The 4th, 5th, and 6th terms of a sequence are 9261, 9724.05, and 10 210.2525, respectively.
a) What type of sequence is this? Justify your reasoning.
b) Determine the recursive formula.
c) State the general term.
d) Determine the 10th term.

4. Determine the sum of the first 10 terms of each series.
a) $3 + 5 + 7 + \ldots$
b) $-27 - 21 - 15 - \ldots$
c) $48 + 31 + 14 + \ldots$
d) $8\,192\,000 - 4\,096\,000 + 2\,048\,000 - \ldots$

5. The population of a city is 200 000 and increases by 5% per year.
a) Determine the expected population at the end of each of the next 3 years.
b) What will be the expected population 10 years from now?

6. Determine the value of x that makes the equation $2^x = 4096$ true.

7. Solve each equation by graphing the corresponding functions on a graphing calculator. Round your answers to two decimal places.
a) $2^x = 1\,000\,000$
b) $5 \times 3^x = 228$
c) $14\,000 \times 1.07^x = 30\,000$
d) $250 \times 1.0045^{12x} = 400$

8. Complete the chart shown by writing what you know about exponential functions.

Example:	Visual representation:
Definition in your own words:	Personal association:

APPLYING What You Know

Saving for a Trip

Mark is saving for an overseas trip that costs $1895. Each week, he deposits $50 into a savings account that pays him 0.35% on the minimum monthly balance. He already has $200 in his account at the start of the month.

? Assuming that the cost of the trip stays the same, how long will it take Mark to save enough money to pay for it?

A. How do you know it will take less than three years for Mark to meet his goal?

B. When Mark pays for the trip, he will have to pay the Goods and Services Tax (GST). Determine the additional costs he will incur. Calculate the total price of the trip.

C. What do you think earning 0.35% on the minimum monthly balance means? How will earning interest affect the amount of time he will have to save?

D. Determine how much Mark will have in his account at the end of the 1st, 2nd, and 3rd months. Record your values in a table as shown.

Month	Starting Balance	Deposits	Interest Earned	Final Balance
1				
2				
3				

E. Use a spreadsheet or the lists on a graphing calculator, or continue the table in part D to determine how long it will take Mark to save for the trip.

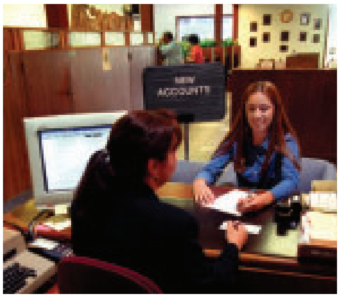
YOU WILL NEED

- graphing calculator
- spreadsheet software (optional)

8.1 Simple Interest

YOU WILL NEED

- graphing calculator
- spreadsheet software



principal

a sum of money that is borrowed or invested

simple interest

interest earned or paid only on the original sum of money that was invested or borrowed

interest

the money earned from an investment or the cost of borrowing money

amount

the total value of an investment or loan. The amount is given by $A = P + I$, where A is the amount, P is the principal, and I is the interest.

Tech Support

For help using a graphing calculator to enter lists and to create scatter plots, see Technical Appendix, B-11.

GOAL

Calculate simple interest.

INVESTIGATE the Math

Amanda wants to invest \$2000. Her bank will pay 6% of the **principal** per year each year the money is kept in a savings account that earns **simple interest**.

? What function can be used to model the growth of Amanda's money?

A. Calculate the **interest** earned and the **amount** of the investment at the end of the first year. Record your results in a table as shown.

Year	Interest Earned	Amount
0	—	\$2000
1		
2		
3		

B. Calculate the interest earned and the amount of the investment at the end of the second and third years. Record your results in your table.

C. Enter your data for Year and Amount into either lists on a graphing calculator or columns in a spreadsheet.

D. Create a scatter plot from your two lists or columns, using Year as the independent variable.

E. What type of function best models the growth of Amanda's money? You may need to calculate more data points before you decide. Explain your reasoning.

F. Determine the equation of the function that models the amount of Amanda's investment over time.

Reflecting

G. What type of sequence could you use to represent the amount of Amanda's investment for successive years? How do you know?

H. How does the recursive formula for this sequence relate to her investment?

I. How do the principal, interest, and amount of Amanda's investment relate to
- the sequence from part G that represents the amount of the investment over time?
- the function from part F that represents the amount of the investment over time?

APPLY the Math

EXAMPLE 1 Representing any situation earning simple interest as a function

Allen invests \$3240 at 2.4%/a simple interest.

a) Calculate the interest earned each year.

b) Calculate the amount and the total interest earned after 20 years.

c) Determine the total amount, A, and the interest, I, earned if he invested a principal, $\$P$, for t years at r%/a simple interest.

Communication *Tip*

Interest rates are often advertised as a certain percent per year. So a rate of 5% means that 5% interest is earned each year. These rates are sometimes abbreviated to 5%/a, which means 5% per *annum*, or year.

Jasmine's Solution

a) $I = 0.024 \times \$3240$

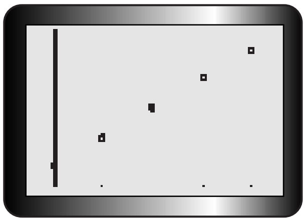
$= \$77.76$

Each year, Allen earns 2.4% of the principal as interest. 2.4% as a decimal is 0.024. To calculate the interest earned each year, I multiplied the principal by the interest rate.

b)

Year	Interest (\$)	Amount (\$)
0	0	3240
1	77.76	3240 + 77.76 = 3317.76
2	77.76	3317.76 + 77.76 = 3395.52
3	77.76	3395.52 + 77.76 = 3473.28
4	77.76	3473.28 + 77.76 = 3551.04

I set up a table to calculate the amount at the end of each year. I added the interest earned each year to the previous amount. Then I entered the year and amount into separate lists on a graphing calculator.



I used the calculator to create a scatter plot of amount versus time. The graph is linear. I used two points to calculate the slope and I found that it was $77.76. That is the rate of change of the amount. The *y*-intercept is Allen's principal of $3240.

$$f(t) = 77.76t + 3240$$

I used the slope and *y*-intercept to create a linear function in terms of *t*, the time in years, and $f(t)$, the amount.

$$\begin{aligned} f(20) &= 77.76(20) + 3240 \\ &= 4795.20 \end{aligned}$$

To determine the amount after 20 years, I substituted $t = 20$.

$$\begin{aligned} I &= 4795.20 - 3240 \\ &= 1555.20 \end{aligned}$$

To determine the total interest earned, I subtracted the principal from the amount.

After 20 years, Allen will have $4795.20 and will have earned $1555.20 in interest.

c) end of 1st year:

$$I_1 = Pr$$

After one year, Allen would earn r% of his original investment of $*P*.

end of 2nd year:

$$I_2 = Pr + Pr = 2Pr$$

Each year, the total interest earned would go up by the same amount. The increase would be the interest earned in one year, $P \times r$.

end of 3rd year:

$$I_3 = Pr + Pr + Pr = 3Pr$$

end of *t*th year:

$$I = Prt$$

Since the interest earned at the end of each year depends on time, I wrote a formula for interest in terms of *t*, the time in years.

$$A = P + I$$

$$A = P + Prt$$

The amount is the sum of the principal and the interest. Since interest depends on time, the amount must also depend on time.

$$A = P(1 + rt)$$

I wrote the formula for the amount by factoring out the *P*.

The total amount of an investment of $*P* for *t* years at r%/a simple interest is $A = P(1 + rt)$, and the total interest earned is $I = Prt$.

EXAMPLE 2 Using a spreadsheet to represent the amount owed

Tina borrows \$15 000 at 6.8%/a simple interest. She plans to pay back the loan in 10 years. Calculate how much she will owe at the end of each year during this period.

Tom's Solution

	A	B	C
1	Time (Years)	Total Interest Charged	Total Amount of Loan
2			\$15 000.00
3	1	"= C2* (6.8/100)"	"= C2 + B3"
4	2	"= C2* (6.8/100)"	"= C3 + B4"

I set up a spreadsheet to calculate the interest charged every year and the loan amount. Each year, 6.8% of \$15 000, or \$1020, will be charged in interest.

	A	B	C
1	Time (Years)	Total Interest Charged	Total Amount of Loan
2			\$15 000.00
3	1	\$1 020.00	\$16 020.00
4	2	\$1 020.00	\$17 040.00
5	3	\$1 020.00	\$18 060.00
6	4	\$1 020.00	\$19 080.00
7	5	\$1 020.00	\$20 100.00
8	6	\$1 020.00	\$21 120.00
9	7	\$1 020.00	\$22 140.00
10	8	\$1 020.00	\$23 160.00
11	9	\$1 020.00	\$24 180.00
12	10	\$1 020.00	\$25 200.00

I used the spreadsheet to calculate the amount Tina would need to pay back at the end of each year during the 10-year period.

Tech Support

For help using a spreadsheet to enter values and formulas, and to fill down, see Technical Appendix, B-21.

EXAMPLE 3 Selecting a strategy to calculate the amount owed after less than a year

Philip borrows \$540 for 85 days by taking a cash advance on his credit card. The interest rate is 26%/a simple interest. How much will he need to pay back at the end of the loan period, and how much interest will he have paid?

Lara's Solution

$$t = \frac{85}{365}$$

Philip isn't borrowing the money for a full year, so I expressed the time as a fraction of 365 days in a year. I knew the principal, *P*, and I wrote the interest rate, *r*, as a decimal.

$$P = \$540$$

$$r = 26\% = 0.26$$

$$\begin{aligned} A &= P(1 + rt) \\ &= (\$540)\left(1 + 0.26 \times \frac{85}{365}\right) \end{aligned}$$

To calculate the amount at the end of the loan period, I substituted the values of *P*, *r*, and *t* into the formula.

$$\begin{aligned} &\doteq (\$540)(1.061) \\ &= \$572.70 \end{aligned}$$

I rounded to the nearest cent.

Of the \$572.70 that Philip has to pay back, \$32.70 is interest.

I subtracted the principal to get the interest.

EXAMPLE 4 Calculating the time needed to earn a specific amount on an investment

Tanya invests \$4850 at 7.6%/a simple interest. If she wants the money to increase to \$8000, how long will she need to invest her money?

Josh's Solution

$$P = \$4850$$
$$r = 7.6\% = 0.076$$
$$A = \$8000$$

I knew the principal, the interest rate, and the total amount.

$$A = P + Prt$$
$$8000 = 4850 + (4850)(0.076)t$$
$$8000 = 4850 + 368.6t$$
$$3150 = 368.6t$$

I substituted the values of P, r, and A into the formula and solved for t.

$$t = \frac{3150}{368.6}$$
$$\doteq 8.546$$

When I solved for t, I got a value greater than 8.

$$0.546 \times 365 \text{ days} \doteq 199.2 \text{ days}$$

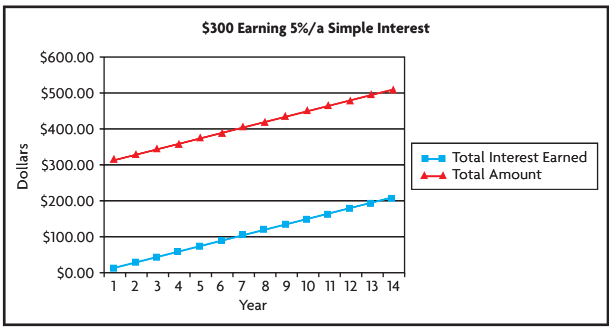
The 8 meant 8 years, so I had to figure out what 0.546 of a year was.

Tanya would have to invest her money for 8 years and 200 days to get \$8000.

In Summary

Key Ideas

- Simple interest is calculated only on the principal.
- The total amount, A, and interest earned, I, are linear functions in terms of time, so their graphs are straight lines (see graph below). The values of A and I at the end of each interest period form the terms of two arithmetic sequences.



(*continued*)

Need to Know

- Simple interest can be calculated using the formula $I = Prt$, where I is the interest; P is the principal; r is the interest rate, expressed as a decimal; and t is time, expressed in the same period as the interest rate, usually per year.
- The total amount, A, of an investment earning simple interest can be calculated using the formula $A = P + Prt$ or $A = P(1 + rt)$.
- Unless otherwise stated, an interest rate is assumed to be per year.

CHECK *Your Understanding*

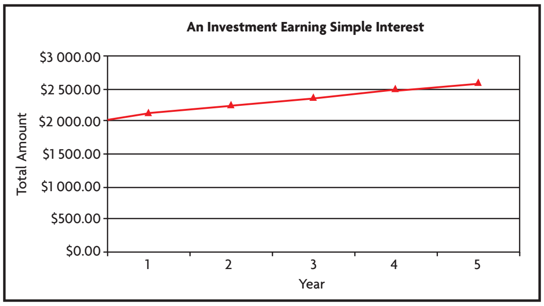
1. Each situation represents an investment earning simple interest. Calculate the total amount at the end of each period.

i) 1st, 2nd, and 3rd years **ii)** 15 years

a) \$500 at 6.4%/a
b) \$1250 at 4.1%/a
c) \$25 000 at 5%/a
d) \$1700 at 2.3%/a

2. The graph at the right represents the total amount of an investment of principal \$$P$ earning a fixed rate of simple interest over a period of 5 years.

a) What is the principal?
b) How much interest is earned in 5 years?
c) What interest rate is being applied?
d) State the equation that represents the amount as a function of time.



3. Michel invests \$850 at 7%/a simple interest. How long will he have to leave his investment in the bank before earning \$200 in interest?

4. Sally has a balance of \$2845 on her credit card. What rate of simple interest is she being charged if she must pay \$26.19 interest for the 12 days her payment is late?

PRACTISING

5. For each investment, calculate the interest earned and the total amount.
K

	Principal	Rate of Simple Interest per Year	Time
a)	\$500	4.8%	8 years
b)	\$3 200	9.8%	12 years
c)	\$5 000	3.9%	16 months
d)	\$128	18%	5 months
e)	\$50 000	24%	17 weeks
f)	\$4 500	12%	100 days

6. Mario borrows $4800 for 8.5 years at a fixed rate of simple interest. At the end of that time, he owes $8000. What interest rate is he being charged?

7. How much money must be invested at 6.3%/a simple interest to earn $250 in interest each month?

8. Nina deposits $3500 into a savings account. The rate of simple interest is 5.5%/a.

a) By how much does the amount in her account increase each year?

b) Determine the amount in her account at the end of each of the first 5 years.

c) State the total amount as the general term of a sequence.

d) Graph this sequence.

9. Ahmad deposits an amount on September 1, 2005, into an account that earns simple interest quarterly. His bank sends him statements after each quarter. The amounts for the first four quarters are shown.

a) How much did Ahmad invest?

b) What rate of simple interest is he earning?

Statement	Date	Balance
1	Dec. 1, 2005	$3994.32
2	Mar. 1, 2006	$4248.64
3	Jun. 1, 2006	$4502.96
4	Sept. 1, 2006	$4757.28

Year	Amount Owed
1	$2081.25
2	$2312.50
3	$2543.75
4	$2775.00
5	$3006.25

10. **A** Anita borrows some money at a fixed rate of simple interest. The amount she owes at the end of each of the first five years is shown at the left.

a) How much did Anita borrow?

b) State the total amount as the general term of a sequence.

c) How much time will have passed before Anita owes $7500?

11. **T** Len invests $5200 at 3%/a simple interest, while his friend Dave invests $3600 at 5%/a simple interest. How long will it take for Dave's investment to be worth more than Len's?

12. **C** Lotti invests some money at a fixed rate of simple interest. She uses the function $A(t) = 750 + 27.75t$ to calculate how much her investment will be worth after t years. How much did she invest and what interest rate is she earning? Explain your reasoning.

Extending

13. The doubling time of an investment is the length of time it takes for the total amount invested to become double the original amount invested. Determine a formula for the doubling time, D, of an investment of principal $\$P$ earning a rate of simple interest of r%/a.

14. Sara's parents decide to invest $500 on each of her birthdays from the day she is born until she becomes 25. Each investment earns 6.4%/a simple interest. What will be the total amount of the investments when Sara is 25?

8.2 Compound Interest: Future Value

GOAL

Determine the future value of a principal being charged or earning compound interest.

YOU WILL NEED

- graphing calculator
- spreadsheet software (optional)

LEARN ABOUT the Math

Mena invests $2000 in a bank account that pays 6%/a compounded annually. The savings account is called the "Accumulator" and pays compound interest.

compound interest

interest that is added to the principal *before* new interest earned is calculated. So interest is calculated on the principal *and* on interest already earned. Interest is paid at regular time intervals called the **compounding period.**

compounding period

the intervals at which interest is calculated; for example,
annually ⇒ 1 time per year
semi-annually ⇒ 2 times per year
quarterly ⇒ 4 times per year
monthly ⇒ 12 times per year

? What type of function will model the growth of Mena's money?

A. Calculate the interest earned and amount at the end of the first year. Record your answers in a table as shown.

Year	Balance at Start of Year	Interest Earned	Balance at End of Year
0	—	—	$2000
1	$2000		
2			
3			
4			
5			

B. Complete the table for the 2nd to 5th years.

C. Enter your data for Year and Balance at End of Year into either lists on a graphing calculator or columns in a spreadsheet.

D. Create a scatter plot, using Year as the independent variable.

E. What type of function best models the growth of Mena's money? You may need to calculate more data points before you decide. Explain how you know.

F. Determine the function that models the amount of her investment over time.

Reflecting

G. Compare the total amount of Mena's investment with that based on the same principal earning simple interest. What is the advantage of earning compound interest over simple interest?

H. How are compound interest, exponential functions, and geometric sequences related?

APPLY the Math

EXAMPLE 1 Representing any situation earning compound interest as a function

future value

the total amount, A, of an investment after a certain length of time

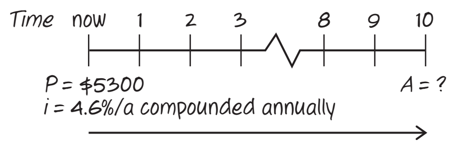
Tim borrows \$5300 at 4.6%/a compounded annually.

a) How much will he have to pay back if he borrows the money for 10 years?

b) Determine the future value, A, and interest earned, I, if he invested a principal of \$$P$ for n years at i%/a compounded annually.

Shelley's Solution

a)



I drew a timeline for the situation.

end of 1st year:

$$A = P(1 + rt)$$
$$= 5300[1 + 0.046(1)]$$
$$= 5300(1.046)$$
$$= \$5543.80$$

At the end of the first year, Tim gets charged 4.6% of his original \$5300 loan. So I calculated the amount he would owe at the end of that year.

end of 2nd year:

$$A = 5543.80[1 + 0.046(1)]$$
$$= 5543.80(1.046)$$
$$\doteq \$5798.81$$

At the end of the second year, he gets charged 4.6% of the amount he owed at the end of the first year, \$5543.80. So I calculated the amount he would owe at the end of that year.

end of 3rd year:

$$A = 5798.81[1 + 0.046(1)]$$
$$= 5798.81(1.046)$$
$$= \$6065.56$$

I used the same method to calculate the amount at the end of the third year. Each time, I rounded to the nearest cent.

$$t_1 = 5300 \times 1.046^1 = \$5543.80$$
$$t_2 = 5300 \times 1.046^2 \doteq \$5798.81$$
$$t_3 = 5300 \times 1.046^3 \doteq \$6065.56$$
$$\vdots$$
$$t_n = 5300 \times 1.046^n$$

I noticed that I was multiplying by 1.046 each time. This is a geometric sequence with common ratio 1.046. The general term of the sequence is $t_n = 5300 \times 1.046^n$. I used this formula to calculate the first three terms, and I got the same numbers as in my previous calculations.

$$t_{10} = 5300 \times 1.046^{10}$$
$$\doteq \$8309.84$$

To determine how much Tim would owe after 10 years, I substituted $n = 10$ into the formula for the general term.

Tim would have to pay back \$8309.84 after 10 years.

b) end of 1st year:

$$A = P(1 + in)$$
$$A_1 = P[1 + i(1)]$$
$$= P(1 + i)$$

end of 2nd year:

$$A_2 = [P(1 + i)](1 + i)$$
$$= P(1 + i)^2$$

end of 3rd year:

$$A_3 = [P(1 + i)^2](1 + i)$$
$$= P(1 + i)^3$$

end of nth year:

$$A = P(1 + i)^n$$

For compound interest, the amount or future value depends on time, as it did for simple interest. Since the interest rate is i%/a, I substituted i for r into the formula for the total amount and calculated the future value for the 1st, 2nd, and 3rd years.

This is a geometric sequence with first term $P(1 + i)$ and common ratio $1 + i$, so I wrote the general term, which gave me the amount after n years. The amount is an exponential function in terms of time.

$$I = A - P$$
$$= P(1 + i)^n - P$$
$$= P[(1 + i)^n - 1]$$

To determine the interest earned over a period of n years, I subtracted the principal from the total amount.

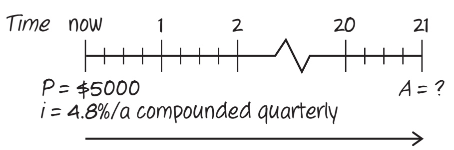
I factored out the common factor P.

The future value of an investment of $\$P$ for n years at i%/a compounded annually will be $A = P(1 + i)^n$, and the total interest earned will be $I = P[(1 + i)^n - 1]$.

EXAMPLE 2 Selecting a strategy to determine the amount when the compounding period is less than a year

Lara's grandparents invested $5000 at 4.8%/a compounded quarterly when she was born. How much will the investment be worth on her 21st birthday?

Herman's Solution



I drew a timeline for the situation.

$P = \$5000$

$i = 0.048 \div 4$

$= 0.012$

Since interest is paid quarterly for each compounding period, I divided the annual interest rate by 4 to get the interest rate.

$n = 21 \times 4$

$= 84$

Interest is paid 4 times per year, so I calculated the number of compounding periods. This is the total number of times that the interest would be calculated over the 21 years.

$A = P(1 + i)^n$

I used the formula $A = P(1 + i)^n$, where i is the interest rate per compounding period and n is the number of compounding periods.

$A = 5000(1 + 0.012)^{84}$

$\doteq \$13\,618.62$

I noticed that solving this problem is the same as solving a problem in which the money earns 1.2%/a compounded annually for 84 years.

The \$5000 investment will be worth \$13 618.62 on Lara's 21st birthday.

EXAMPLE 3 Calculating the difference of the amounts of two different investments

On her 15th birthday, Trudy invests \$10 000 at 8%/a compounded monthly. When Lina turns 45, she invests \$10 000 at 8%/a compounded monthly. If both women leave their investments until they are 65, how much more will Trudy's investment be worth?

Henry's Solution

$P = \$10\,000$

$i = \frac{0.08}{12}$

To calculate how much Trudy's investment will be worth when she turns 65, I first determined the interest rate per month as a fraction.

$n = (65 - 15) \times 12$

$= 600$

Since interest is compounded monthly and she is investing for 50 years, there will be $50 \times 12 = 600$ compounding periods.

$A = 10\,000\left(1 + \frac{0.08}{12}\right)^{600}$

$\doteq \$538\,781.94$

Trudy's investment will be worth \$538 781.94 when she turns 65.

$P = \$10\,000$

$i = \dfrac{0.08}{12}$

Lina's investment has the same principal and interest rate per month as Trudy's, but fewer compounding periods.

$n = (65 - 45) \times 12$
$= 240$

Since Lina invested for 20 years, there will be $20 \times 12 = 240$ compounding periods.

$A = 10\,000\left(1 + \dfrac{0.08}{12}\right)^{240}$

$\doteq \$49\,268.03$

Lina's investment will be worth \$49 268.03 when she turns 65.

$\$538\,781.94 - \$49\,268.03 = \$489\,513.91$

I subtracted Lina's amount from Trudy's amount.

Trudy's investment will be worth \$489 513.91 more than Lina's.

EXAMPLE 4 Comparing simple interest and compound interest

Nicolas invests \$1000. How long would it take for his investment to double for each type of interest earned?

a) 5%/a simple interest
b) 5%/a compounded semi-annually

Jesse's Solution

a) $P = \$1000$

$r = 5\% = 0.05$

I knew the principal and the interest rate.

$I = \$1000$

$I = Prt$

Since Nicolas's investment will double and he is earning simple interest, the interest earned must be the same as the principal.

$1000 = 1000(0.05)t$

I substituted the values of *P*, *r*, and *I* into the formula for the interest earned.

$$\frac{1000}{1000(0.05)} = \left(\frac{1000(0.05)}{1000(0.05)}\right)t$$

To solve for *t*, I divided both sides of the equation by 1000(0.05).

$20 = t$

It will take 20 years for Nicolas's investment to double at 5%/a simple interest.

b) $i = \frac{0.05}{2} = 0.025$

$A = P(1 + i)^n$

Since interest is paid semi-annually, I divided the annual interest rate by 2 to get the interest rate per half year.

$A = 1000(1 + 0.025)^{40}$

$\doteq \$2685.06$

I substituted $P = 1000$ and $i = 0.025$ into the formula for the amount. Then I used guess-and-check to determine n. I tried 20 years, or $n = 20 \times 2 = 40$ compounding periods.

$A = 1000(1 + 0.025)^{20}$

$\doteq \$1638.62$

The amount after 20 years was too much.

Next I tried 10 years, or $n = 10 \times 2 = 20$ compounding periods, but that wasn't enough.

$A = 1000(1 + 0.025)^{28}$

$\doteq \$1996.50$

Since my second guess was slightly closer to \$2000 than my first guess, I tried 14 years, or $n = 14 \times 2 = 28$ compounding periods. The result was close to double.

	A	B	C
1	Year	Simple Interest	Compound Interest
2	0	$1 000.00	$1 000.00
3	0.5	"=B2 + 1000*0.05/2"	"=C2*(1 + 0.025)"
4	1	"=B3 + 1000*0.05/2"	"=C3*(1 + 0.025)"

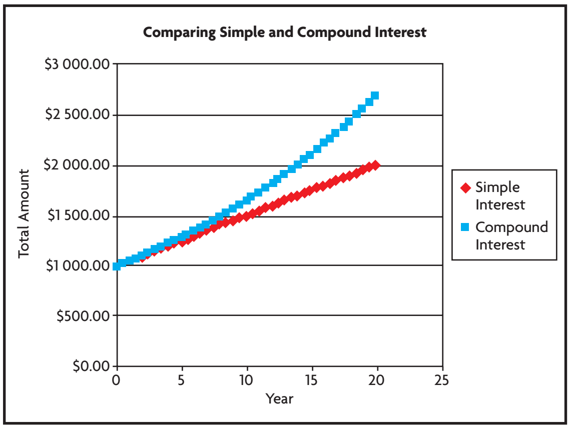
I then used a spreadsheet to check my result.

	A	B	C
1	Year	Simple Interest	Compound Interest
2	0	$1 000.00	$1 000.00
3	0.5	$1 025.00	$1 025.00
4	1	$1 050.00	$1 050.63
5	1.5	$1 075.00	$1 076.89
6	2	$1 100.00	$1 103.81
7	2.5	$1 125.00	$1 131.41
8	3	$1 150.00	$1 159.69

28	13	$1 650.00	$1 900.29
29	13.5	$1 675.00	$1 947.80
30	14	$1 700.00	$1 996.50
31	14.5	$1 725.00	$2 046.41

40	19	$1 950.00	$2 555.68
41	19.5	$1 975.00	$2 619.57
42	20	$2 000.00	$2 685.06

The investment earning simple interest took 20 years to double. If Nicolas earns compound interest, he gets interest on interest previously earned, so his investment grows faster. I used the spreadsheet to compare the two possibilities. With compound interest, the investment almost doubles after 14 years.



I graphed the amount of the investment for both cases. From the graph, the simple-interest situation is modelled by a linear function growing at a constant rate, while the compound-interest situation is modelled by an exponential function growing at a changing rate.

It will take about 14 years for Nicolas's investment to double at 5%/a compounded semi-annually.

Tech Support

For help using a spreadsheet to graph functions, see Technical Appendix, B-21.

In Summary

Key Ideas

- Compound interest is calculated by applying the interest rate to the principal and any interest already earned.
- The total amounts at the end of each interest period form a geometric sequence. So compound interest results in exponential growth.
- The total amount, A, of an investment after a certain period is called the future value of the investment.

Need to Know

- Banks pay or charge compound interest at regular intervals called the compounding period. If interest is compounded annually, then at the end of the first year, interest is calculated and added to the principal. At the end of the second year, interest is calculated on the new balance (principal plus interest earned from the previous year). This pattern continues every year the investment is kept.
- The future value of an investment earning compound interest can be calculated using the formula $A = P(1 + i)^n$, where A is the future value; P is the principal; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.
- The most common compounding periods are:

annually	1 time per year	i = annual interest rate	n = number of years
semi-annually	2 times per year	i = annual interest rate ÷ 2	n = number of years × 2
quarterly	4 times per year	i = annual interest rate ÷ 4	n = number of years × 4
monthly	12 times per year	i = annual interest rate ÷ 12	n = number of years × 12

- Compound interest can be calculated using the formula $I = A - P$ or $I = P[(1 + i)^n - 1]$, where I is the total interest.

CHECK Your Understanding

1. Copy and complete the table.

	Rate of Compound Interest per Year	Compounding Period	Time	Interest Rate per Compounding Period, *i*	Number of Compounding Periods, *n*
a)	5.4%	semi-annually	5 years		
b)	3.6%	monthly	3 years		
c)	2.9%	quarterly	7 years		
d)	2.6%	weekly	10 months		

2. **i)** Determine the amount owed at the end of each of the first five compounding periods.

ii) Determine the general term for the amount owed at the end of the *n*th compounding period.

	Amount Borrowed	Rate of Compound Interest per Year	Compounding Period
a)	$10 000	7.2%	annually
b)	$10 000	3.8%	semi-annually
c)	$10 000	6.8%	quarterly
d)	$10 000	10.8%	monthly

3. Calculate the future value of each investment. Draw a timeline for each.

	Principal	Rate of Compound Interest per Year	Compounding Period	Time
a)	$258	3.5%	annually	10 years
b)	$5 000	6.4%	semi-annually	20 years
c)	$1 200	2.8%	quarterly	6 years
d)	$45 000	6%	monthly	25 years

PRACTISING

4. For each investment, determine the future value and the total interest earned.
K

	Principal	Rate of Compound Interest per Year	Compounding Period	Time
a)	$4 000	3%	annually	4 years
b)	$7 500	6%	monthly	6 years
c)	$15 000	2.4%	quarterly	5 years
d)	$28 200	5.5%	semi-annually	10 years
e)	$850	3.65%	daily	1 year
f)	$2 225	5.2%	weekly	47 weeks

5. Sima invests some money in an account that earns a fixed rate of interest compounded annually. The amounts of the investment at the end of the first three years are shown at the right.
   a) Determine the annual rate of compound interest earned.
   b) How much did Sima invest?

Year	Total Amount
1	$4240.00
2	$4494.40
3	$4764.06

6. Chris invests $10 000 at 7.2%/a compounded monthly. How long will it take for his investment to grow to $25 000?

7. **A** Serena wants to borrow $15 000 and pay it back in 10 years. Interest rates are high, so the bank makes her two offers:
   - Option 1: Borrow the money at 10%/a compounded quarterly for the full term.
   - Option 2: Borrow the money at 12%/a compounded quarterly for 5 years and then renegotiate the loan based on the new balance for the last 5 years.

   If, in 5 years, the interest rate will be 6%/a compounded quarterly, how much will Serena save by choosing the second option?

8. Ted used the exponential function $A(n) = 5000 \times 1.0075^{12n}$ to represent the future value, A, in dollars, of an investment. Determine the principal, the annual interest rate, and the compounding period. Explain your reasoning.

9. Margaret can finance the purchase of a $949.99 refrigerator one of two ways:
   - Plan A: 10%/a simple interest for 2 years
   - Plan B: 5%/a compounded quarterly for 2 years

   Which plan should she choose? Justify your answer.

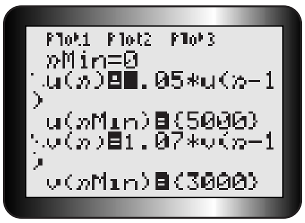
10. Eric bought a $1000 Canada Savings Bond that earns 5%/a compounded annually. Eric can redeem the bond in 7 years. Determine the future value of the bond.

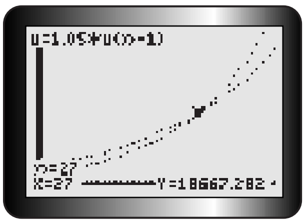
11. Dieter deposits $9000 into an account that pays 10%/a compounded quarterly. After three years, the interest rate changes to 9%/a compounded semi-annually. Calculate the value of his investment two years after this change.

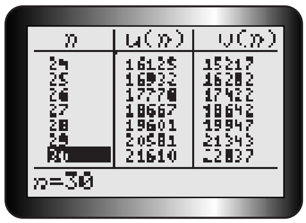
12. Cliff has some money he wants to invest for his retirement. He is offered two options:
    - 10%/a simple interest
    - 5%/a compounded annually

    Under what conditions should he choose the first option?

13. **T** Noreen used her graphing calculator to investigate two sequences. Three screenshots from her investigation are shown. Create a problem for this situation and solve it.







14. You are searching different banks for the best interest rate on an investment, and you find these rates:
- 6.6%/a compounded annually
- 6.55%/a compounded semi-annually
- 6.5%/a compounded quarterly
- 6.45/a compounded monthly

Rank the rates from most to least return on your investment.

15. On July 1, 1996, Anna invested \$2000 in an account that earned 6%/a compounded monthly. On July 1, 2001, she moved the total amount to a new account that paid 8%/a compounded quarterly. Determine the balance in her account on January 1, 2008.

16. Bernie deposited \$4000 into an account that pays 4%/a compounded quarterly during the first year. The interest rate on this account is then increased by 0.2% each year. Calculate the balance in Bernie's account after three years.

17. On the day Rachel was born, her grandparents deposited \$500 into a savings account that earns 4.8%/a compounded monthly. They deposited the same amount on her 5th, 10th, and 15th birthdays. Determine the balance in the account on Rachel's 18th birthday.

18. C Create a mind map for the concept of *interest*. Show how the calculations of simple and compound interest are related to functions and sequences.

Extending

19. Liz decides to save money to buy an electric car. She invests \$500 every 6 months at 6.8%/a compounded semi-annually. What total amount of money will she have at the end of the 10th year?

20. An effective annual interest rate is the interest rate that is equivalent to the given one, assuming that compounding occurs annually. Calculate the effective annual interest rate for each loan. Round to two decimal places.

	Rate of Compound Interest per Year	Compounding Period
a)	6.3%	semi-annually
b)	4.2%	monthly
c)	3.2%	quarterly

8.3 Compound Interest: Present Value

GOAL

Determine the present value of an amount being charged or earning compound interest.

YOU WILL NEED

- graphing calculator
- spreadsheet software

LEARN ABOUT the Math

Anton's parents would like to put some money away so that he will have \$15 000 to study music professionally in 10 years. They can earn 6%/a compounded annually on their investment.

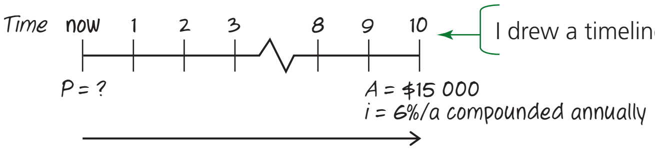
? How much money should Anton's parents invest now so that it will grow to \$15 000 in 10 years at 6%/a compounded annually?

EXAMPLE 1 Selecting a strategy to determine the principal for a given amount

Determine the present value of Anton's parents' investment if it must be worth \$15 000 ten years from now.

present value
the principal that would have to be invested now to get a specific future value in a certain amount of time. *PV* is used for present value instead of *P*, since *P* is used for principal.

Tina's Solution: Working Backward



I drew a timeline for the situation.

end of 1st year:

$I = 0.06P$

$A = P + 0.06P$

$= 1.06P$

The interest rate is 6%/a compounded annually. I calculated the interest earned and the amount at the end of the first year.

end of 2nd year:

$I = 0.06(1.06P)$

$A = 1.06P + 0.06(1.06P)$

$= 1.06P(1 + 0.06)$

$= 1.06P(1.06)$

$= 1.06^2P$

For the second year, interest is earned on the amount at the end of the first year. I calculated the interest earned and the amount at the end of the second year.

$1.06P,\ 1.06^2P,\ 1.06^3P,\ \ldots,\ 1.06^{10}P$

Since 6% is added at the end of each year, I got 106% of what I started with. So multiplying by 1.06 gives the next term of the sequence. The amounts at the end of each year form a geometric sequence with common ratio 1.06. The amount at the end of the 10th year has an exponent of 10.

end of 10th year:

$$15\,000 = 1.06^{10}P$$

Since Anton's parents want \$15 000 at the end of 10 years, I set the 10th term of the sequence equal to \$15 000 and solved for *P*.

$$P = \frac{15\,000}{1.06^{10}}$$

$$\doteq \$8375.92$$

Anton's parents would need to invest \$8375.92 now to get \$15 000 in 10 years.

Jamie's Solution: Representing the Formula for the Amount in a Different Way

$$A = \$15\,000$$

$$i = 6\% = 0.06$$

$$n = 10$$

Anton's parents invest a certain amount and let it grow to \$15 000 at 6%/a compounded annually.

$$A = PV(1 + i)^n$$

An investment earning compound interest grows like an exponential function, so I wrote the amount as an exponential formula.

$$PV = \frac{A}{(1 + i)^n}$$

$$= \frac{15\,000}{(1 + 0.06)^{10}}$$

$$\doteq \$8375.92$$

To calculate the principal that Anton's parents would have to invest, I rearranged the formula, substituted, and solved for *PV*.

Anton's parents would need to invest \$8375.92 now to get \$15 000 in 10 years.

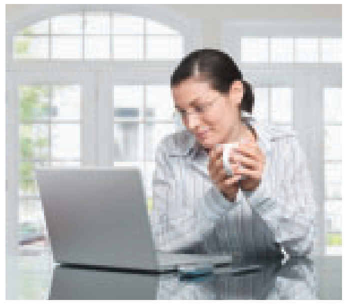
Reflecting

A. How are the problems of determining the present value and the amount of an investment related?

B. Based on Example 1, which method do you prefer to use to calculate the present value? Why?

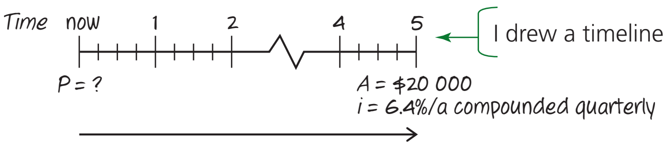
APPLY the Math

EXAMPLE 2 Solving a problem involving present value



Monica wants to start a business and needs to borrow some money. Her bank will charge her 6.4%/a compounded quarterly. Monica wants to repay the loan in 5 years, but doesn't want the amount she pays back to be more than \$20 000. What is the maximum amount that she can borrow and how much interest will she pay if she doesn't pay anything back until the end of the 5 years?

Kwok's Solution



I drew a timeline for the situation.

$$i = \frac{0.064}{4} = 0.016$$

$$n = 5 \times 4 = 20$$

I calculated the interest rate Monica would be charged each compounding period and how many periods the loan would last.

$$PV = \frac{A}{(1 + i)^n}$$

$$= \frac{20\ 000}{(1 + 0.016)^{20}}$$

$$\doteq \$14\ 559.81$$

Next I calculated the present value of the \$20 000 at the given interest rate.

$$I = A - PV$$

$$= \$20\ 000 - \$14\ 559.81$$

$$= \$5440.19$$

I determined the interest by subtracting the present value from the amount.

The most Monica can borrow is \$14 559.81; she will pay \$5440.19 in interest.

EXAMPLE 3 Selecting a strategy to determine the interest rate

Tony is investing $5000 that he would like to grow to at least $50 000 by the time he retires in 40 years. What annual interest rate, compounded annually, will provide this? Round your answer to two decimal places.

Philip's Solution: Using a Graphing Calculator

$PV = \$5000$

$n = 40$

$A = 50\ 000$

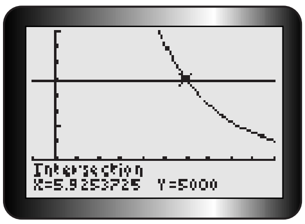
I knew the principal, the amount (or future value), and the number of years.

$$PV = \frac{A}{(1 + i)^n}$$

$$5000 = \frac{50\ 000}{(1 + i)^{40}}$$

I wrote the formula for the present value and substituted the given information. To calculate *i*, I thought of the intersection of two functions: $\text{Y1} = \frac{50\ 000}{\left(1 + \frac{X}{100}\right)^{40}}$ and Y2 = 5000.

I entered these into my graphing calculator. I used Y1 and Y2 for the present value and X for the interest rate.



I graphed both functions on the same graph and used the calculator to find the point of intersection.

Tony would need to get at least 5.93%/a compounded annually to reach his goal.

Tech Support

For help using a graphing calculator to find the point of intersection of two functions, see Technical Appendix, B-12.

Derek's Solution: Using the Formula

$$PV = \$5000$$
$$n = 40$$
$$A = \$50\ 000$$

I knew the principal, the amount (or future value), and the number of years.

$$PV = \frac{A}{(1 + i)^n}$$
$$5000 = \frac{50\ 000}{(1 + i)^{40}}$$

I wrote the formula for the present value, then substituted the given information, and rearranged to solve for *i*.

$$5000(1 + i)^{40} = 50\ 000$$
$$(1 + i)^{40} = 10$$

I multiplied both sides of the equation by $(1 + i)^{40}$, then I divided both sides by 5000.

$$1 + i = \sqrt[40]{10}$$
$$1 + i \doteq 1.0593$$
$$i = 0.0593$$
$$i = 5.93\%$$

To calculate *i*, I used the inverse operation of raising something to the 40th power, which is determining the 40th root.

Tony would need to get at least 5.93%/a compounded annually to reach his goal.

In Summary

Key Idea

- The principal, *PV,* that must be invested now to grow to a specific future value, *A,* is called the present value.

Need to Know

- The present value of an investment earning compound interest can be calculated using the formula $PV = \frac{A}{(1 + i)^n}$ or $PV = A(1 + i)^{-n}$, where *PV* is the present value; *A* is the total amount, or future value; *i* is the interest rate per compounding period, expressed as a decimal; and *n* is the number of compounding periods.

CHECK Your Understanding

1. Calculate the present value of each investment.

	Rate of Compound Interest per Year	Compounding Period	Time	Future Value
a)	4%	annually	10 years	$10 000
b)	6.2%	semi-annually	5 years	$100 000
c)	5.2%	quarterly	15 years	$23 000
d)	6.6%	monthly	100 years	$2 500

2. Kevin and Lui both want to have $10 000 in 20 years. Kevin can invest at 5%/a compounded annually and Lui can invest at 4.8%/a compounded monthly. Who has to invest more money to reach his goal?

PRACTISING

3. For each investment, determine the present value and the interest earned.
K

	Rate of Compound Interest per Year	Compounding Period	Time	Future Value
a)	6%	annually	4 years	$10 000
b)	8.2%	semi-annually	3 years	$6 200
c)	5.6%	quarterly	15 years	$20 000
d)	4.2%	monthly	9 years	$12 800

4. Chandra borrows some money at 7.2%/a compounded annually. After 5 years, she repays $12 033.52 for the principal and the interest. How much money did Chandra borrow?

5. Nazir saved $900 to buy a plasma TV. He borrowed the rest at an interest rate of 18%/a compounded monthly. Two years later, he paid $1429.50 for the principal and the interest. How much did the TV originally cost?

6. Rico can invest money at 10%/a compounded quarterly. He would like $15 000 in 10 years. How much does he need to invest now?

7. Colin borrowed some money at 7.16%/a compounded quarterly. Three years later, he paid $5000 toward the principal and the interest. After another two years, he paid another $5000. After another five years, he paid the remainder of the principal and the interest, which totalled $5000. How much money did he originally borrow?

8. Tia is investing $2500 that she would like to grow to $6000 in 10 years. At what annual interest rate, compounded quarterly, must Tia invest her money? Round your answer to two decimal places.

9. **A** Franco invests some money at 6.9%/a compounded annually and David invests some money at 6.9%/a compounded monthly. After 30 years, each investment is worth $25 000. Who made the greater original investment and by how much?

10. **T** Sally invests some money at 6%/a compounded annually. After 5 years, she takes the principal and interest and reinvests it all at 7.2%/a compounded quarterly for 6 more years. At the end of this time, her investment is worth $14 784.56. How much did Sally originally invest?

11. Steve wants to have $25 000 in 25 years. He can get only 3.2%/a interest compounded quarterly. His bank will guarantee the rate for either 5 years or 8 years.
    - In 5 years, he will probably get 4%/a compounded quarterly for the remainder of the term.
    - In 8 years, he will probably get 5%/a compounded quarterly for the remainder of the term.

    a) Which guarantee should Steve choose, the 5-year one or the 8-year one?
    b) How much does he need to invest?

12. **C** Describe how determining the present value of an investment is similar to solving a radioactive decay problem.

Extending

13. Louise invests $5000 at 5.4%/a compounded semi-annually. She would like the money to grow to $12 000. How long will she have to wait?

14. What annual interest rate, compounded quarterly, would cause an investment to triple every 10 years? Round your answer to two decimal places.

15. You buy a home entertainment system on credit. You make monthly payments of $268.17 for $2\frac{1}{2}$ years and are charged 19.2%/a interest compounded monthly. How much interest will you have paid on your purchase?

16. Determine a formula for the present value of an investment with future value, A, earning simple interest at a rate of i% per interest period for n interest periods.

Curious *Math*

The Rule of 72

Working with compound interest can be difficult if you don't have a calculator handy. For years, banks, investors, and the general public have used "the rule of 72" to help approximate calculations involving compound interest. Here is how the rule works:

If an investment is earning r%/a compounded annually, then it will take $72 \div r$ years to double in value.

So if you are earning 8%/a compounded annually, the rule indicates that it will take $72 \div 8 = 9$ years for the money to double in value. Suppose you invested \$1000. Then the formula for the future value gives

$$\begin{aligned} A &= P(1 + i)^n \\ &= 1000(1.08)^9 \\ &\doteq \$1999.00 \end{aligned}$$

which is very close to double your money. The spreadsheet below shows how the time, in years, predicted by the rule of 72 is very close to the actual doubling time.

	A	B	C
1	Interest Rate	Double Time Using the Rule of 72	Actual Double Time
2	1	72.00	69.66
3	2	36.00	35.00
4	3	24.00	23.45
5	4	18.00	17.67
6	5	14.40	14.21
7	6	12.00	11.90
8	7	10.29	10.24
9	8	9.00	9.01
10	9	8.00	8.04
11	10	7.20	7.27
12	11	6.55	6.64
13	12	6.00	6.12

1. How could you use the rule of 72 to determine how much a \$1000 investment earning 8%/a compounded annually would be worth after 45 years?

2. How close is your estimate to the actual amount after 45 years?

8 Mid-Chapter Review

FREQUENTLY ASKED Questions

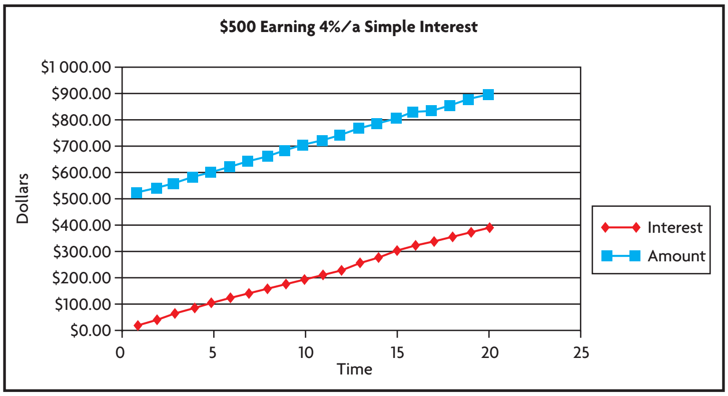
Q: What strategies can you use to solve problems involving simple interest?

A1: Since simple interest is paid only on the principal, both the interest, I, and the amount, A, grow at a constant rate. So I and A can be modelled by linear functions in terms of time or arithmetic sequences.

EXAMPLE

If \$500 is invested at 4%/a simple interest, then 4% of \$500, or \$20, interest is earned each year. You can model the interest and the amount by the functions $I(t) = 20t$ and $A(t) = 500 + 20t$, where I and A are in dollars and t is time in years.

	A	B	C
1	Year	Interest	Amount
2	1	\$20.00	\$520.00
3	2	\$40.00	\$540.00
4	3	\$60.00	\$560.00
5	4	\$80.00	\$580.00
6	5	\$100.00	\$600.00
7	6	\$120.00	\$620.00
8	7	\$140.00	\$640.00
9	8	\$160.00	\$660.00
10	9	\$180.00	\$680.00
11	10	\$200.00	\$700.00
12	11	\$220.00	\$720.00
13	12	\$240.00	\$740.00
14	13	\$260.00	\$760.00
15	14	\$280.00	\$780.00
16	15	\$300.00	\$800.00
17	16	\$320.00	\$820.00
18	17	\$340.00	\$840.00
19	18	\$360.00	\$860.00
20	19	\$380.00	\$880.00
21	20	\$400.00	\$900.00



A2: To calculate the interest, use the formula $I = Prt$, where I is the interest; P is the principal; r is the interest rate, expressed as a decimal; and t is time, expressed in the same period as the interest rate. To calculate the amount, use the formula $A = P + Prt$ or $A = P(1 + rt)$, where A is the total amount. The formulas for I and A are linear in terms of time.

Study *Aid*

- See Lesson 8.1, Examples 1 to 4.
- Try Mid-Chapter Review Questions 1 to 3.

Q: How can you solve problems involving compound interest?

A1: Compound interest is paid at regular intervals, called compounding periods, and is added to the principal. So interest is calculated on the principal *and* interest already earned. After each compounding period, the total amount, A, or future value, grows by a fixed rate. So A can be modelled by an exponential function in terms of time or a geometric sequence.

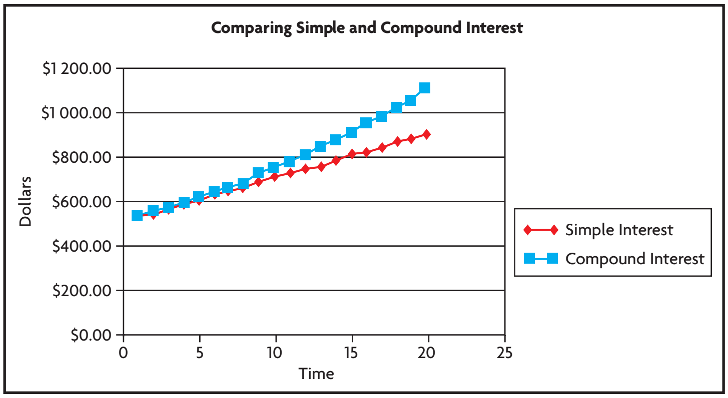
Study *Aid*

- See Lesson 8.2, Examples 1 to 4.
- Try Mid-Chapter Review Questions 4 to 6.

EXAMPLE

If \$500 is invested at 4%/a compounded annually, then 4% of the total amount is earned as interest each year. So the amount can be modelled by $A(t) = 500 \times 1.04^t$, where A is the future value in dollars and t is time in years. The amount grows exponentially, faster than the linear growth of simple interest.

	A	B	C
1	Year	Simple Interest	Compound Interest
2	1	\$520.00	\$520.00
3	2	\$540.00	\$540.80
4	3	\$560.00	\$562.43
5	4	\$580.00	\$584.93
6	5	\$600.00	\$608.33
7	6	\$620.00	\$632.66
8	7	\$640.00	\$657.97
9	8	\$660.00	\$684.28
10	9	\$680.00	\$711.66
11	10	\$700.00	\$740.12
12	11	\$720.00	\$769.73
13	12	\$740.00	\$800.52
14	13	\$760.00	\$832.54
15	14	\$780.00	\$865.84
16	15	\$800.00	\$900.47
17	16	\$820.00	\$936.49
18	17	\$840.00	\$973.95
19	18	\$860.00	\$1012.91
20	19	\$880.00	\$1 053.42
21	20	\$900.00	\$1 095.56



A2: To calculate the total amount, or future value, use the formula $A = P(1 + i)^n$, where A is the future value; P is the principal; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods. This formula is exponential in terms of the number of compounding periods.

Study Aid

- See Lesson 8.3, Examples 1 to 3.
- Try Mid-Chapter Review Questions 7 to 9.

Q: For problems involving compound interest, what is the difference between present value and future value?

A: Present value, PV, is the amount you start with (the principal), while future value, A, is the amount you end up with after the last compounding period. These two values are related through the formulas

$$A = P(1 + i)^n \quad \text{and}$$

$$PV = \frac{A}{(1 + i)^n} \quad \text{or} \quad PV = A(1 + i)^{-n}$$

which are just rearrangements of each other. When you solve for PV, you are determining the principal that should be invested to yield the desired amount.

PRACTICE Questions

Lesson 8.1

1. Each situation represents a loan being charged simple interest. Calculate the interest being charged and the total amount.

	Principal	Rate of Simple Interest per Year	Time
a)	\$5 400	6.7%	15 years
b)	\$400	9.6%	16 months
c)	\$15 000	14.3%	80 weeks
d)	\$2 500	27.1%	150 days

2. How long would you have to invest \$5300 at 7.2%/a simple interest to earn \$1200 interest?

3. Tom borrows some money and is charged simple interest on the principal. The balances from his statements for the first three months are shown.

Statement	Balance
1	\$1014.60
2	\$1079.20
3	\$1143.80

a) How much interest is he being charged each month?
b) How much did Tom borrow?
c) What interest rate is he being charged?

Lesson 8.2

4. For each investment, calculate the future value and the total interest earned.

	Principal	Rate of Compound Interest per Year	Compounding Period	Time
a)	\$6 300	4.9%	annually	7 years
b)	\$14 000	8.8%	semi-annually	10.5 years
c)	\$120 000	4.4%	quarterly	44 years
d)	\$298	22.8%	monthly	1.5 years

5. George invests \$15 000 at 7.2%/a compounded monthly. How long will it take for his investment to grow to \$34 000?

6. Sara buys a washer and dryer for \$2112. She pays \$500 and borrows the remaining amount. A year and a half later, she pays off the loan, which amounted to \$2112. What annual interest rate, compounded semi-annually, was Sara being charged? Round your answer to two decimal places.

Lesson 8.3

7. How much money does Maria need to invest at 9.2%/a compounded quarterly in order to have \$25 000 after 25 years?

8. Clive inherits an investment that his grandparents made at 7.4%/a compounded semi-annually. The investment was worth \$39 382.78 when they took it out 65 years ago. How much did Clive's grandparents invest?

9. Iris borrowed some money at a fixed rate of compound interest, but she forgot what the interest rate was. She knew that the interest was compounded semi-annually. The balances of her first two statements are shown.

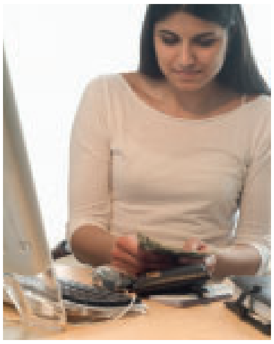
Statement	Time	Balance
1	6 months	\$8715.91
2	1 year	\$9125.56

a) What interest rate is she being charged? Round your answer to two decimal places.
b) How much did Iris borrow?

8.4 Annuities: Future Value

YOU WILL NEED

- graphing calculator
- spreadsheet software



GOAL

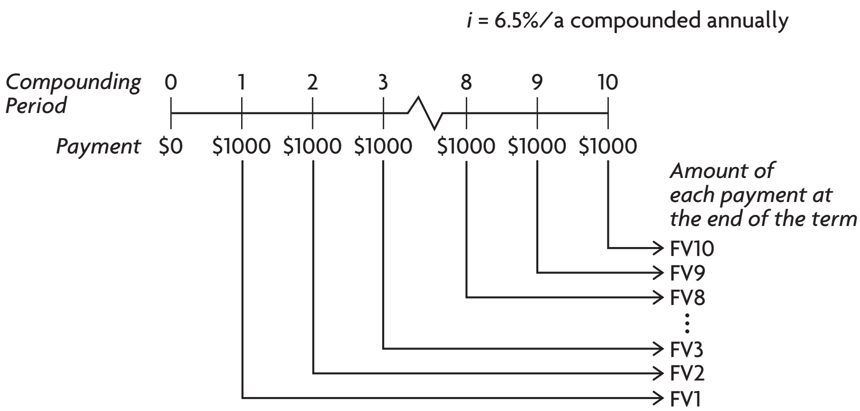
Determine the future value of an annuity earning compound interest.

INVESTIGATE the Math

Christine decides to invest $1000 at the end of each year in a Canada Savings Bond earning 6.5%/a compounded annually. Her first deposit is on December 31, 2007.

? How much will her investments be worth 10 years later, on January 1, 2017?

A. Copy the timeline shown. How would you calculate each of the future values FV1 to FV10?



B. Set up a spreadsheet with columns as shown. Copy the data already entered. Complete the entries under Date Invested up to Dec. 31, 2007.

	A	B	C	D
1	Date Invested	Amount Invested	Number of Years Invested	Value on Jan. 1, 2017
2	Dec. 31, 2016	$1 000.00	0	$1 000.00
3	Dec. 31, 2015	$1 000.00	1	
4	Dec. 31, 2014	$1 000.00	2	

C. Fill in cells D3 and D4 to show what the investments made on Dec. 31, 2015, and Dec. 31, 2014, respectively will be worth on Jan. 1, 2017.

D. How is the value in cell D3 (FV9) related to the value in cell D2 (FV10)? How is the value in cell D4 (FV8) related to the value in cell D3 (FV9)?

E. Use the pattern from part D to complete the rest of the entries under Value on Jan. 1, 2017.

F. What type of sequence do the values on Jan. 1, 2017 form?

G. Calculate the total amount of all the investments at the end of 10 years for this annuity.

annuity
a series of payments or investments made at regular intervals. A **simple** annuity is an annuity in which the payments coincide with the compounding period, or *conversion* period. An **ordinary** annuity is an annuity in which the payments are made at the end of each interval. Unless otherwise stated, each annuity in this chapter is a simple, ordinary annuity.

Reflecting

H. The values of all of the investments at the end of each year for 10 years formed a specific type of sequence. How is the total value of the annuity at the end of 10 years related to a series?

I. How could you use the related series to solve problems involving annuities?

APPLY the Math

EXAMPLE 1 Representing the future value of an annuity earning compound interest as a series

a) Hans plans to invest \$1000 at the end of each 6-month period in an annuity that earns 4.8%/a compounded semi-annually for the next 20 years. What will be the future value of his annuity?

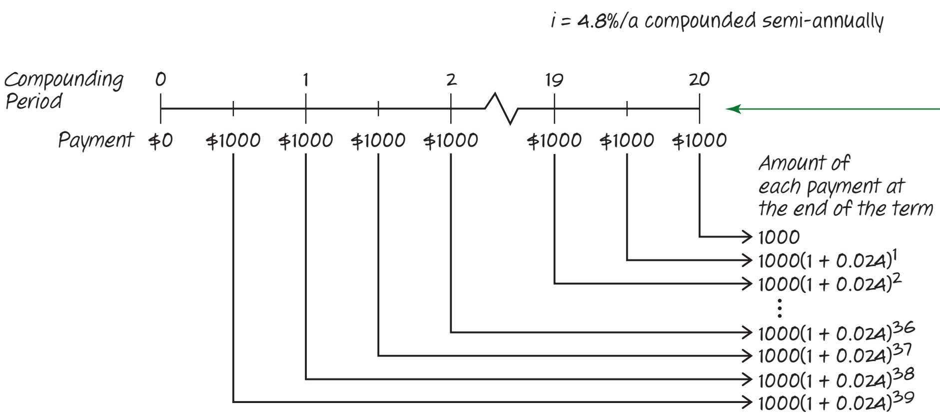
b) You plan to invest $\$R$ at regular intervals in an annuity that earns i% compounded at the end of each interval. What will be the future value, FV, of your annuity after n intervals?

Barbara's Solution

a) $i = \dfrac{0.048}{2} = 0.024$

$n = 20 \times 2 = 40$

Since the interest is paid semi-annually, I calculated the interest rate per compounding period and the number of compounding periods.



I drew a timeline of the investments for each compounding period, and I represented the amount of each investment.

The last \$1000 investment earned no interest because it was deposited at the end of the term.

The first \$1000 investment earned interest over 39 periods. It didn't earn interest during the first compounding period because it was deposited at the end of that period.

$$1000, 1000(1.024), 1000(1.024)^2, \ldots, 1000(1.024)^{38}, 1000(1.024)^{39}$$

The future values of all of the investments form a geometric sequence with first term \$1000 and common ratio 1.024.

$$S_{40} = 1000 + 1000(1.024) + 1000(1.024)^2 + \ldots + 1000(1.024)^{38} + 1000(1.024)^{39}$$

The total amount of all these investments is the first 40 terms of a geometric series.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{40} = \frac{1000(1.024^{40} - 1)}{1.024 - 1}$$

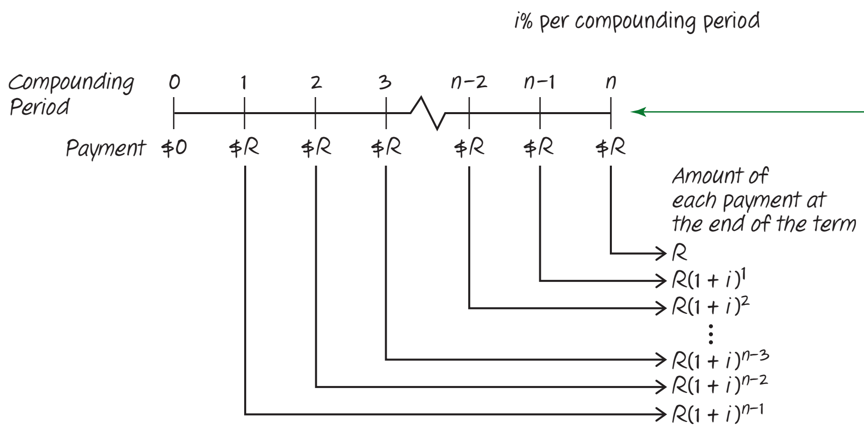
I used the formula for the sum of a geometric series to calculate the total amount of all of Hans's investments.

$$\doteq \$65\,927.08$$

I rounded to the nearest cent.

The future value of Hans's annuity at the end of 20 years is \$65 927.08.

b)



I drew a timeline of the investments for each compounding period to show the amount of each investment.

The last \$$R$ investment earned no interest. The first \$$R$ investment earned interest $n - 1$ times.

$$R, R(1 + i), R(1 + i)^2, \ldots, R(1 + i)^{n-2}, R(1 + i)^{n-1}$$

The values of all of the investments form a geometric sequence with first term R and common ratio $1 + i$.

$$S_n = R + R(1 + i) + R(1 + i)^2 + \ldots + R(1 + i)^{n-2} + R(1 + i)^{n-1}$$

The total amount of all these investments is the first n terms of a geometric series.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{R[(1 + i)^n - 1]}{(1 + i) - 1}$$

$$= R \times \left(\frac{(1 + i)^n - 1}{i}\right)$$

I used the formula for the sum of a geometric series to calculate the total amount of all the investments.

The future value of an annuity in which \$$R$ is invested at the end of each of n regular intervals earning i% of compound interest per interval is

$FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right)$, where i is expressed as a decimal.

EXAMPLE 2 Selecting a strategy to determine the future value of an annuity

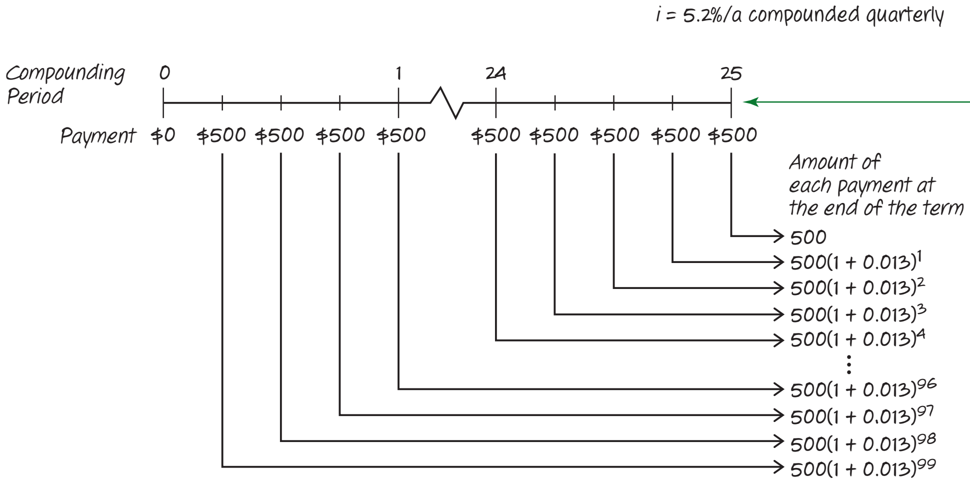
Chie puts away \$500 every 3 months at 5.2%/a compounded quarterly. How much will her annuity be worth in 25 years?

Kew's Solution: Using a Geometric Series

$$i = \frac{0.052}{4} = 0.013$$

$$n = 25 \times 4 = 100$$

First I calculated the interest rate per compounding period and the number of compounding periods.



I drew a timeline of the investments for each quarter to show the amounts of each investment. I calculated the value of each investment at the end of 25 years.

$$500, 500(1.013), 500(1.013)^2, \ldots, 500(1.013)^{98}, 500(1.013)^{99}$$

The values form a geometric sequence with first term \$500 and common ratio 1.013.

$$S_{100} = 500 + 500(1.013) + 500(1.013)^2 + \ldots + 500(1.013)^{98} + 500(1.013)^{99}$$

The total amount of all these investments is the first 100 terms of a geometric series.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{100} = \frac{500(1.013^{100} - 1)}{1.013 - 1}$$

I used the formula for the sum of a geometric series to calculate the total amount of all of Chie's investments.

$$\doteq \$101\,487.91$$

I rounded to the nearest cent.

The total amount of all of Chie's investments at the end of 25 years will be \$101 487.91.

Tina's Solution: Using the Formula for the Future Value of an Annuity

$R = \$500$

$i = \frac{0.052}{4} = 0.013$

I calculated the interest rate per compounding period and the number of compounding periods.

$n = 25 \times 4 = 100$

$FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right)$

I substituted the values of R, i, and n into the formula for the future value of a simple, ordinary annuity.

$= 500 \times \left(\frac{(1 + 0.013)^{100} - 1}{0.013}\right)$

$\doteq \$101\ 487.91$

I rounded to the nearest cent.

The future value of Chie's annuity will be $101 487.91.

EXAMPLE 3 Selecting a strategy to determine the regular payment of an annuity

Sam wants to make monthly deposits into an account that guarantees 9.6%/a compounded monthly. He would like to have $500 000 in the account at the end of 30 years. How much should he deposit each month?

Chantal's Solution

$i = \frac{0.096}{12} = 0.008$

$n = 30 \times 12 = 360$

I calculated the interest rate per compounding period and the number of compounding periods.

$FV = \$500\ 000$

The future value of the annuity is $500 000.

$FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right)$

$500\ 000 = R \times \left(\frac{(1 + 0.008)^{360} - 1}{0.008}\right)$

I substituted the values of FV, i, and n into the formula for the future value of an annuity.

$$500\,000 \doteq R \times 2076.413$$

$$\frac{500\,000}{2076.413} = R \times \frac{2076.413}{2076.413}$$

To solve for R, I divided both sides of the equation by 2076.413.

$$R = \$240.80$$

I rounded to the nearest cent.

Sam would have to deposit \$240.80 into the account each month in order to have \$500 000 at the end of 30 years.

Tech Support

For help using a spreadsheet to enter values and formulas, and to fill down, see Technical Appendix, B-21.

EXAMPLE 4 Selecting a strategy to determine the term of an annuity

Nahid borrows \$95 000 to buy a cottage. She agrees to repay the loan by making equal monthly payments of \$750 until the balance is paid off. If Nahid is being charged 5.4%/a compounded monthly, how long will it take her to pay off the loan?

Zak's Solution

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2					\$95 000.00
3	1	\$750.00	\$427.50	\$322.50	\$94 677.50
4	2	\$750.00	"=E3*0.054/12"	"=B4-C4"	"=E3-D4"

I set up a spreadsheet to calculate the balance after every payment. The interest is always charged on the balance and is $\frac{1}{12}$ of 5.4% since it is compounded monthly. The part of the principal that is paid off with each payment is \$750, less the interest. The new balance is the old balance, less the part of the principal that is paid.

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2					\$95 000.00
3	1	\$750.00	\$427.50	\$322.50	\$94 677.50
4	2	\$750.00	\$426.05	\$323.95	\$94 353.55
5	3	\$750.00	\$424.59	\$325.41	\$94 028.14
6	4	\$750.00	\$423.13	\$326.87	\$93 701.27
7	5	\$750.00	\$421.66	\$328.34	\$93 372.92
188	184	\$750.00	\$16.55	\$733.45	\$2 944.92
187	185	\$750.00	\$13.25	\$736.75	\$2 208.17
188	186	\$750.00	\$9.94	\$740.06	\$1 468.11
189	187	\$750.00	\$6.61	\$743.39	\$724.72
190	188	\$750.00	\$3.26	\$746.74	–\$22.02

I used the FILL DOWN command to complete the spreadsheet until the balance was close to zero.

$$t = \frac{188}{12}$$

$$\doteq 15.667$$

After 188 payments, the balance is close to zero. I calculated the number of years needed to make 188 payments by dividing by 12, since there are 12 payments each year.

$$0.667 \times 12 \text{ months} \doteq 8 \text{ months}$$

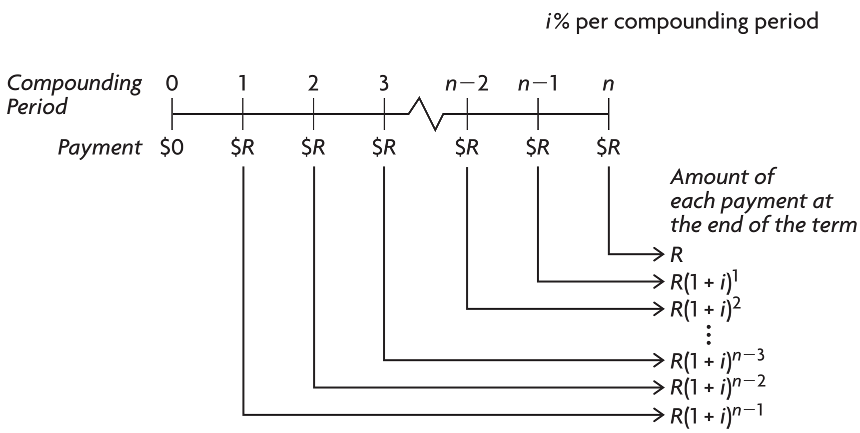
I got a value greater than 15. The 15 meant 15 years, so I had to figure out what 0.667 of a year was.

Nahid can pay off the loan after 188 payments, which would take about 15 years and 8 months.

In Summary

Key Ideas

- The future value of an annuity is the sum of all regular payments and interest earned.



- The future value can be written as the geometric series

$$FV = R + R(1 + i) + R(1 + i)^2 + \ldots + R(1 + i)^{n-2} + R(1 + i)^{n-1}$$

where FV is the future value; R is the regular payment; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

- The formula for the sum of a geometric series can be used to determine the future value of an annuity.

Need to Know

- A variety of technological tools (spreadsheets, graphing calculators) can be used to solve problems involving annuities.
- The payment interval of an annuity is the time between successive payments.
- The term of an annuity is the time from the first payment to the last payment.
- The formula for the future value of an annuity is

$$FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right)$$

where FV is the future value; R is the regular payment each compounding period; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

CHECK Your Understanding

1. Each year, Eric invests $2500 at 8.2%/a compounded annually for 25 years.
   a) Calculate the value of each of the first four investments at the end of 25 years.
   b) What type of sequence do the values form?
   c) Determine the total amount of all of Eric's investments.

2. Calculate the future value of each annuity.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	$100 per month	3.6%	monthly	50 years
b)	$1500 per quarter	6.2%	quarterly	15 years
c)	$500 every 6 months	5.6%	semi-annually	8 years
d)	$4000 per year	4.5%	annually	10 years

3. Lois invests $650 every 6 months at 4.6%/a compounded semi-annually for 25 years. How much interest will she earn after the 25th year?

4. Josh borrows some money on which he makes monthly payments of $125.43 for 3 years. If the interest rate is 5.4%/a compounded monthly, what will be the total amount of all of the payments at the end of the 3 years?

PRACTISING

5. K Calculate the future value of each annuity.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	$1500 per year	6.3%	annually	10 years
b)	$250 every 6 months	3.6%	semi-annually	3 years
c)	$2400 per quarter	4.8%	quarterly	7 years
d)	$25 per month	8%	monthly	35 years

6. A Mike wants to invest money every month for 40 years. He would like to have $1 000 000 at the end of the 40 years. For each investment option, how much does he need to invest each month?
   a) 10.2%/a compounded monthly
   b) 5.1%/a compounded monthly

7. Kiki has several options for investing $1200 per year:

	Regular Payment	Rate of Compound Interest per Year	Compounding Period
a)	$100 per month	7.2%	monthly
b)	$300 per quarter	7.2%	quarterly
c)	$600 every 6 months	7.2%	semi-annually
d)	$1200 per year	7.2%	annually

Without doing any calculations, which investment would be best? Justify your reasoning.

8. Kenny wants to invest $250 every three months at 5.2%/a compounded quarterly. He would like to have at least $6500 at the end of his investment. How long will he need to make regular payments?

9. Sonja and Anita want to make equal monthly payments for the next 35 years. At the end of that time, each person would like to have $500 000. Sonja's bank will give her 6.6%/a compounded monthly. Anita can invest through her work and earn 10.8%/a compounded monthly.
   a) How much more per month does Sonja have to invest?
   b) If Anita decides to invest the same monthly amount as Sonja, how much more money will she have at the end of 35 years?

10. **T** Jamal wants to invest $150 every month for 10 years. At the end of that time, he would like to have $25 000. At what annual interest rate, compounded monthly, does Jamal need to invest to reach his goal? Round your answer to two decimal places.

11. **C** Draw a mind map for the concept of *future value of annuities*. Show how it is related to interest, sequences, and series.

Extending

12. Carmen borrows $10 000 at 4.8%/a compounded monthly. She decides to make monthly payments of $250.
    a) How long will it take her to pay off the loan?
    b) How much interest will she pay over the term of the loan?

13. Greg borrows $123 000 for the purchase of a house. He plans to make regular monthly payments over the next 20 years to pay off the loan. The bank is charging Greg 6.6%/a compounded monthly. What monthly payments will Greg have to make?

14. How many equal monthly payments would you have to make to get 100 times the amount you are investing each month if you are earning 8.4%/a compounded monthly?

8.5 Annuities: Present Value

YOU WILL NEED

- graphing calculator
- spreadsheet software

GOAL

Determine the present value of an annuity earning compound interest.

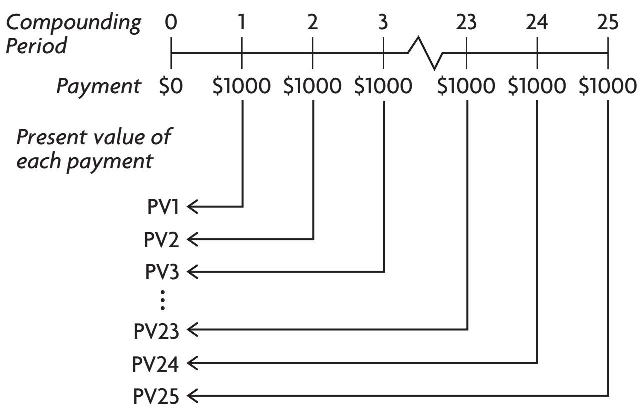
INVESTIGATE the Math

Kew wants to invest some money at 5.5%/a compounded annually. He would like the investment to provide $1000 for scholarships at his old high school at the end of each year for the next 25 years.

? How much should Kew invest now?

A. Copy the timeline shown. How would you calculate each of the present values PV1 to PV25?

i = 5.5%/a compounded annually



B. How much would Kew need to invest now if he wanted to provide $1000 at the end of the 1st year?

C. How much would Kew need to invest now if he wanted to provide $1000 at the end of the 2nd, 3rd, and 4th years, respectively?

D. How is the present value after 2 years (PV2) related to the present value after 1 year (PV1)?

E. Set up a spreadsheet with columns as shown at the right. Enter your values of PV1 and PV2 in the Present Value column.

	A	B	C
1	Year	Scholarship Payment	Present Value
2	1	$1 000.00	
3	2	$1 000.00	
4	3	$1 000.00	
5	4	$1 000.00	
6	5	$1 000.00	
7	6	$1 000.00	
8	7	$1 000.00	
9	8	$1 000.00	
10	9	$1 000.00	
11	10	$1 000.00	

F. Use the relationship among the present values to complete the rest of the entries under Present Value.

G. Use the values in the Present Value column to determine how much Kew would need to invest now in order to provide the scholarships for the next 25 years.

Reflecting

H. What type of sequence do the present values in part F form?

I. Describe a method that you could have used to solve this problem without using a spreadsheet.

APPLY the Math

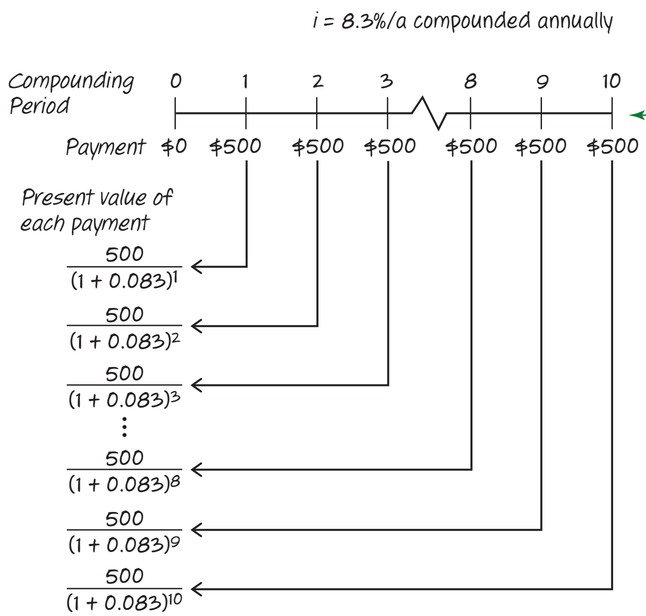
EXAMPLE 1 Representing the present value of an annuity earning compound interest as a series

a) How much would you need to invest now at 8.3%/a compounded annually to provide \$500 per year for the next 10 years?

b) How much would you need to invest now to provide n regular payments of $\$R$ if the money is invested at a rate of i% per compounding period?

Tara's Solution

a)



I drew a timeline showing the \$500 payments for the next 10 years.

$$PV = \frac{A}{(1 + i)^n}$$

$$PV_1 = \frac{500}{(1.083)}$$

$$PV_2 = \frac{500}{(1.083)^2}$$

$$PV_3 = \frac{500}{(1.083)^3}$$

$$\vdots$$

I considered each payment separately and used the present-value formula to determine how much would need to be invested now to provide each \$500 payment.

$$PV_9 = \frac{500}{(1.083)^9}$$

$$PV_{10} = \frac{500}{(1.083)^{10}}$$

$$a = \frac{500}{(1.083)} = 500 \times 1.083^{-1}$$

$$r = \frac{1}{1.083} = 1.083^{-1}$$

$$n = 10$$

The present values for each payment are the first 10 terms of a geometric sequence with first term 500×1.083^{-1} and common ratio 1.083^{-1}.

$$S_{10} = 500 \times 1.083^{-1} + 500 \times 1.083^{-2} + 500 \times 1.083^{-3} + \ldots + 500 \times 1.083^{-9} + 500 \times 1.083^{-10}$$

The total amount of money invested now has to provide each of the \$500 future payments. So I had to calculate the sum of all of the present values.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

The sum of the present values forms a geometric series, so I used the formula for the sum of a geometric series.

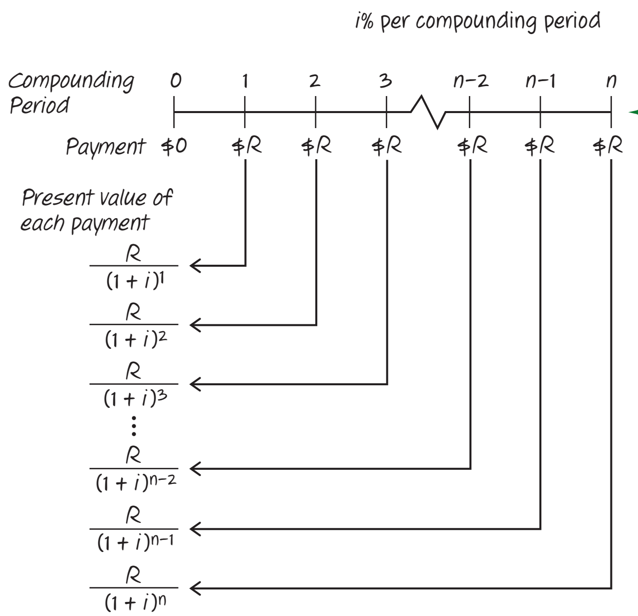
$$S_{10} = \frac{500 \times 1.083^{-1}[(1.083^{-1})^{10} - 1]}{1.083^{-1} - 1}$$

$$\doteq \$3310.11$$

I rounded to the nearest cent.

A sum of \$3310.11 invested now would provide a payment of \$500 for each of the next 10 years.

b)



I drew a timeline showing the $\$R$ payments for n regular intervals.

$$PV = \frac{A}{(1 + i)^n}$$

I considered each payment separately and used the present-value formula to determine how much would need to be invested now to provide each specific $R payment.

$$PV_1 = \frac{R}{1 + i}$$

$$PV_2 = \frac{R}{(1 + i)^2}$$

$$PV_3 = \frac{R}{(1 + i)^3}$$

$$\vdots$$

$$PV_n = \frac{R}{(1 + i)^n}$$

$$a = R \times (1 + i)^{-1}$$

$$r = (1 + i)^{-1}$$

I used negative exponents, since I was dividing by $1 + i$ each time.

The present values for each payment are the first n terms of a geometric sequence with first term $R \times (1 + i)^{-1}$ and common ratio $(1 + i)^{-1}$.

$$S_n = R \times (1 + i)^{-1} + R \times (1 + i)^{-2} + R \times (1 + i)^{-3} + \ldots + R \times (1 + i)^{-n}$$

I needed to determine the total amount of money invested now to provide each of the $R future payments. So I had to calculate the sum of all of the present values.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

The sum of the present values forms a geometric series, so I used the formula for the sum of a geometric series.

$$= \frac{R \times (1 + i)^{-1}[(1 + i)^{-n} - 1]}{(1 + i)^{-1} - 1}$$

$$= \frac{R \times (1 + i)^{-1}[(1 + i)^{-n} - 1]}{(1 + i)^{-1} - 1} \times \frac{1 + i}{1 + i}$$

The numerator and denominator each have a factor of $(1 + i)^{-1}$, so I multiplied them both by $1 + i$ to simplify.

$$= \frac{R[(1 + i)^{-n} - 1]}{1 - (1 + i)}$$

$$= \frac{R[(1 + i)^{-n} - 1]}{-i}$$

$$= R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$$

I multiplied the numerator and denominator by -1 to simplify.

The present value of an annuity in which $R is paid at the end of each of n regular intervals earning i% compound interest per interval is $PV = R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$.

EXAMPLE 2 Selecting a strategy to determine the present value of an annuity

Sharon won a lottery that offers \$50 000 a year for 20 years or a lump-sum payment now. If she can invest the money at 5%/a compounded annually, how much should the lump-sum payment be to be worth the same amount as the annuity?

Joel's Solution: Using a Spreadsheet

	A	B	C
1	Year	Payment	Present Value
2	1	\$50 000.00	"= B2/1.05"
3	2	\$50 000.00	"= B3/(1.05)^A3"
4	3	\$50 000.00	"= B4/(1.05)^A4"

I set up a spreadsheet to determine the present value of each of the payments for the next 20 years. The present value of each payment is given by the formula $PV = \frac{A}{(1 + i)^n}$, so the present value of the payments form a geometric sequence with $r = \frac{1}{1 + i}$. Since Sharon is earning 5%/a, the present value of each following year is equal to 1.05 times the present value of the previous year.

	A	B	C
1	Year	Payment	Present Value
2	1	\$50 000.00	\$47 619.05
3	2	\$50 000.00	\$45 351.47
4	3	\$50 000.00	\$43 191.88
5	4	\$50 000.00	\$41 135.12
6	5	\$50 000.00	\$39 176.31
7	6	\$50 000.00	\$37 310.77
8	7	\$50 000.00	\$35 534.07
9	8	\$50 000.00	\$33 841.97
10	9	\$50 000.00	\$32 230.45
11	10	\$50 000.00	\$30 695.66
12	11	\$50 000.00	\$29 233.96
13	12	\$50 000.00	\$27 841.87
14	13	\$50 000.00	\$26 516.07
15	14	\$50 000.00	\$25 253.40
16	15	\$50 000.00	\$24 050.85
17	16	\$50 000.00	\$22 905.58
18	17	\$50 000.00	\$21 814.83
19	18	\$50 000.00	\$20 776.03
20	19	\$50 000.00	\$19 786.70
21	20	\$50 000.00	\$18 844.47
22			\$623 110.52

I used the FILL DOWN command to determine the present values for the remaining payments. I then used the SUM command to determine the sum of all the present values.

The lump-sum payment should be \$623 110.52.

EXAMPLE 3 Selecting a strategy to determine the regular payment and total interest of an annuity

Len borrowed \$200 000 from the bank to purchase a yacht. If the bank charges 6.6%/a compounded monthly, he will take 20 years to pay off the loan.

a) How much will each monthly payment be?
b) How much interest will he have paid over the term of the loan?

Jasmine's Solution: Using the Formula

a)

$$i = \frac{0.066}{12} = 0.0055$$
$$n = 20 \times 12 = 240$$

I calculated the interest rate per compounding period and the number of compounding periods.

$$PV = \$200\ 000$$
$$PV = R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$$
$$200\ 000 = R \times \left(\frac{1 - (1 + 0.0055)^{-240}}{0.0055}\right)$$

I substituted the values of *PV*, *i*, and *n* into the formula for the present value of an annuity.

$$200\ 000 \doteq R \times 133.072$$
$$\frac{200\ 000}{133.072} = R \times \frac{133.072}{133.072}$$

To solve for *R*, I divided both sides of the equation by 133.072.

$$R \doteq 1502.94$$

I rounded to the nearest cent.

Len will have to pay \$1502.94 per month for 20 years to pay off the loan.

b)

$$A = 1502.94 \times 240$$
$$= \$360\ 706.60$$

I calculated the total amount that Len will have paid over the 20-year term.

$$I = A - PV$$
$$= \$360\ 706.60 - \$200\ 000$$
$$= \$160\ 706.60$$

I determined the interest by subtracting the present value from the total amount that Len will have paid.

Over the 20-year term of the loan, Len will have paid \$160 706.60 in interest.

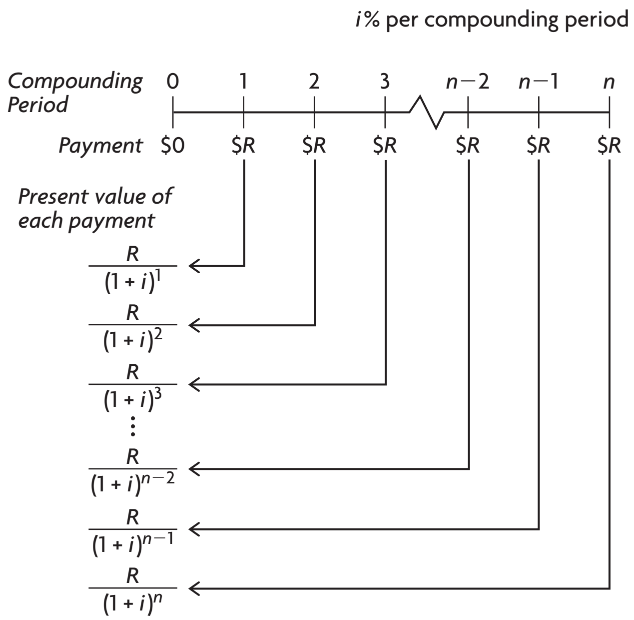
In Summary

Key Ideas

- The present value of an annuity is the value of the annuity at the beginning of the term. It is the sum of all present values of the payments and can be written as the geometric series

$$PV = R \times (1 + i)^{-1} + R \times (1 + i)^{-2} + R \times (1 + i)^{-3} + \ldots + R \times (1 + i)^{-n}$$

where PV is the present value; R is the regular payment; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.



- The formula for the sum of a geometric series can be used to determine the present value of an annuity.

Need to Know

- The formula for the present value of an annuity is

$$PV = R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$$

where PV is the present value; R is the regular payment each compounding period; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

CHECK Your Understanding

1. Each situation represents a loan.

i) Draw a timeline to represent the amount of the original loan.
ii) Write the series that represents the amount of the original loan.
iii) Calculate the amount of the original loan.
iv) Calculate the interest paid.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	\$650 per year	3.7%	annually	5 years
b)	\$1200 every 6 months	9.4%	semi-annually	9 years
c)	\$84.73 per quarter	3.6%	quarterly	$3\frac{1}{2}$ years
d)	\$183.17 per month	6.6%	monthly	10 years

2. Each situation represents a simple, ordinary annuity.

i) Calculate the present value of each payment.
ii) Write the present values of the payments as a series.
iii) Calculate the present value of the annuity.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	\$8000 per year	9%	annually	7 years
b)	\$300 every 6 months	8%	semi-annually	3.5 years
c)	\$750 per quarter	8%	quarterly	2 years

PRACTISING

3. Calculate the present value of each annuity.
K

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	\$5000 per year	7.2%	annually	5 years
b)	\$250 every 6 months	4.8%	semi-annually	12 years
c)	\$25.50 per week	5.2%	weekly	100 weeks
d)	\$48.50 per month	23.4%	monthly	$2\frac{1}{2}$ years

4. You want to buy a \$1300 stereo on credit and make monthly payments over 2 years. If the store is charging you 18%/a compounded monthly, what will be your monthly payments?

5. Lily wants to buy a snowmobile. She can borrow \$7500 at 10%/a compounded quarterly if she repays the loan by making equal quarterly payments for 4 years.

a) Draw a timeline to represent the annuity.
b) Write the series that represents the present value of the annuity.
c) Calculate the quarterly payment that Lily must make.

6. Rocco pays \$50 for a DVD/CD player and borrows the remaining amount. He plans to make 10 monthly payments of \$40 each. The first payment is due next month.
a) The interest rate is 18%/a compounded monthly. What was the selling price of the player?
b) How much interest will he have paid over the term of the loan?

7. Emily is investing \$128 000 at 7.8%/a compounded monthly. She wants to withdraw an equal amount from this investment each month for the next 25 years as spending money. What is the most she can take out each month?

8. The Peca family wants to buy a cottage for \$69 000. The Pecas can pay \$5000 and finance the remaining amount with a loan at 9%/a compounded monthly. The loan payments are monthly, and they may choose either a 7-year or a 10-year term.
a) Calculate the monthly payment for each term.
b) How much would they save in interest by choosing the shorter term?
c) What other factors should the Pecas consider before making their financing decision?

9. A Charles would like to buy a new car that costs \$32 000. The dealership offers to finance the car at 2.4%/a compounded monthly for five years with monthly payments. The dealer will reduce the selling price by \$3000 if Charles pays cash. Charles can get a loan from his bank at 5.4%/a compounded monthly. Which is the best way to buy the car? Justify your answer with calculations.

10. To pay off \$35 000 in loans, Nina's bank offers her a rate of 8.4%/a compounded monthly. She has a choice between a 5-, 10-, or 15-year term.
a) Determine the monthly payment for each term.
b) Calculate how much interest Nina would pay in each case.

11. Pedro pays \$45 for a portable stereo and borrows the remaining amount. The loan payments are \$25 per month for 1 year. The interest rate is 18.6%/a compounded monthly.
a) What was the selling price of the stereo?
b) How much interest will Pedro have paid over the term of the loan?

12. Suzie buys a new computer for $2500. She pays $700 and finances the rest at $75.84 per month for $2\frac{1}{2}$ years. What annual interest rate, compounded monthly, is Suzie being charged? Round your answer to two decimal places.

13. Leo invests $50 000 at 11.2%/a compounded quarterly for his retirement. Leo's financial advisor tells him that he should take out a regular amount quarterly when he retires. If Leo has 20 years until he retires and wants to use the investment for recreation for the first 10 years of retirement, what is the maximum quarterly withdrawal he can make?

14. Charmaine calculates that she will require about $2500 per month for the first 15 years of her retirement. If she has 25 years until she retires, how much should she invest each month at 9%/a compounded monthly for the next 25 years if she plans to withdraw $2500 per month for the 15 years after that?

15. **T** A lottery has two options for winners collecting their prize:
- Option A: $1000 each week for life
- Option B: $660 000 in one lump sum

The current interest rate is 6.76%/a compounded weekly.

a) Which option would you suggest to a winner who expects to live for another 25 years?

b) When is option A better than option B?

16. **C** Classify situations and factors that show the differences between each pair of terms. Give examples.

a) a lump sum or an annuity

b) future value or present value

Extending

17. Stefan claims that he has found a different method for calculating the present value of an annuity. Instead of calculating the present value of each payment, he calculates the future value of each payment. Then he calculates the sum of the future values of the payments. Finally, he calculates the present value of this total sum.

a) Use Stefan's method to solve Example 1 (a).

b) Create another example to show that his claim is true. Include timelines.

c) Use the formula for present value to prove that Stefan's claim works for all annuities.

18. Kyla must repay student loans that total $17 000. She can afford to make $325 monthly payments. The bank is charging an interest rate of 7.2%/a compounded monthly. How long will it take Kyla to repay her loans?

19. In question 14, Charmaine invested a fixed amount per month so that her annuity would provide her with another monthly amount in her retirement. Derive a formula for the regular payment $\$R$ that must be made for m payments at an interest rate of i% per compounding period to provide for a regular withdrawal of $\$W$ after all the payments are made for n withdrawals.

8.6 Using Technology to Investigate Financial Problems

GOAL

Use technology to investigate the effects of changing the conditions in financial problems.

YOU WILL NEED

- graphing calculator
- spreadsheet software

INVESTIGATE the Math

Tina wants to buy a stereo that costs \$566.47 after taxes. The store allows her to buy the stereo by making payments of \$29.95 per month for 2 years.

? What annual interest rate, compounded monthly, is the store charging?

A. Draw a timeline for this situation. Will you be calculating present values or future values?

B. Use a spreadsheet to set up an **amortization schedule** as shown.

	A	B	C	D	E	F
1	Interest Rate	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2						\$566.47
3	0.01	1	\$29.95	"=F2*A3"	"=C3-D3"	"=F2-E3"
4		"=B3+1"	\$29.95	"=F3*A3"	"=C4-D4"	"=F3-E4"

amortization schedule

a record of payments showing the interest paid, the principal, and the current balance on a loan or investment

Tech Support

For help creating an amortization schedule using a spreadsheet, see Technical Appendix, B-22.

C. Use the COPY DOWN command to complete the spreadsheet so that 24 payments are showing. The spreadsheet shown here is set up with an interest rate of 1% per compounding period. Adjust the value of the interest rate to solve the problem.

D. Enter the formula for the present value of the annuity into a graphing calculator, where Y is the (unknown) present value and X is the annual interest rate compounded monthly.

E. Graph the equation in part D, as well as $y = 566.47$. Use these graphs to solve the problem.

F. On your graphing calculator, activate the TVM Solver.

G. Enter the corresponding values and then solve the problem.

Tech Support

For help using the TVM Solver on a graphing calculator, see Technical Appendix, B-19.

Reflecting

H. Why could you not solve this problem easily with pencil and paper?

I. Which of the three methods (the spreadsheet in parts B and C, the graphs in parts D and E, or the TVM Solver in parts F and G) used to solve the problem do you prefer? Why?

APPLY the Math

EXAMPLE 1 Selecting a tool to investigate the effects of varying the interest rate

Jamal has \$10 000 to invest. Bank of North America offers an interest rate of 4.2%/a compounded monthly. Key Bank offers an interest rate of 5%/a compounded quarterly. How much longer will it take the money invested to grow to \$50 000 if Jamal chooses Bank of North America?

Lina's Solution: Using Guess-and-Check

$$i = \frac{0.042}{12} = 0.0035$$

I first looked at the Bank of North America. Since interest is paid monthly, I divided the annual interest rate by 12 to get the interest rate per month.

$$P = \$10\,000$$

$$A = P(1 + i)^n$$
$$= 10\,000(1.0035)^n$$

I substituted the values of i and P into the compound-interest formula. I thought 10 years might be a good guess. That would give $n = 120$ compounding periods.

$$A = 10\,000(1.0035)^{120}$$
$$\doteq 15\,208.46$$

$$A = 10\,000(1.0035)^{480}$$
$$\doteq 53\,498.41$$

My guess was way too small, so I tried 40 years, which gives $n = 480$ compounding periods.

Try $n = 460$:
$$A = 10\,000(1.0035)^{460}$$
$$\doteq 49\,887.68$$

Try $n = 461$:
$$A = 10\,000(1.0035)^{461}$$
$$\doteq 50\,062.29$$

That guess was much closer. Eventually, I tried 460 months. It was a little low, so I tried 461 months.

$$\frac{461}{12} \doteq 38.417$$

I determined how long 461 months is in terms of years. First I divided 461 by 12 to get 38 years.

$$0.417 \times 12 \text{ months} \doteq 5 \text{ months}$$

Then I multiplied 0.417 by 12 to get 5 months.

$$n = 38 \text{ years and } 5 \text{ months}$$

$$i = \frac{0.05}{4} = 0.0125$$

Next, I looked at Key Bank. Since interest is paid quarterly, I divided the annual interest rate by 4 to get the interest rate per quarter.

$$P = \$10\,000$$

$$A = P(1 + i)^n$$
$$= 10\,000(1.0125)^n$$

I substituted the values of i and P into the compound-interest formula.

$$A = 10\,000(1.0125)^{140}$$
$$\doteq 56\,925.19$$

Since it took Bank of North America 38 years to grow to \$50 000, I used 35 years as my first guess for Key Bank because the interest rate is higher.
35 years is $35 \times 4 = 140$ quarters.

Try $n = 129$:

$A = 10\,000(1.0125)^{129}$

$\doteq 49\,654.56$

Try $n = 130$:

$A = 10\,000(1.0125)^{130}$

$\doteq 50\,275.24$

This result was close, but a bit high. Eventually, I tried 129 quarters and then 130 quarters.

$$\frac{130}{4} = 32.5$$

I determined how long 130 quarters is in terms of years. I divided 130 by 4 to get 32 years.

$n = 32$ years and 6 months

I knew that 0.5 years is 6 months.

It will take 38 years and 5 months to get $50 000 if Jamal chooses Bank of North America. It will take 32 years and 6 months if he chooses Key Bank. So it will take almost 6 years longer to reach his goal if Jamal chooses Bank of North America.

George's Solution: Using a Graphing Calculator

$A = P(1 + i)^n$

Bank of North America:

$$i = \frac{0.042}{12} = 0.0035$$

$A = 10\,000(1.0035)^n$

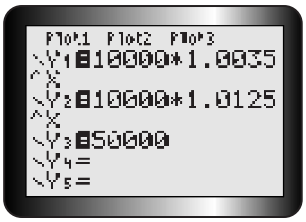
Key Bank:

$$i = \frac{0.05}{4} = 0.0125$$

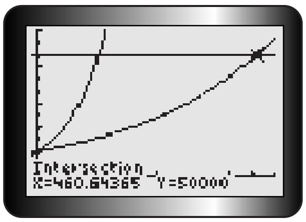
$A = 10\,000(1.0125)^n$

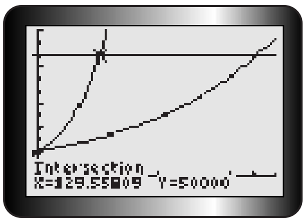
At Bank of North America, interest is compounded monthly. At Key Bank, interest is compounded quarterly. I calculated the interest rate per compounding period at each bank.

Then I used the compound-interest formula to calculate the amounts.



I entered the equations for the amounts into my graphing calculator, using Y1 and Y2 for the amounts for Bank of North America and Key Bank, respectively, and X for the number of compounding periods. I entered Y3 = 50 000.





I graphed the three equations and used the calculator to find the point of intersection of each exponential function with the horizontal line, which indicated when the amount of the investment had reached $50 000.

It will take about 460 months, or 38 years and 5 months, to get $50 000 if Jamal chooses Bank of North America. It will take about 129 quarters, or 32 years and 6 months, if he chooses Key Bank. So it will take almost 6 years longer to reach his goal if Jamal chooses Bank of North America.

Coco's Solution: Using the TVM Solver

I entered the information on the investment with Bank of North America into the TVM Solver. Jamal pays into the account at the start, so the present value is −$10 000. Also, no regular payments are being made. This is a lump-sum investment, so I set PMT on my calculator to 0. I determined that it would take a bit more than 460 months, or 38 years and 5 months, to reach his goal with Bank of North America.

I entered the information on the investment with Key Bank into the TVM Solver. I determined that it would take a bit more than 129 quarters, or 32 years and 6 months, to reach his goal with Key Bank.

If Jamal chooses Bank of North America, it will take about 6 years longer to reach his goal.

EXAMPLE 2 Selecting a tool to investigate the effects of increasing the monthly payment

Lia borrows $180 000 to open a restaurant. She can afford to make monthly payments between $1000 and $1500 at 4.8%/a compounded monthly. How much sooner can she pay off the loan if she makes the maximum monthly payment?

Teresa's Solution: Using a Spreadsheet

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2					$180 000.00
3	1	$1 000.00	"=E2*(0.048/12)"	"=B3-C3"	"=E2-D3"
4	"=A3+1"	$1 000.00	"=E3*(0.048/12)"	"=B4-C4"	"=E3-D4"

I set up a spreadsheet to solve the problem. Since the interest is compounded monthly, I divided 4.8% by 12 to get the interest rate per month. For the $1000 minimum payment, I calculated the proportion of the principal paid for each payment. Then I subtracted that proportion from the previous balance to get the balance at the end of the next month.

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2					$180 000.00
3	1	$1 000.00	$720.00	$280.00	$179 720.00
4	2	$1 000.00	$718.88	$281.12	$179 438.88
5	3	$1 000.00	$717.76	$282.24	$179 156.64
6	4	$1 000.00	$716.63	$283.37	$178 873.26
7	5	$1 000.00	$715.49	$284.51	$178 588.76
8	6	$1 000.00	$714.36	$285.64	$178 303.11
9	7	$1 000.00	$713.21	$286.79	$178 016.32
10	8	$1 000.00	$712.07	$287.93	$177 728.39
11	9	$1 000.00	$710.91	$289.09	$177 439.30
12	10	$1 000.00	$709.76	$290.24	$177 149.06

Next, I used the FILL DOWN command to complete the other rows.

314	312	$1 000 .00	$30.96	$969.04	$6 770.39
315	313	$1 000 .00	$27.08	$972.92	$5 797.48
316	314	$1 000 .00	$23.19	$976.81	$4 820.67
317	315	$1 000 .00	$19.28	$980.72	$3 839.95
318	316	$1 000 .00	$15.36	$984.64	$2 855.31
319	317	$1 000 .00	$11.42	$988.58	$1 866.73
320	318	$1 000 .00	$7.47	$992.53	$874.20
321	319	$1 000 .00	$3.50	$996.50	–$122.31

I continued until the balance became negative, indicating that the loan was paid off.

	A	B	C	D	E
1	**Payment Number**	**Payment**	**Interest Paid**	**Principal Paid**	**Balance**
2					$180 000.00
3	1	$1 500 .00	$720.00	$780.00	$179 220.00
4	2	$1 500 .00	$716.88	$783.12	$178 436.88
5	3	$1 500 .00	$713.75	$786.25	$177 650.63
6	4	$1 500 .00	$710.60	$789.40	$176 861.23
7	5	$1 500 .00	$707.44	$792.56	$176 068.67
8	6	$1 500 .00	$704.27	$795.73	$175 272.95
9	7	$1 500 .00	$701.09	$798.91	$174 474.04
10	8	$1 500 .00	$697.90	$802.10	$173 671.94
11	9	$1 500 .00	$694.69	$805.31	$172 866.63
12	10	$1 500 .00	$691.47	$808.53	$172 058.09

I replaced the $1000 minimum payment with the $1500 maximum payment and used the FILL DOWN command in all the cells under Payment.

159	157	$1 500 .00	$46.04	$1 453.96	$10 054.91
160	158	$1 500 .00	$40.22	$1 459.78	$8 595.13
161	159	$1 500 .00	$34.38	$1 465.62	$7 129.51
162	160	$1 500 .00	$28.52	$1 471.48	$5 658.03
163	161	$1 500 .00	$22.63	$1 477.37	$4 180.66
164	162	$1 500 .00	$16.72	$1 483.28	$2 697.39
165	163	$1 500 .00	$10.79	$1 489.21	$1 208.17
166	164	$1 500 .00	$4.83	$1 495.17	–$286.99

I continued until the balance became negative, indicating that the loan was paid off.

At the minimum payment of $1000, Lia's loan would be paid off after 319 months, or 26 years and 7 months. At the maximum payment of $1500, the loan would be paid off after 164 months, or 13 years and 8 months. So Lia can pay off the loan almost 13 years sooner if she makes the maximum payment.

Mike's Solution: Using the TVM Solver

I entered the information on the loan into the TVM Solver on a graphing calculator. I entered the minimum monthly payment of $1000 and then used the calculator to determine the number of payments needed.

I then changed the monthly payment to the maximum amount of $1500, and used the calculator to determine the number of payments needed.

At the minimum payment of $1000, Lia's loan would be paid off after 319 months, or 26 years and 7 months. At the maximum payment of $1500, the loan would be paid off after 164 months, or 13 years and 8 months. So Lia can pay off the loan almost 13 years sooner if she makes the maximum payment.

Communication *Tip*

Sometimes you can make a large purchase by paying a small portion of the cost right away and financing the rest with a loan. The portion paid right away is called a **down payment**.

EXAMPLE 3 Selecting a tool to investigate the effects of paying more frequently

Sarah and John are both saving for a down payment on their first home. Both plan to save \$2400 each year by depositing into an account that earns 4%/a.

- John makes monthly deposits of \$200 into an account on which the interest is compounded monthly.
- Sarah makes annual payments of \$2400 into an account on which the interest is compounded annually.

Determine the difference in their account balances at the end of 5 years.

Jason's Solution

I used the TVM Solver on my graphing calculator and entered the information on John. I found that his balance would be \$13 259.80 at the end of 5 years.

I repeated the same type of calculation, but this time with the information on Sarah. I found that her balance would be \$12 999.17 at the end of 5 years.

\$13 259.80 − \$12 999.17 = \$260.63

I subtracted to calculate the difference in the amounts.

John's account will have \$260.63 more than Sarah's after 5 years.

EXAMPLE 4 Connecting the interest paid on a loan with time

You borrow \$100 000 at 8.4%/a compounded monthly. You make monthly payments of \$861.50 to pay off the loan after 20 years. How long will it take to pay off

a) the first \$25 000?
b) the next \$25 000?
c) the next \$25 000?
d) the last \$25 000?
e) Why are the answers to parts (a) through (d) all different?

Mena's Solution

a)

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2					$100 000.00
3	1	$861.50	$700.00	$161.50	$99 838.50
4	2	$861.50	$698.87	$162.63	$99 675.87
5	3	$861.50	$697.73	$163.77	$99 512.10
6	4	$861.50	$696.58	$164.92	$99 347.19
7	5	$861.50	$695.43	$166.07	$99 181.12
104	102	$861.50	$534.80	$326.70	$76 073.31
105	103	$861.50	$532.51	$328.99	$75 744.32
106	104	$861.50	$530.21	$331.29	$75 413.03
107	105	$861.50	$527.89	$333.61	$75 079.42
108	106	$861.50	$525.56	$335.94	$74 743.48

I used a spreadsheet to create an amortization schedule. I then used the FILL DOWN feature to complete the spreadsheet.

I noticed that the balance is reduced to $74 743.48 after 106 months, so it took 8 years and 10 months to pay off the first $25 000.

b)

164	162	$861.50	$365.00	$496.50	$51 646.68
165	163	$861.50	$361.53	$499.97	$51 146.71
166	164	$861.50	$358.03	$503.47	$50 643.23
167	165	$861.50	$354.50	$507.00	$50 136.24
168	166	$861.50	$350.95	$510.55	$49 625.69

The balance is reduced to $49 625.69 after 166 months, so it took 60 months, or 5 years, to pay off the next $25 000.

c)

206	204	$861.50	$195.99	$665.51	$27 332.96
207	205	$861.50	$191.33	$670.17	$26 662.79
208	206	$861.50	$186.64	$674.86	$25 987.93
209	207	$861.50	$181.92	$679.58	$25 308.35
210	208	$861.50	$177.16	$684.34	$24 624.00

The balance is reduced to $24 624.00 after 208 months, so it took 42 months, or 3 years and 6 months, to pay off the next $25 000.

d)

239	237	$861.50	$23.72	$837.78	$2 551.46
240	238	$861.50	$17.86	$843.64	$1 707.82
241	239	$861.50	$11.95	$849.55	$858.28
242	240	$861.50	$6.01	$855.49	$2.78
243	241	$861.50	$0.02	$861.48	−$858.70

The loan is paid off after 240 months, or 20 years. It takes 208 months to pay about $75 000, so I subtracted 208 from 240 to determine how long it takes to pay the last $25 000 of the loan. The last $25 000 takes 32 months, or 2 years and 8 months, to pay off.

e) It takes different lengths of time to pay off the same amount of money because the interest paid is greater when the balance owed is greater. Less of the payment goes toward the principal.

In Summary

Key Idea

- Spreadsheets and graphing calculators are just two of the technological tools that can be used to investigate and solve financial problems involving interest, annuities, and amortization schedules.

Need to Know

- The advantage of an amortization schedule is that it provides the history of all payments, interest paid, and balances on a loan.
- More interest can be earned if
  - the interest rate is higher
  - there are more compounding periods per year
- If you increase the amount of the regular payment of a loan, you can pay it off sooner and save a significant amount in interest charges.
- Early in the term of a loan, the major proportion of each regular payment is interest, with only a small amount going toward paying off the principal. As time progresses, a larger proportion of each regular payment goes toward the principal.

CHECK Your Understanding

1. Use technology to determine how long it will take to reach each investment goal.

	Principal	Rate of Compound Interest per Year	Compounding Period	Future Value
a)	\$5 000	8.3%	annually	\$13 000
b)	\$2 500	6.8%	semi-annually	\$4 000
c)	\$450	12.4%	quarterly	\$4 500
d)	\$15 000	3.6%	monthly	\$20 000

2. Use technology to determine the annual interest rate, to two decimal places, being charged in each loan. The compounding period corresponds to when the payments are made.

	Principal	Regular Payment	Time
a)	\$2 500	\$357.59 per year	10 years
b)	\$15 000	\$1497.95 every 6 months	6 years
c)	\$3 500	\$374.56 per quarter	3 years
d)	\$450	\$29.62 per month	18 months

PRACTISING

3. Trevor wants to save \$3500. How much will he have to put away each month at 12.6%/a compounded monthly in order to have enough money in $2\frac{1}{2}$ years?

4. **A** Nadia borrows \$120 000 to buy a house. The current interest rate is 6.6%/a compounded monthly, and Nadia negotiates the term of the loan to be 25 years.

a) What will be each monthly payment?

b) After paying for 3 years, Nadia receives an inheritance and makes a one-time payment of \$15 000 against the outstanding balance of the loan. How much earlier can she pay off the loan because of this payment?

c) How much will she save in interest charges by making the \$15 000 payment?

5. **K** Lisa and Karl are deciding to invest \$750 per month for the next 7 years.

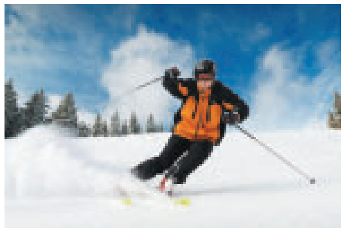
- Bank A has offered them 6.6%/a compounded monthly.
- Bank B has offered them 7.8%/a compounded monthly.

How much more will they end up with by choosing the second offer?

6. Mario decides to pay \$250 per month at 5%/a compounded monthly to pay off a \$25 000 loan. After 2 years, he is making a bit more money and decides to increase the monthly payment. If he pays \$50 extra per month at the end of each 2-year period, how long will it take him to pay off the loan?

7. Natalie borrows \$150 000 at 4.2%/a compounded monthly for a period of 20 years to start a business. She is guaranteed that interest rate for 5 years and makes monthly payments of \$924.86. After 5 years, she renegotiates her loan, but interest rates have gone up to 7.5%/a compounded monthly.
   a) If Natalie would like to have the loan paid off after the original 20-year period, what should her new monthly payment be?
   b) If she keeps her payments the same, how much extra time will it take her to pay off the loan?

8. Peter buys a ski vacation package priced at \$2754. He pays \$350 down and finances the balance at \$147 per month for $1\frac{1}{2}$ years. Determine the annual interest rate, compounded monthly, being charged. Round your answer to two decimal places.



9. **T** a) Suppose you have a loan where the interest rate doubles. If you want to keep the same amortization period, should you double the payment? Justify your reasoning with examples.
   b) Suppose you are borrowing money. If you decide to double the amount borrowed, should you double the payment if you want to keep the same amortization period? Justify your reasoning with examples.

10. Laurie borrows \$50 000 for 10 years at 6.6%/a compounded monthly. How much sooner can she pay off the loan if she doubles the monthly payment after 4 years?

11. **C** What are the advantages and disadvantages of using each technology to solve financial problems?
    - a spreadsheet
    - a graphing calculator

Extending

12. A music store will finance the purchase of a rare guitar at 3.6%/a compounded monthly over 5 years, but offers a \$250 reduction if the payment is cash. If you can get a loan from a bank at 4.8%/a compounded annually, how much would the guitar have to sell for to make it worthwhile to take out the loan?

13. The interest on all mortgages is charged semi-annually. You are given a choice of monthly, semi-monthly, bi-weekly, and weekly payments. Suppose you have a mortgage at 8%/a, the monthly payments are \$1000, and the amortization period is 20 years. Investigate the effect on the time to pay off the mortgage if you made each of these payments.
    a) \$500 semi-monthly
    b) \$500 bi-weekly
    c) \$250 weekly

14. Steve decides to pay \$150 per month to pay off a \$6800 loan. In the beginning, the interest rate is 13%/a compounded monthly. The bank guarantees the interest rate for one year at a time. The rate for the next year is determined by the going rate at the time. Assuming that each year the rate drops by 0.5%/a, how long will it take Steve to pay off his loan?

8 Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 8.4, Examples 1 to 4.
- Try Chapter Review Questions 11 and 12.

Q: How do you determine the future value of an annuity?

A1: An annuity is a series of payments or investments made at regular intervals. The future value of an annuity is the sum of all regular payments and interest earned. You can determine the future value of each payment or investment by using the formula $A = P(1 + i)^n$.

Since an annuity consists of regular payments, the future values of the investments, starting from the last, will be P, $P(1 + i)$, $P(1 + i)^2$, ... These form a geometric sequence with common ratio $1 + i$. So the future value of all of the investments is the geometric series $P + P(1 + i) + P(1 + i)^2 + \ldots$, which can be calculated with the formula for the sum of a geometric series.

A2: You can use technology such as a spreadsheet or the TVM Solver on a graphing calculator to calculate the future value of an annuity.

EXAMPLE

The spreadsheet below is set up for an annuity in which 40 regular investments of $250 are made at the end of each compounding period. The annuity earns 2% interest per compounding period.

Since the last $250 investment was deposited at the end of the term, it earned no interest. The first $250 investment earned interest 39 times, but didn't earn interest during the first compounding period because it was deposited at the end of that period.

	A	B	C
1	Number of Compounding Periods Invested	Amount Invested	Future Value
2	0	$250.00	"=B2"
3	"=A2+1"	$250.00	"=B3*(1+0.02)"
4	"=A3+1"	$250.00	"=B4*(1+0.02)*2"

	A	B	C
1	Number of Compounding Periods Invested	Amount Invested	Future Value
2	0	$250.00	$250.00
3	1	$250.00	$255.00
4	2	$250.00	$260.10
5	3	$250.00	$265.30
6	4	$250.00	$270.61
7	5	$250.00	$276.02
8	6	$250.00	$281.54
37	35	$250.00	$499.97
38	36	$250.00	$509.97
39	37	$250.00	$520.17
40	38	$250.00	$530.57
41	39	$250.00	$541.19
42			$15 100.50

The future value of this annuity is $15 100.50

A3: Use the formula for the future value of an annuity, $FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right)$, where FV is the future value; R is the regular payment each compounding period; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

Q: How can you determine the present value of an annuity?

A1: The present value of an annuity is the amount of money you have to invest to get a specific amount some time in the future. You can determine the present value of each investment by using the formula $PV = A(1 + i)^{-n}$.

Since an annuity consists of regular payments, the present values of the investments, starting from the first, will be $A(1 + i)^{-1}$, $A(1 + i)^{-2}$, $A(1 + i)^{-3}$, ... These form a geometric sequence with common ratio $(1 + i)^{-1}$. So the present value of all of the investments is the geometric series $A(1 + i)^{-1} + A(1 + i)^{-2} + A(1 + i)^{-3} + \ldots$, which can be calculated with the formula for the sum of a geometric series.

A2: You can use technology such as a spreadsheet or the TVM Solver on a graphing calculator to calculate the present value of an annuity.

EXAMPLE

The spreadsheet below is set up for an annuity earning 0.5% interest per compounding period and providing 20 regular payments of $50.

	A	B	C
1	Number of Compounding Periods Invested	Payment	Present Value
2	1	$50.00	"=B2/1.005"
3	"=A2+1"	$50.00	"=B3/(1.005)^A3"
4	"=A3+1"	$50.00	"=B4/(1.005)^A4"

	A	B	C
1	Number of Compounding Periods Invested	Payment	Present Value
2	1	$50.00	$49.75
3	2	$50.00	$49.50
4	3	$50.00	$49.26
5	4	$50.00	$49.01
6	5	$50.00	$48.77
7	6	$50.00	$48.53
17	16	$50.00	$46.17
18	17	$50.00	$45.94
19	18	$50.00	$45.71
20	19	$50.00	$45.48
21	20	$50.00	$45.25
22			$949.37

The present value of all of the investements in this annuity is $949.37.

A3: Use the formula for the present value of an annuity,

$PV = R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$, where PV is the present value; R is the regular payment each compounding period; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

Study Aid

- See Lesson 8.5, Examples 1, 2, and 3.
- Try Chapter Review Questions 13 to 17.

PRACTICE Questions

Lesson 8.1

1. Each situation represents an investment earning simple interest. Calculate the interest earned and the total amount.

	Principal	Rate of Simple Interest per Year	Time
a)	$3 500	6%	10 years
b)	$15 000	11%	3 years
c)	$280	3.2%	34 months
d)	$850	29%	100 weeks
e)	$21 000	7.3%	42 days

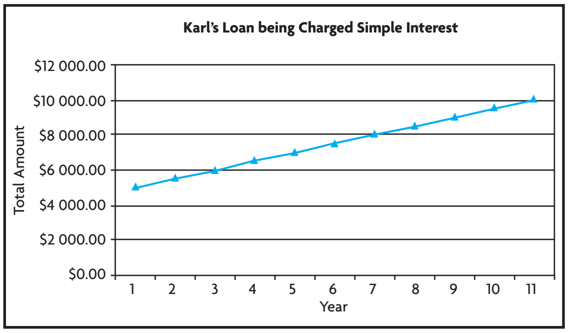
2. Pia invests $2500 in an account that earns simple interest. At the end of each month, she earns $11.25 in interest.

a) What annual rate of simple interest is Pia earning? Round your answer to two decimal places.

b) How much money will be in her account after 7 years?

c) How long will it take for her money to double?

3. Karl borrows some money and is charged simple interest. The graph below shows how the amount he owes grows over time.



a) How much did Karl borrow?

b) What annual interest rate is he being charged?

c) How long will it take before he owes $20 000?

Lesson 8.2

4. Isabelle invests $4350 at 7.6%/a compounded quarterly. How long will it take for her investment to grow to $10 000?

5. Each situation represents a loan being charged compound interest. Calculate the total amount and the interest being charged.

	Principal	Rate of Compound Interest per Year	Compounding Period	Time
a)	$4 300	9.1%	annually	8 years
b)	$500	10.4%	semi-annually	11.5 years
c)	$25 000	6.4%	quarterly	3 years
d)	$307	27.6%	monthly	2.5 years

6. Deana invests some money that earns interest compounded annually. At the end of the first year, she earns $400 in interest. At the end of the second year, she earns $432 in interest.

a) What interest rate, compounded annually, is Deana earning? Round your answer to two decimal places.

b) How much did she invest?

7. Vlad purchased some furniture for his apartment. The total cost was $2942.37. He paid $850 down and financed the rest for 18 months. At the end of the finance period, Vlad owed $2147.48. What annual interest rate, compounded monthly, was he being charged? Round your answer to two decimal places.

Lesson 8.3

8. Calculate the present value of each investment.

	Rate of Compound Interest per Year	Compounding Period	Time	Future Value
a)	6.7%	annually	5 years	$8 000
b)	8.8%	semi-annually	2.5 years	$1 280
c)	5.6%	quarterly	8 years	$100 000
d)	24.6%	monthly	1.5 years	$850

9. Roberto financed a purchase at 9.6%/a compounded monthly for 2.5 years. At the end of the financing period, he still owed $847.53. How much money did Roberto borrow?

10. Marisa invests \$1650 for 3 years, at which time her investment is worth \$2262.70. What interest rate, compounded annually, would yield the same results? Round your answer to two decimal places.

Lesson 8.4

11. For each annuity, calculate the future value and the interest earned.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	\$2500 per year	7.6%	annually	12 years
b)	\$500 every 6 months	7.2%	semi-annually	9.5 years
c)	\$2500 per quarter	4.3%	quarterly	3 years

12. Naomi wants to save \$100 000, so she makes quarterly payments of \$1500 into an account that earns 4.4%/a compounded quarterly. How long will it take her to reach her goal?

Lesson 8.5

13. Ernie wants to invest some money each month at 9%/a compounded monthly for 6 years. At the end of that time, he would like to have \$25 000. How much money does he have to put away each month?

14. For each loan, calculate the amount of the loan and the interest being charged.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	\$450 per year	5.1%	annually	6 years
b)	\$2375 every 6 months	9.2%	semi-annually	4.5 years
c)	\$185.73 per quarter	12.8%	quarterly	3.5 years
d)	\$105.27 per month	19.2%	monthly	1.5 years

15. Paul borrows \$136 000. He agrees to make monthly payments for the next 20 years. The interest rate being charged is 6.6%/a compounded monthly.
    a) How much will Paul have to pay each month?
    b) How much interest is he being charged over the term of the loan?

16. Eden finances a purchase of \$611.03 by making monthly payments of \$26.17 for $2\frac{1}{2}$ years. What annual interest rate, compounded monthly, is she being charged? Round your answer to two decimal places.

17. Chantal purchases a moped for \$1875.47 with \$650 down. She finances the balance at 6.6%/a compounded monthly over 4 years. How much will Chantal have to pay each month?

Lesson 8.6

18. Starting at age 20, Ken invests \$100 per month in an account that earns 5.4%/a compounded monthly. Starting at age 37, his twin brother, Adam, starts saving money in an account that pays 7.2%/a compounded monthly. How much more money will Adam need to invest each month if he wants his investment to be worth the same as Ken's by the time they are 55 years old?

19. Jenny starts a business and borrows \$100 000 at 4.2%/a compounded monthly. She can afford to make payments between \$1000 and \$1500 per month. How much sooner can she pay off the loan if she pays the maximum amount compared with the minimum amount?

20. Kevin purchases a guitar on a payment plan of \$17.85 per week for $2\frac{1}{2}$ years at 13%/a compounded weekly. What was the selling price of the guitar?

8 Chapter Self-Test

1. For each investment, determine the total amount and the interest earned.

	Principal	Rate of Interest per Year	Time
a)	\$850	9% simple interest	6 years
b)	\$5460	8.4% compounded semi-annually	13 years
c)	\$230 per month	4.8% compounded monthly	$6\frac{1}{2}$ years

Loan #1	
Year	**Amount Owed**
1	\$3796
2	\$3942
3	\$4088

Loan #2	
Year	**Amount Owed**
2	\$977.53
3	\$1036.18
4	\$1098.35

2. The amounts owed for two different loans are shown at the left.

a) For each loan, determine whether simple interest or compound interest is being charged. Justify your answer.

b) What annual interest rate is each loan being charged? Round your answer to two decimal places.

c) How much was each loan originally?

d) Determine the future value of each loan after 10 years.

3. Betsy inherits \$15 000 and would like to put some of it away for a down payment on a house in 8 years. If she would like to have \$25 000 for the down payment, how much of her inheritance must she invest at 9.2%/a compounded quarterly?

4. Derek invests \$250 per month for $6\frac{1}{2}$ years at 4.8%/a compounded monthly. How much will his investment be worth at the end of the $6\frac{1}{2}$ years?

5. Simone wants to save money for her retirement. Her two best options are 5.88%/a compounded monthly or 6%/a compounded annually. Which option should she choose? Why?

6. Anand's parents have been paying \$450 per month into a retirement fund for the last 30 years. The fund is now worth \$450 000. What annual interest rate, compounded monthly, are Anand's parents earning? Round your answer to two decimal places.

7. Yvette wants to invest some money under these conditions:

- Each quarter for the next 17 years, she wants to earn 8.4%/a compounded quarterly.
- After 17 years, she plans to reinvest the money at 7.2%/a compounded monthly.
- She wants to withdraw \$5000 per month for the 10 years after the initial 17 years.

How much more would she have to invest per quarter if she earned 7.2%/a compounded quarterly for the first 17 years and 8.4%/a compounded monthly for the next 10 years?

8 Chapter Task

Saving for Retirement

Steve, Carol, and Lisa get their first full-time jobs and talk about saving for retirement. They are each 22 years old and plan to work until they are 55.

Steve starts investing immediately and puts aside $150 per month. Carol wants to enjoy life a bit and decides to start contributing when she is 30. Lisa thinks that they are both starting too early and decides to wait until she is 42 before starting to save.

Assume that Steve, Carol, and Lisa are each earning 9%/a compounded monthly. Carol and Lisa want to accumulate the same amount as Steve upon retirement. When they retire, Steve wants his investment to last 10 years, Carol wants hers to last 15 years, and Lisa wants hers to last 20 years.

? How much will Steve, Carol, and Lisa be able to withdraw monthly upon retirement?

A. What strategies will you use to solve this problem? Justify your strategies.

B. How much money will Steve have accumulated by the time he is 55?

C. For how many months will Carol and Lisa be making payments?

D. How much will Carol and Lisa have to put away each month to meet their goals?

E. For how many months will each person withdraw money?

F. How much will each person be able to withdraw from his or her nest egg each month?

Task *Checklist*

- ✔ Did you explain and justify your strategies?
- ✔ Did you show your work?
- ✔ Did you support your calculations with appropriate reasoning?
- ✔ Did you explain your thinking clearly?

Chapters **7–8** **Cumulative Review**

Multiple Choice

1. Determine S_{21} for the series $2.8 + 3.2 + 3.6 + 4.0 + ...$

a) 142.8 **c)** 10.8
b) 104 **d)** 142.4

2. Identify the sequence that is not geometric.

a) 4, 16, 64, 256, …
b) 30, 6, 1.2, 0.24, …
c) 2, 6, 7, 21, 22, …
d) 5, 5, 5, 5, …

3. Consider the sequence $1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, ...$ Determine t_8.

a) $\frac{128}{2187}$ **c)** $-\frac{128}{2187}$
b) $\frac{64}{79}$ **d)** $-\frac{64}{79}$

4. The first three terms of the sequence 8, a, b, 36 form an arithmetic sequence, but the last three terms form a geometric sequence. Determine all possible values of a and b.

a) $(a, b) = (1, -6)$ **c)** $(a, b) = (16, 24)$
b) $(a, b) = (-1, 6)$ **d)** $(a, b) = (12, 24)$

5. The fifth term of a geometric series is 405 and the sixth term is 1215. Calculate the sum of the first nine terms.

a) 147 615 **c)** 49 205
b) 8100 **d)** 36 705

6. Determine the first six terms of the sequence defined by $t_1 = -5$ and $t_n = -3t_{n-1} + 8$.

a) $-5, 23, -61, 191, -565, 1703$
b) $-5, 26, -70, 202, -598, 1786$
c) $-5, 23, 61, 191, 565, 1703$
d) $-5, 9, -51, 129, -411, 1209$

7. Choose the correct simplified expansion for the binomial $(x - 3)^5$.

a) $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$
b) $x^4 - 15x^3 + 90x^2 - 270x + 405$
c) $x^6 - 15x^5 + 90x^4 - 270x^3 + 405x^2 - 243x$
d) $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x$

8. After 15 days, 90% of a radioactive material has decayed. What is the half-life of the material?

a) 1.45 days **c)** 7.5 days
b) 4.52 days **d)** 11.45 days

9. Determine the annual interest rate, compounded annually, that would result in an investment doubling in seven years.

a) 10.4% **c)** 7%
b) 14% **d)** 11.45%

10. How long will it take for $5000 invested at 6%/a compounded monthly to grow to $6546.42?

a) 4.5 years **c)** 40 months
b) 3 years **d)** 48 months

11. Marisa has just won a contest. She must decide between two prize options.

- Collect a lump-sum payment of $50 000.
- Receive $800 at the end of every quarter for 10 years from an investment.

The investment earns 8%/a, compounded quarterly. How much more money would she have if she chooses the lump sum?

a) $3009.89 **c)** $348.92
b) $1678.41 **d)** $30.99

12. An annuity written as a geometric series has r equal to 1.005. Determine the annual interest rate for the annuity if the interest is compounded monthly.

a) 12% **c)** 5%
b) 0.5% **d)** 6%

13. Lee wants to buy a plasma television. The selling price is $1894. The finance plan includes $150 down with payments of $113 at the end of each month for $1\frac{1}{2}$ years. Determine the annual interest rate being charged, if the interest is compounded monthly.

a) 3.25% **c)** 20.06%
b) 1% **d)** 24%

14. Mr. Los is planning to buy a sailboat. He decides to deposit \$300 at the end of each month into an account that earns 6%/a interest, compounded monthly. At the end of four years, he uses the balance in the account as a down payment on a \$56 000 sailboat. He gets financing for the balance at a rate of 8%/a, compounded monthly. He can afford payments of \$525 per month. If interest rates remain constant, how long will it take him to repay the loan?

a) 10 years and 6 months
b) 12 years and 9 months
c) 9 years
d) 8 years and 10 months

15. Which option, if any, would allow you to repay a loan in less time?

a) decrease the regular payment
b) increase the regular payment and decrease the interest rate
c) decrease the regular payment and increase the interest rate
d) none of the above

16. Determine which best describes the regular payment on an amortized loan.

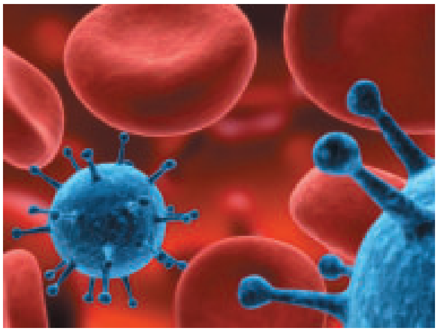
a) the average of all interest payments
b) the fixed periodic payment made up of interest and principal
c) the average of all principal payments
d) the payment of principal only

Investigations

17. Medicine Dosage

Marcus has a bacterial infection and must take 350 mg of medication every 6 h. By the time he takes his next dose, 32% of the medication remains in his body.

a) Determine a recursive formula that models this situation.
b) What will the amount of medication in his body level off to?
c) How long will it take for the medication to reach this level?



18. Financial Planner

You are a financial planner with a new client. Mr. Cowan, who just turned 37, is celebrating his son's fourth birthday.

As his financial planner, he asks you to develop a financial plan for him. Mr. Cowan wants to set up an education fund for his son, Bart, by depositing \$25 at the end of each month until Bart turns 18. The fund earns interest at 6%/a, compounded monthly.

a) Show why the sequence of the monthly amounts in the fund is a geometric sequence. Determine an expression for t_n, the value of the first deposit after n months.
b) How many payments will take place by the time Bart turns 18? Determine the balance in the fund on Bart's 18th birthday.
c) Create a spreadsheet to represent the monthly balance in the fund. Use it to verify your answer in part (b).
d) How much would there be in the fund if Mr. Cowan deposits \$50 per month instead of \$25?

Appendix **A**

Review of Essential Skills and Knowledge

A–1 Operations with Integers

Set of integers $\mathbf{I} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Addition

To add two integers,

- if the signs are the same, then the sum has the same sign as well: $(-12) + (-5) = -17$
- if the signs are different, then the sum takes the sign of the larger number: $18 + (-5) = 13$

Subtraction

Add the opposite:

$$\begin{aligned} -15 - (-8) &= -15 + 8 \\ &= -7 \end{aligned}$$

Multiplication and Division

To multiply or divide two integers,

- if the two integers have the same sign, then the answer is positive: $6 \times 8 = 48$, $(-36) \div (-9) = 4$
- if the two integers have different signs, then the answer is negative: $(-5) \times 9 = -45$, $54 \div (-6) = -9$

More Than One Operation

Follow the order of operations.

B	Brackets	
E	Exponents	
D	Division	from left to right
M	Multiplication	
A	Addition	from left to right
S	Subtraction	

EXAMPLE

Evaluate.

a) $-10 + (-12)$

b) $(-12) + 7$

c) $(-11) + (-4) + 12 + (-7) + 18$

d) $(-6) \times 9 \div 3$

e) $\dfrac{20 + (-12) \div (-3)}{(-4 + 12) \div (-2)}$

Solution

a) $-10 + (-12) = -22$

b) $(-12) + 7 = -5$

c) $(-11) + (-4) + 12 + (-7) + 18$
$= (-22) + 30$
$= 8$

d) $(-6) \times 9 \div 3$
$= -54 \div 3$
$= -18$

e) $\dfrac{20 + (-12) \div (-3)}{(-4 + 12) \div (-2)}$
$= \dfrac{20 + 4}{8 \div (-2)}$
$= \dfrac{24}{-4}$
$= -6$

Practising

1. Evaluate.

a) $6 + (-3)$
b) $12 - (-13)$
c) $-17 - 7$
d) $(-23) + 9 - (-4)$
e) $24 - 36 - (-6)$
f) $32 + (-10) + (-12) - 18 - (-14)$

2. Which choice would make each statement true: $>$, $<$, or $=$?

a) $-5 - 4 - 3 + 3$ ■ $-4 - 3 - 1 - (-2)$
b) $4 - 6 + 6 - 8$ ■ $-3 - 5 - (-7) - 4$
c) $8 - 6 - (-4) - 5$ ■ $5 - 13 - 7 - (-8)$
d) $5 - 13 + 7 - 2$ ■ $4 - 5 - (-3) - 5$

3. Evaluate.

a) $(-11) \times (-5)$
b) $(-3)(5)(-4)$
c) $35 \div (-5)$
d) $(-72) \div (-9)$
e) $(5)(-9) \div (-3)(7)$
f) $56 \div [(8)(7)] \div 49$

4. Evaluate.

a) $(-3)^2 - (-2)^2$
b) $(-5)^2 - (-7) + (-12)$
c) $-4 + 20 \div (-4)$
d) $-3(-4) + 8^2$
e) $(-16) - [(-8) \div 2]$
f) $8 \div (-4) + 4 \div (-2)^2$

5. Evaluate.

a) $\dfrac{-12 - 3}{-3 - 2}$

b) $\dfrac{-18 + 6}{(-3)(-4)}$

c) $\dfrac{(-16 + 4) \div 2}{8 \div (-8) + 4}$

d) $\dfrac{-5 + (-3)(-6)}{(-2)^2 + (-3)^2}$

A–2 Operations with Rational Numbers

Set of rational numbers $\mathbf{Q} = \left\{\frac{a}{b} \middle| a, b \in \mathbf{I}, b \neq 0\right\}$

Addition and Subtraction
To add or subtract rational numbers, you need to find a common denominator.

Division
To divide by a rational number, multiply by the reciprocal.

$$\begin{aligned}\frac{a}{b} \div \frac{c}{d} &= \frac{a}{b} \times \frac{d}{c}\\ &= \frac{ad}{bc}\end{aligned}$$

Multiplication
$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, but first reduce to lowest terms where possible.

More Than One Operation
Follow the order of operations.

EXAMPLE 1

Simplify $\frac{-2}{5} + \frac{3}{-2} - \frac{3}{10}$.

Solution

$$\begin{aligned}\frac{-2}{5} + \frac{3}{-2} - \frac{3}{10} &= \frac{-4}{10} + \frac{-15}{10} - \frac{3}{10}\\ &= \frac{-4 - 15 - 3}{10}\\ &= \frac{-22}{10}\\ &= -\frac{11}{5} \text{ or } -2\frac{1}{5}\end{aligned}$$

EXAMPLE 2

Simplify $\frac{3}{4} \times \frac{-4}{5} \div \frac{-3}{7}$.

Solution

$$\begin{aligned}\frac{3}{4} \times \frac{-4}{5} \div \frac{-3}{7} &= \frac{3}{4} \times \frac{-4}{5} \times \frac{7}{-3}\\ &= \frac{{}^{1}\cancel{3}}{{}_{1}\cancel{4}} \times \frac{\cancel{-4}^{-1}}{5} \times \frac{7}{\cancel{-3}_{-1}}\\ &= \frac{7}{5} \text{ or } 1\frac{2}{5}\end{aligned}$$

Appendix A

Practising

1. Evaluate.

a) $\frac{1}{4}+\frac{-3}{4}$

b) $\frac{1}{2}-\frac{-2}{3}$

c) $\frac{-1}{4}-1\frac{1}{3}$

d) $-8\frac{1}{4}-\frac{-1}{-3}$

e) $\frac{-3}{5}+\frac{-3}{4}-\frac{7}{10}$

f) $\frac{2}{3}-\frac{-1}{2}-\frac{1}{6}$

2. Evaluate.

a) $\frac{4}{5}\times\frac{-20}{25}$

b) $\frac{3}{-2}\times\frac{6}{5}$

c) $\left(\frac{-1}{3}\right)\left(\frac{2}{-5}\right)$

d) $\left(\frac{9}{4}\right)\left(\frac{-2}{-3}\right)$

e) $\left(-1\frac{1}{10}\right)\left(3\frac{1}{11}\right)$

f) $-4\frac{1}{6}\times\left(-7\frac{3}{4}\right)$

3. Evaluate.

a) $\frac{-4}{3}\div\frac{2}{-3}$

b) $-7\frac{1}{8}\div\frac{3}{2}$

c) $\frac{-2}{3}\div\frac{-3}{8}$

d) $\frac{-3}{-2}\div\left(\frac{-1}{3}\right)$

e) $-6\div\left(\frac{-4}{5}\right)$

f) $\left(-2\frac{1}{3}\right)\div\left(-3\frac{1}{2}\right)$

4. Simplify.

a) $\frac{-2}{5}-\left(\frac{-1}{10}+\frac{1}{-2}\right)$

b) $\frac{-3}{5}\left(\frac{-3}{4}-\frac{-1}{4}\right)$

c) $\left(\frac{3}{5}\right)\left(\frac{1}{-6}\right)\left(\frac{-2}{3}\right)$

d) $\left(\frac{-2}{3}\right)^2\left(\frac{1}{-2}\right)^3$

e) $\left(\frac{-2}{5}+\frac{1}{-2}\right)\div\left(\frac{5}{-8}-\frac{-1}{2}\right)$

f) $\dfrac{\frac{-4}{5}-\frac{-3}{5}}{\frac{1}{3}-\frac{-1}{5}}$

A–3 Exponent Laws

3^4 and a^n are called powers

exponent

base

4 factors of 3	n factors of a
$3^4 = (3)(3)(3)(3)$	$a^n = (a)(a)(a)\ldots(a)$

Operations with powers follow a set of procedures or rules.

Rule	Description	Algebraic Description	Example
Zero as an Exponent	When an exponent is zero, the value of the power is 1.	$a^0 = 1$	$120^0 = 1$
Negative Exponents	A negative exponent is the reciprocal of the power with a positive exponent.	$a^{-n} = \frac{1}{a^n}$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
Multiplication	When the bases are the same, keep the base the same and add exponents.	$(a^m)(a^n) = a^{m+n}$	$(5^4)(5^{-3}) = 5^{4+(-3)} = 5^{4-3} = 5^1 = 5$
Division	When the bases are the same, keep the base the same and subtract exponents.	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{4^6}{4^{-2}} = 4^{6-(-2)} = 4^{6+2} = 4^8$
Power of a Power	Keep the base and multiply the exponents.	$(a^m)^n = a^{mn}$	$(3^2)^4 = 3^{(2)(4)} = 3^8$

EXAMPLE

Simplify and evaluate.

$3(3^7) \div (3^3)^2$

Solution

$$\begin{aligned}3(3^7) \div (3^3)^2 &= 3^{1+7} \div 3^{3\times 2}\\ &= 3^8 \div 3^6\\ &= 3^{8-6}\\ &= 3^2\\ &= 9\end{aligned}$$

Practising

1. Evaluate to three decimal places where necessary.

a) 4^2
b) 5^0
c) 3^2
d) -3^2
e) $(-5)^3$
f) $\left(\frac{1}{2}\right)^3$

2. Evaluate.

a) $3^0 + 5^0$
b) $2^2 + 3^3$
c) $5^2 - 4^2$
d) $\left(\frac{1}{2}\right)^3\left(\frac{2}{3}\right)^2$
e) $-2^5 + 2^4$
f) $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2$

3. Evaluate to an exact answer.

a) $\frac{9^8}{9^7}$
b) $\frac{2(5^5)}{5^3}$
c) $(4^5)(4^2)^3$
d) $\frac{(3^2)(3^3)}{(3^4)^2}$

4. Simplify.

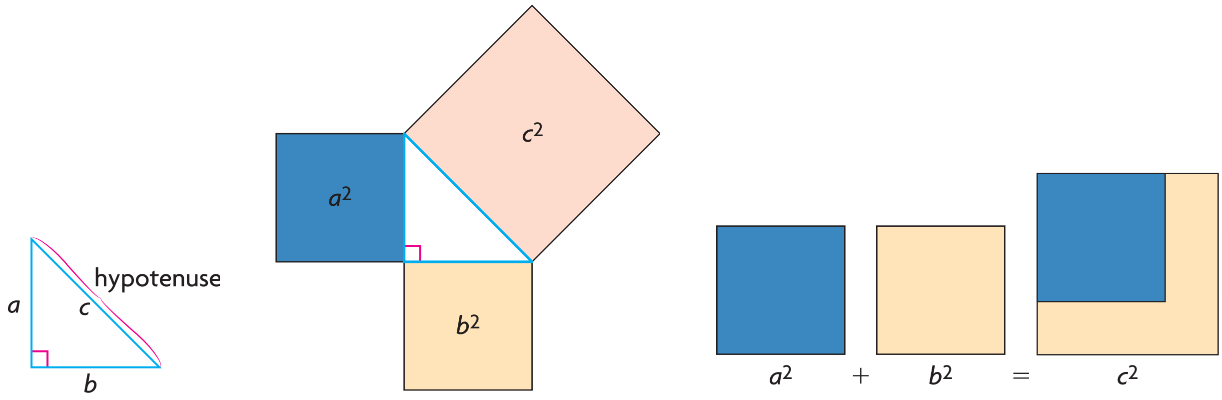
a) $(x)^5(x)^3$
b) $(m)^2(m)^4(m)^3$
c) $(y)^5(y)^2$
d) $(a^b)^c$
e) $\frac{(x^5)(x^3)}{x^2}$
f) $\left(\frac{x^4}{y^3}\right)^3$

5. Simplify.

a) $(x^2y^4)(x^3y^2)$
b) $(-2m^3)^2(3m^2)^3$
c) $\frac{(5x^2)^2}{(5x^2)^0}$
d) $(4u^3v^2)^2 \div (-2u^2v^3)^2$

A–4 The Pythagorean Theorem

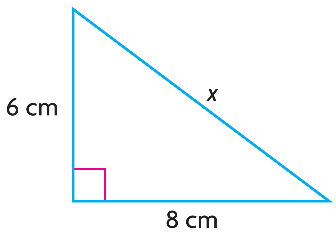
The three sides of a right triangle are related to each other in a unique way. Every right triangle has a longest side, called the **hypotenuse**, which is always opposite the right angle. One of the important relationships in mathematics is known as the **Pythagorean theorem**. It states that the area of the square of the hypotenuse is equal to the sum of the areas of the squares of the other two sides.



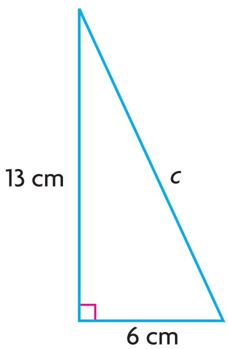
Practising

1. For each right triangle, write the equation for the Pythagorean theorem.

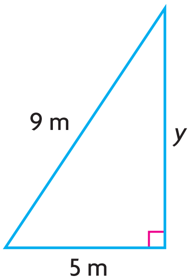
a)



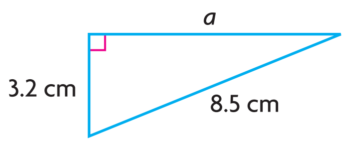
b)



c)



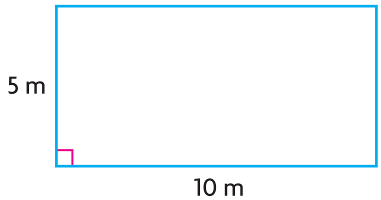
d)

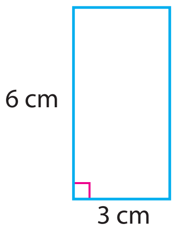


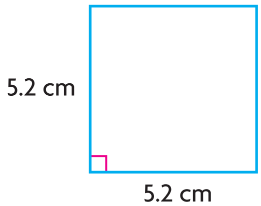
2. Calculate the length of the unknown side of each triangle in question 1. Round all answers to one decimal place.

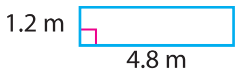
3. Find the value of each unknown measure to the nearest hundredth.
   a) $a^2 = 5^2 + 13^2$
   b) $10^2 = 8^2 + m^2$
   c) $26^2 = b^2 + 12^2$
   d) $2.3^2 + 4.7^2 = c^2$

4. Determine the length of the diagonals of each rectangle to the nearest tenth.

a)


b)


c)


d)


5. An isosceles triangle has a hypotenuse 15 cm long. Determine the length of the two equal sides.

6. An apartment building casts a shadow. From the tip of the shadow to the top of the building is 100 m. The tip of the shadow is 72 m from the base of the building. How tall is the building?

A–5 Graphing Linear Relationships

The graph of a linear relationship ($Ax + By + C = 0$) is a straight line. The graph can be drawn if at least two ordered pairs of the relationship are known. This information can be determined in several different ways.

EXAMPLE 1 Table of Values

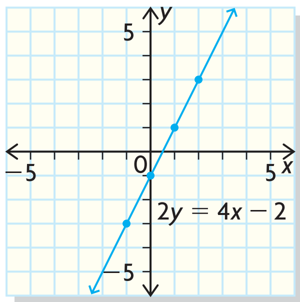
Sketch the graph of $2y = 4x - 2$.

Solution

A table of values can be created. Express the equation in the form $y = mx + b$.

$$\frac{2y}{2} = \frac{4x - 2}{2}$$
$$y = 2x - 1$$

x	y
-1	$2(-1) - 1 = -3$
0	$2(0) - 1 = -1$
1	$2(1) - 1 = 1$
2	$2(2) - 1 = 3$



EXAMPLE 2 Using Intercepts

Sketch the graph of $2x + 4y = 8$.

Solution

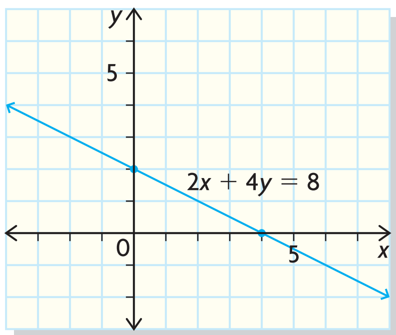
The intercepts of the line can be found.
For the x-intercept, let $y = 0$.

$$2x + 4(0) = 8$$
$$2x = 8$$
$$x = 4$$

x	y
4	0
0	2

For the y-intercept, let $x = 0$.

$$2(0) + 4y = 8$$
$$4y = 8$$
$$y = 2$$

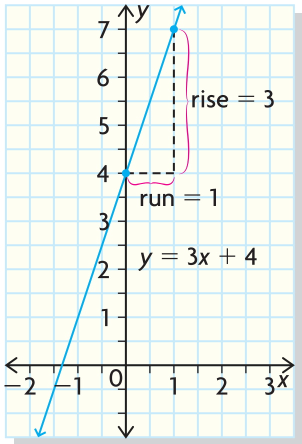


EXAMPLE 3 Using the Slope and y-intercept

Sketch the graph of $y = 3x + 4$.

Solution

When the equation is in the form $y = mx + b$, the slope, m, and y-intercept, b, can be determined. For $y = 3x + 4$, the line has a slope of 3 and a y-intercept of 4.



Practising

1. Express each equation in the form $y = mx + b$.
   a) $3y = 6x + 9$
   b) $2x - 4y = 8$
   c) $3x + 6y - 12 = 0$
   d) $5x = y - 9$

2. Graph each equation, using a table of values where $x \in \{-2, -1, 0, 1, 2\}$.
   a) $y = 3x - 1$
   b) $y = \frac{1}{2}x + 4$
   c) $2x + 3y = 6$
   d) $y = 4$

3. Determine the x- and y-intercepts of each equation.
   a) $x + y = 10$
   b) $2x + 4y = 16$
   c) $50 - 10x - y = 0$
   d) $\frac{x}{2} + \frac{y}{4} = 1$

4. Graph each equation by determining the intercepts.
   a) $x + y = 4$
   b) $x - y = 3$
   c) $2x + 5y = 10$
   d) $3x - 4y = 12$

5. Graph each equation, using the slope and y-intercept.
   a) $y = 2x + 3$
   b) $y = \frac{2}{3}x + 1$
   c) $y = -\frac{3}{4}x - 2$
   d) $2y = x + 6$

6. Graph each equation. Use the most suitable method.
   a) $y = 5x + 2$
   b) $3x - y = 6$
   c) $y = -\frac{2}{3}x + 4$
   d) $4x = 20 - 5y$

A–6 Solving Linear Systems

Many kinds of situations can be modelled with linear equations. When two or more linear equations are used to model a problem, they are called a linear system of equations. Point $P(x, y)$ is the intersection point of the linear equations in the system. Point P is called the solution of the linear system and satisfies all equations in the system.

Solving a Linear System Graphically

Linear systems can be solved graphically, although this method does not always yield an exact solution.

EXAMPLE 1

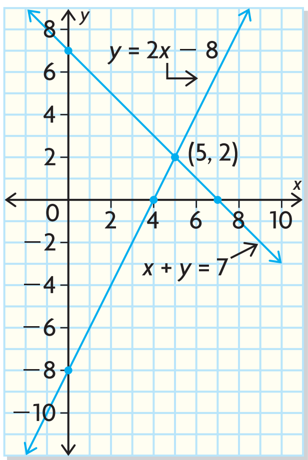
Solve the system graphically.

$y = 2x + 8$ ①

$x + y = 7$ ②

Solution

Draw both graphs on the same axes and locate the point of intersection. Point (5, 2) appears to be the point of intersection. Verify this result algebraically by substituting (5, 2) into equations ① and ②.



In equation ①,

L.S.	R.S.
$y = 2$	$2x - 8$
	$= 2(5) - 8$
	$= 2$

Therefore, L.S. = R.S.

In equation ②,

L.S.	R.S.
$x + y$	7
$= 5 + 2$	
$= 7$	

Therefore, L.S. = R.S.

Solving a System of Linear Equations by Substitution

Linear systems can also be solved by using algebra. Algebraic methods always yield exact solutions. One such method is called substitution.

EXAMPLE 2

Solve the system of linear equations by substitution.

$3x + 2y + 24 = 0 \quad ①$

$5y + 2x = -38 \quad ②$

Solution

Solve for y in equation ①. $$\begin{aligned} 3x + 2y + 24 &= 0 \quad ① \\ 2y &= -24 - 3x \\ y &= -12 - \frac{3}{2}x \end{aligned}$$	Choose one of the equations and isolate one of its variables by expressing that variable in terms of the other variable.
Substitute $y = -12 - \frac{3}{2}x$ into equation ②. $$\begin{aligned} 5y + 2x &= -38 \quad ② \\ 5\left(-12 - \frac{3}{2}x\right) + 2x &= -38 \\ -60 - \frac{15}{2}x + 2x &= -38 \\ \frac{-15x + 4x}{2} &= -38 + 60 \\ \frac{-11x}{2} &= 22 \\ -11x &= 44 \\ x &= -4 \end{aligned}$$	Substitute the expression that you determined for the corresponding variable in the other equation.
Substitute $x = -4$ into equation ①. $$\begin{aligned} 3x + 2y + 24 &= 0 \quad ① \\ 3(-4) + 2y + 24 &= 0 \\ -12 + 2y + 24 &= 0 \\ 2y + 12 &= 0 \\ 2y &= -12 \\ y &= -6 \end{aligned}$$	Determine the other value by substituting the solved value into equation ① or ②.

The solution of the system is $(-4, -6)$.

Practising

1. Determine which ordered pair satisfies both equations.
   a) $x + y = 5$ $\quad$ (4, 1), (2, 3),
      $x = y + 1$ $\quad$ (3, 2), (5, 4)
   b) $x + y = -5$ $\quad$ (3, −6), (10, −5),
      $y = -2x$ $\quad$ (5, −10), (−3, −2)

2. Solve the system by drawing the graph.
   a) $3x + 4y = 12$
      $2x + 3y = 9$
   b) $x + y = -4$
      $2x - y = 4$
   c) $x - 3y + 1 = 0$
      $2x + y - 4 = 0$
   d) $x = 1 - 2y$
      $y = 2x + 3$

3. Using substitution to determine the coordinates of the point of intersection.
   a) $3p + 2q - 1 = 0$
      $p = q + 2$
   b) $2m - n = 3$
      $m + 2n = 24$
   c) $2x + 5y + 18 = 0$
      $x + 2y + 6 = 0$
   d) $6g - 3h = 9$
      $4g = 5 + 3h$
   e) $10x + 15y = 30$
      $15x - 5y = -25$
   f) $13a - 7b = -11$
      $a + 5b = 13$

A–7 Evaluating Algebraic Expressions and Formulas

Algebraic expressions and formulas are evaluated by substituting the given numbers for the variables. Then follow the order of operations to calculate the answer.

EXAMPLE 1

Find the value of $2x^2 - y$ if $x = -2$ and $y = 3$.

Solution

$$\begin{aligned} 2x^2 - y &= 2(-2)^2 - 3 \\ &= 2(4) - 3 \\ &= 8 - 3 \\ &= 5 \end{aligned}$$

EXAMPLE 2

The formula for finding the volume of a cylinder is $V = \pi r^2 h$. Find the volume of a cylinder with a radius of 2.5 cm and a height of 7.5 cm.

Solution

$$\begin{aligned} V &= \pi r^2 h \\ &\doteq (3.14)(2.5)^2(7.5) \\ &= (3.14)(6.25)(7.5) \\ &\doteq 147 \text{ cm}^3 \end{aligned}$$

Practising

1. Find the value of each expression for $x = -5$ and $y = -4$.

a) $-4x - 2y$

b) $-3x - 2y^2$

c) $(3x - 4y)^2$

d) $\left(\frac{x}{y}\right) - \left(\frac{y}{x}\right)$

2. If $x = -\frac{1}{2}$ and $y = \frac{2}{3}$, find the value of each expression.

a) $x + y$

b) $x + 2y$

c) $3x - 2y$

d) $\frac{1}{2}x - \frac{1}{2}y$

3. a) The formula for the area of a triangle is $A = \frac{1}{2}bh$. Find the area of a triangle when $b = 13.5$ cm and $h = 12.2$ cm.

b) The area of a circle is found using the formula $A = \pi r^2$. Find the area of a circle with a radius of 4.3 m.

c) The hypotenuse of a right triangle, c, is found using the formula $c = \sqrt{a^2 + b^2}$. Find the length of the hypotenuse when $a = 6$ m and $b = 8$ m.

d) A sphere's volume is calculated using the formula $V = \frac{4}{3}\pi r^3$. Determine the volume of a sphere with a radius of 10.5 cm.

A–8 Expanding and Simplifying Algebraic Expressions

Type	Description	Example
Collecting Like Terms $2a + 3a = 5a$	Add or subtract the coefficients of the terms that have the same variables and exponents.	$3a - 2b - 5a + b$ $= 3a - 5a - 2b + b$ $= -2a - b$
Distributive Property $a(b + c) = ab + ac$	Multiply each term of the binomial by the monomial.	$-4a(2a - 3b)$ $= -8a^2 + 12ab$
Product of Two Binomials $(a + b)(c + d)$ $= ac + ad + bc + bd$	Multiply the first term of the first binomial by the second binomial, and then multiply the second term of the first binomial by the second binomial. Collect like terms if possible.	$(2x^2 - 3)(5x^2 + 2)$ $= 10x^4 + 4x^2 - 15x^2 - 6$ $= 10x^4 - 11x^2 - 6$

Practising

1. Simplify.
- **a)** $3x + 2y - 5x - 7y$
- **b)** $5x^2 - 4x^3 + 6x^2$
- **c)** $(4x - 5y) - (6x + 3y) - (7x + 2y)$
- **d)** $m^2n + p - (2p - 3m^2n)$

2. Expand.
- **a)** $3(2x + 5y - 2)$
- **b)** $5x(x^2 - x + y)$
- **c)** $m^2(3m^2 - 2n)$
- **d)** $x^5y^3(4x^2y^4 - 2xy^5)$

3. Expand and simplify.
- **a)** $3x(x + 2) + 5x(x - 2)$
- **b)** $-7h(2h + 5) - 4h(5h - 3)$
- **c)** $2m^2n(m^3 - n) - 5m^2n(3m^3 + 4n)$
- **d)** $-3xy^3(5x + 2y + 1) + 2xy^3(-3y - 2 + 7x)$

4. Expand and simplify.
- **a)** $(3x - 2)(4x + 5)$
- **b)** $(7 - 3y)(2 + 4y)$
- **c)** $(5x - 7y)(4x + y)$
- **d)** $(3x^3 - 4y^2)(5x^3 + 2y^2)$

A–9 Factoring Algebraic Expressions

Factoring is the opposite of expanding.

$$\xrightarrow{\text{expanding}}$$
$$2x(3x - 5) = 6x^2 - 10x$$
$$\xleftarrow{\text{factoring}}$$

Type	Example	Comment
Common Factoring $ab + ac = a(b + c)$ Factor out the largest common factor of each term.	$10x^4 - 8x^3 + 6x^5$ $= 2x^3(5x - 4 + 3x^2)$	Each term has a common factor of $2x^3$.
Factoring Trinomials $ax^2 + bx + c$, when $a = 1$ Write as the product of two binomials. Determine two numbers whose sum is b and whose product is c.	$x^2 + 4x - 21$ $= (x + 7)(x - 3)$	$(-21) = 7(-3)$ and $4 = 7 + (-3)$
Factoring Trinomials $ax^2 + bx + c$, when $a \neq 1$ Look for a common factor. If none exists, use decomposition and write as the product of two binomials. Check by expanding and simplifying.	$3x^2 + 4x - 4$ $= 3x^2 - 2x + 6x - 4$ $= (3x^2 - 2x) + (6x - 4)$ $= x(3x - 2) + 2(3x - 2)$ $= (3x - 2)(x + 2)$ Check: $(3x)(x) + (3x)(2)$ $+ (-2)(x) + (-2)(2)$ $= 3x^2 + 6x - 2x - 4$ $= 3x^2 + 4x - 4$	Multiply: $3(-4) = -12$ Find two numbers whose product is -12 and whose sum is 4. In this case, the numbers are 6 and -2. Using these numbers, decompose the x-term. Group the terms and factor out the common factors.

Practising

1. Factor each expression.

a) $4 - 8x$

b) $6x^2 - 5x$

c) $3m^2n^3 - 9m^3n^4$

d) $28x^2 - 14xy$

2. Factor each expression.

a) $x^2 - x - 6$

b) $x^2 + 7x + 10$

c) $x^2 - 9x + 20$

d) $3y^2 + 18y + 24$

3. Factor.

a) $6y^2 - y - 2$

b) $12x^2 + x - 1$

c) $5a^2 + 7a - 6$

d) $12x^2 - 18x - 12$

A–10 Solving Quadratic Equations Algebraically

To solve a quadratic equation, first rewrite it in the form $ax^2 + bx + c = 0$. Then factor the left side, if possible, or use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A quadratic equation can have no roots, one root, or two roots. Not all quadratic equations can be solved by factoring.

EXAMPLE 1

Solve $x^2 + 3x = 10$.

Solution

$$\begin{aligned} x^2 + 3x &= 10 \\ x^2 + 3x - 10 &= 0 \\ (x + 5)(x - 2) &= 0 \end{aligned}$$

Then $x + 5 = 0$ or $x - 2 = 0$

$x = -5$ or $x = 2$

EXAMPLE 2

Solve $-x = 3 - 2x^2$.

Solution

$$-x = 3 - 2x^2$$

$$2x^2 - x - 3 = 0$$

$a = 2$, $b = -1$, and $c = -3$

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-3)}}{2(2)} \\ &= \frac{1 \pm \sqrt{1 + 24}}{4} \\ &= \frac{1 \pm \sqrt{25}}{4} \\ &= \frac{1 \pm 5}{4} \end{aligned}$$

$x = \frac{1 + 5}{4}$ or $x = \frac{1 - 5}{4}$

$x = \frac{6}{4} = \frac{3}{2}$ or $x = \frac{-4}{4} = -1$

Practising

1. Solve.
   a) $(x - 3)(x - 2) = 0$
   b) $(2x - 5)(3x - 1) = 0$
   c) $(m - 4)(m - 3) = 0$
   d) $(3 - 2x)(4 - 3x) = 0$
   e) $(2y + 5)(3y - 7) = 0$
   f) $(5n - 3)(4 - 3n) = 0$

2. Determine the roots.
   a) $x^2 - x - 2 = 0$
   b) $x^2 + x - 20 = 0$
   c) $m^2 + 2m - 15 = 0$
   d) $6x^2 - x - 2 = 0$
   e) $6t^2 + 5t - 4 = 0$
   f) $2x^2 + 4x - 30 = 0$

3. Solve.
   a) $4x^2 = 8x - 1$
   b) $4x^2 = 9$
   c) $6x^2 - x = 1$
   d) $5x^2 - 6 = -7x$
   e) $3x^2 + 5x - 1 = 2x^2 + 6x + 5$
   f) $7x^2 + 2(2x + 3) = 2(3x^2 - 4) + 13x$

4. A model rocket is shot straight into the air. Its height in metres at t seconds is given by $h = -4.9t^2 + 29.4t$. When does the rocket reach the ground?

5. The population of a city is modelled by $P = 0.5t^2 + 10t + 200$, where P is the population in thousands and t is the time in years, with $t = 0$ corresponding to the year 2000. When is the population 350 000?

A–11 Creating Scatter Plots and Lines or Curves of Good Fit

A **scatter plot** is a graph that shows the relationship between two sets of numeric data. The points in a scatter plot often show a general pattern, or **trend**. A line that approximates a trend for the data in a scatter plot is called a **line of best fit**. A line of best fit passes through as many points as possible, with the remaining points grouped equally above and below the line.

Data that have a **positive correlation** have a pattern that slopes up and to the right. Data that have a **negative correlation** have a pattern that slopes down and to the right. If the points nearly form a line, then the correlation is strong. If the points are dispersed, but still form some linear pattern, then the correlation is weak.

EXAMPLE 1

a) Make a scatter plot of the data and describe the kind of correlation the scatter plot shows.

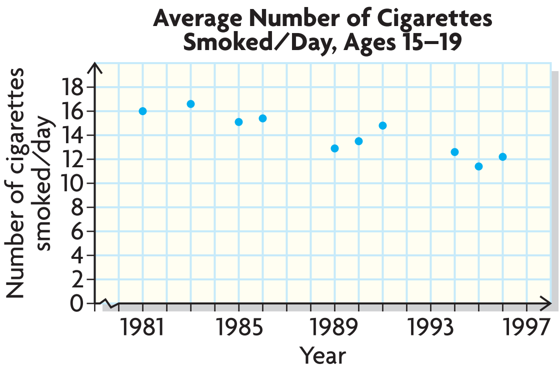
b) Draw the line of best fit.

Long-Term Trends in Average Number of Cigarettes Smoked per Day by Smokers Aged 15–19

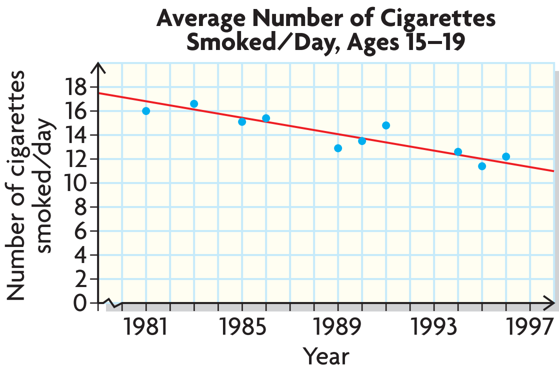
Year	1981	1983	1985	1986	1989	1990	1991	1994	1995	1996
Number Per Day	16.0	16.6	15.1	15.4	12.9	13.5	14.8	12.6	11.4	12.2

Solution

a)



b)



The scatter plot shows a negative correlation.

EXAMPLE 2

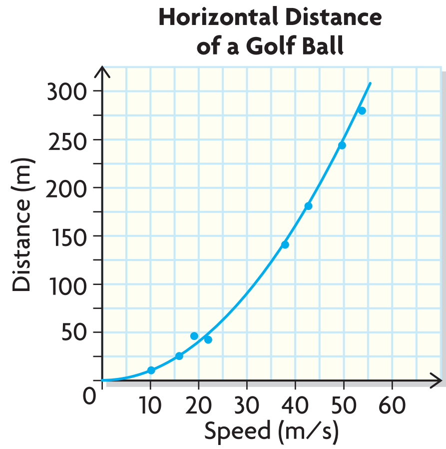
A professional golfer is taking part in a scientific investigation. Each time she drives the ball from the tee, a motion sensor records the initial speed of the ball. The final horizontal distance of the ball from the tee is also recorded. Here are the results:

Speed (m/s)	10	16	19	22	38	43	50	54
Distance (m)	10	25	47	43	142	182	244	280

Draw the line or curve of good fit.

Solution

The scatter plot shows that a line of best fit does not fit the data as well as an upward-sloping curve does. Therefore, sketch a curve of good fit.



Practising

1. For each set of data,

i) create a scatter plot and draw the line of best fit

ii) describe the type of correlation the trend in the data displays

a) **Population of the Hamilton–Wentworth, Ontario, Region**

Year	1966	1976	1986	1996	1998
Population	449 116	529 371	557 029	624 360	618 658

b) **Percent of Canadians with Less than Grade 9 Education**

Year	1976	1981	1986	1991	1996
Percent of the Population	25.4	20.7	17.7	14.3	12.4

2. In an experiment for a physics project, marbles are rolled up a ramp. A motion sensor detects the speed of the marble at the start of the ramp, and the final height of the marble is recorded. However, the motion sensor may not be measuring accurately. Here are the data:

Speed (m/s)	1.2	2.1	2.8	3.3	4.0	4.5	5.1	5.6
Final Height (m)	0.07	0.21	0.38	0.49	0.86	1.02	1.36	1.51

a) Draw a curve of good fit for the data.

b) How consistent are the motion sensor's measurements? Explain.

A–12 Using Properties of Quadratic Relations to Sketch Their Graphs

If the algebraic expression of a relation can be identified as quadratic, its graph can be sketched without making a table or using graphing technology.

Quadratic Relations in Standard Form

The standard form of a quadratic relation is $y = ax^2 + bx + c$. The graph is a parabola. If $a > 0$, the parabola opens upward; if $a < 0$, the parabola opens downward.

EXAMPLE 1 Graphing Using Symmetry

Sketch the graph of $y = -3x^2 - 2x + 7$.

Solution

$y = -3x^2 - 2x + 7$ $\quad$ $a = -3$, so the parabola opens downward.

$y = x(-3x - 2) - 7$ $\quad$ Factor partially.

Let $x = 0$ or $-3x - 2 = 0$ to find two points on the curve.

When $x = 0$, then $y = 7$.

When $-3x - 2 = 0$, then $x = -\frac{2}{3}$ and $y = 7$.

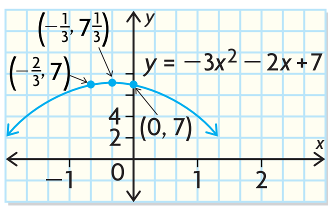
The axis of symmetry is halfway between $(0, 7)$ and $\left(-\frac{2}{3}, 7\right)$.

Therefore, $x = \dfrac{0 + \left(-\frac{2}{3}\right)}{2} = -\frac{1}{3}$.

To find the vertex, substitute $x = -\frac{1}{3}$ into $y = -3x^2 - 2x + 7$.

$$y = -3\left(-\frac{1}{3}\right)^2 - 2\left(-\frac{1}{3}\right) + 7 = 7\frac{1}{3}$$

The curve opens downward and the vertex is $\left(-\frac{1}{3}, 7\frac{1}{3}\right)$.



Quadratic Relations in Vertex Form

The vertex form of a quadratic relation is $y = a(x - h)^2 + k$, where (h, k) is the vertex. These are also the coordinates of the maximum point when $a < 0$ and of the minimum point when $a > 0$.

EXAMPLE 2 **Graphing Using the Vertex Form**

Sketch the graph of $y = 2(x + 2)^2 + 3$.

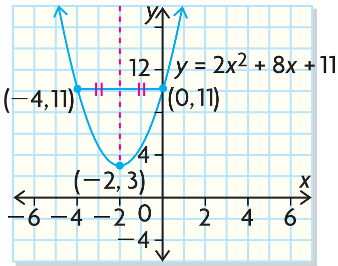
Solution

The equation is in vertex form, and we see that the vertex is $(-2, 3)$. Determine one point on the curve and use symmetry to find a second point.

When $x = 0$,

$$\begin{aligned} y &= 2(0 + 2)^2 + 3 \\ &= 11 \end{aligned}$$

So, $(0, 11)$ is a point on the curve. Another point, $(-4, 11)$, is symmetric to the axis of symmetry. Now sketch the graph.



In this case, $(-2, 3)$ is the minimum point. The relation has a minimum value of 3 when $x = -2$.

Practising

1. Sketch the graphs, using partial factoring.
   a) $y = 2x^2 - 6x + 5$
   b) $y = -3x^2 + 9x - 2$
   c) $y = 5x^2 - 3 + 5x$
   d) $y = 3 + 4x - 2x^2$
2. Sketch the graphs, using the zeros of the curve.
   a) $y = x^2 + 4x - 12$
   b) $y = x^2 - 7x + 10$
   c) $y = 2x^2 - 5x - 3$
   d) $y = 6x^2 - 13x - 5$
3. Sketch the graphs.
   a) $y = (x - 2)^2 + 3$
   b) $y = (x + 4)^2 - 10$
   c) $y = 2(x - 1)^2 + 3$
   d) $y = -3(x + 1)^2 - 4$

A–13 Completing the Square to Convert to the Vertex Form of a Parabola

A quadratic relation in standard form, $y = ax^2 + bx + c$, can be rewritten in vertex form as $y = a(x - h)^2 + k$ by creating a perfect square in the original and then factoring the square.

Steps Used to Complete the Square	Example: $y = 2x^2 + 12x - 5$
• Divide out the common constant factor from both the x^2- and x-terms.	$y = 2(x^2 + 6x) - 5$
• Determine the constant that must be added (and subtracted) to create a perfect square. This is half the coefficient of the x-term, squared.	$y = 2(x^2 + 6x + 9 - 9) - 5$
• Group the three terms of the perfect square. Multiply the subtracted value and move it outside the bracket.	$y = 2(x^2 + 6x + 9) - 2(9) - 5$
• Factor the perfect square and collect like terms.	$y = 2(x + 3)^2 - 23$

Practising

1. Write each trinomial as a perfect square.

a) $x^2 + 2x + 1$
b) $x^2 + 4x + 4$
c) $x^2 + 6x + 9$
d) $x^2 + 10x + 25$

2. Complete the square, and write in vertex form.

a) $y = x^2 + 2x + 2$
b) $y = x^2 + 4x + 6$
c) $y = x^2 - 12x + 40$
d) $y = x^2 - 18x + 80$

3. Express in vertex form by completing the square. State the equation of the axis of symmetry and the coordinates of the vertex.

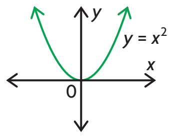
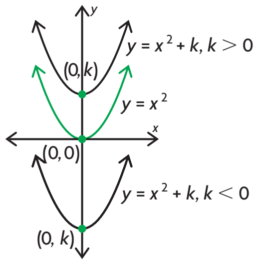
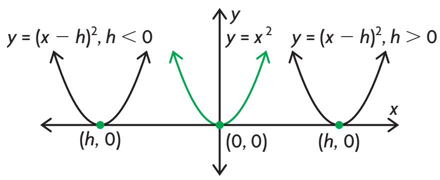
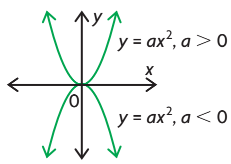
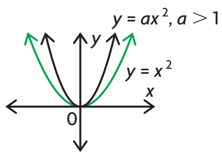
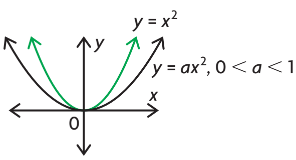
a) $y = 2x^2 - 4x + 7$
b) $y = 5x^2 + 10x + 6$
c) $y = -3x^2 - 12x + 2$
d) $y = -2x^2 + 6x + 2$

4. A baseball is hit from a height of 1 m. Its height in metres, h, after t seconds is $h = -5t^2 + 10t + 1$.

a) What is the maximum height of the ball?
b) When does the ball reach this height?

A–14 Transformations of Quadratic Relations

The graph of any quadratic relation can be created by altering or repositioning the graph of the base curve, $y = x^2$. To do so, write the relation in vertex form, $y = a(x - h)^2 + k$. The base curve, $y = x^2$, is translated vertically or horizontally, stretched or compressed vertically, or reflected about the x-axis, depending on the values of a, h, and k.

Relation	Type of Transformation	Graph	Explanation
$y = x^2$	Base parabola		This is the base curve upon which other transformations are applied.
$y = x^2 + k$	Vertical Translation The curve shifts up if $k > 0$ and down if $k < 0$.		Add k to the y-coordinate of every point on the base curve. The resultant curve is congruent to the base curve.
$y = (x - h)^2$	Horizontal Translation The curve shifts to the right if $h > 0$ and to the left if $h < 0$.		Subtract h from the x-coordinate of every point on the base curve.
$y = ax^2$	Reflection The curve is reflected about the x-axis if $a < 0$.		Multiply the y-coordinate of every point on the base curve by a.
	Vertical Stretch		The curve has a narrow opening if $a > 1$.
	Vertical Compression		The curve has a wide opening if $0 < a < 1$.

EXAMPLE

Use transformations to graph $y = -2(x + 3)^2 - 4$.

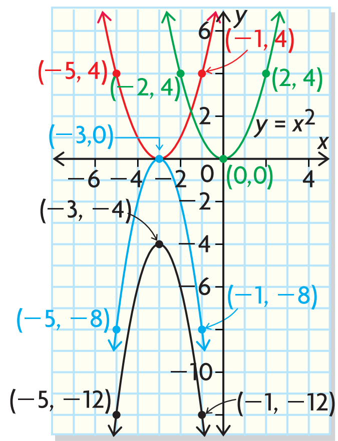
Solution

Step 1: Begin with the graph of the base curve, $y = x^2$ (green). Select three points on the curve to help define its shape. From the form $y = a(x - h)^2 + k$, note that $a = -2$, $h = -3$, and $k = -4$.

Step 2: $h = -3$, so shift the entire parabola 3 units to the left (red).

Step 3: $a = -2$, so multiply every y-coordinate by -2. The parabola is reflected and its opening becomes narrower (blue).

Step 4: $k = -4$, so shift the entire parabola 4 units down. This is the final graph (black).



Practising

In questions 1 and 2, the coordinates of a point on the parabola $y = x^2$ are given. State the new coordinates under each transformation.

1. **a)** (2, 4); shift up 3
   b) (−2, 4); shift down 5
   c) (−1, 1); shift left 4
   d) (1, 1); shift right 6
   e) (3, 9); vertical stretch of $-\frac{1}{3}$
   f) (2, 4); vertical stretch of 2

2. **a)** (−2, 4); shift left 2 and up 5
   b) (−1, 1); shift right 4, vertical stretch 3, and reflect about x-axis
   c) (3, 9); vertical compression by 3, shift left 5, and shift down 2
   d) (0, 0); shift left 5, vertical stretch 3, shift up 4, and reflect about x-axis

3. Points (−2, 4), (0, 0), and (2, 4) are on the parabola $y = x^2$. Use your knowledge of transformations to determine the equation of the parabola using these coordinates.
   a) (−2, 6), (0, 2), (2, 6)
   b) (−2, 12), (0, 0), (2, 12)
   c) (2, 4), (4, 0), (6, 4)
   d) (−2, −6), (0, −2), (2, −6)

In questions 4 and 5, sketch each graph, using transformations on $y = x^2$.

4. **a)** $y = +x^2$
   b) $y = -x^2$
   c) $y = (x - 3)^2$
   d) $y = (x + 4)^2$
   e) $y = 2x^2$
   f) $y = \frac{1}{2}x^2$

5. **a)** $y = 2(x - 1)^2$
   b) $y = -3(x + 1)^2$
   c) $y = \frac{1}{2}(x - 2)^2$
   d) $y = -\frac{1}{3}(x + 2)^2$

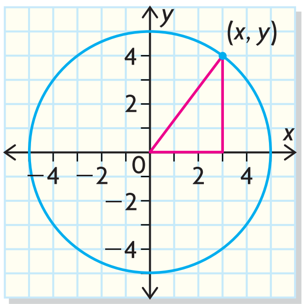
A–15 Equations of Circles Centred at the Origin

A circle can be described by an equation. If the circle is centred at the origin, the equation has a simple form.

Applying the Pythagorean theorem, the coordinates x and y satisfy

$$x^2 + y^2 = 5^2$$

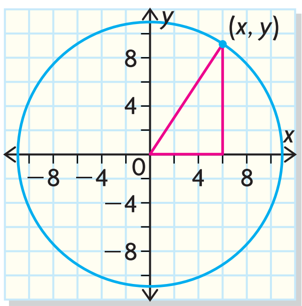
$$x^2 + y^2 = 25$$



EXAMPLE 1

Write the equation of a circle centred at the origin, with radius 11.

Solution



Applying the Pythagorean theorem,

$$x^2 + y^2 = 11^2$$

$$x^2 + y^2 = 121$$

EXAMPLE 2

What is the radius of a circle with equation $x^2 + y^2 = 30$? Give your answer to the nearest hundredth.

Solution

If the radius is r, the equation must be $x^2 + y^2 = r^2$. Therefore,

$$r^2 = 30$$

$$r = \sqrt{30}$$

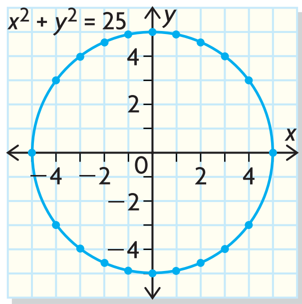
$$r \doteq 5.48$$

Suppose you know the x-coordinate of a point on the circle $x^2 + y^2 = 25$. The possible values of the y-coordinate can be found by substituting for x in the equation. For example, if $x = 3$,

$$\begin{aligned} x^2 + y^2 &= 25 \\ (3)^2 + y^2 &= 25 \\ 9 + y^2 &= 25 \\ y^2 &= 16 \\ y &= \pm\sqrt{16} = \pm 4 \end{aligned}$$

The circle $x^2 + y^2 = 25$ can be plotted from a table of values. Since the radius is $\sqrt{25} = 5$, start at $x = -5$ and go through to $x = 5$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	0	± 3	± 4	± 4.6	± 4.9	± 5	± 4.9	± 4.6	± 4	± 3	0



Notice that there are two y-values for each x-value. The equation of a circle defines a relation that is not a function.

Practising

1. Write the equation of a circle centred at the origin, with each radius.
   - **a)** 3
   - **b)** 7
   - **c)** 8
   - **d)** 1

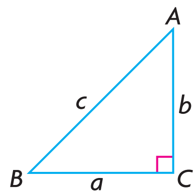
2. What is the radius of a circle with each equation? Round your answer to the nearest hundredth, if necessary.
   - **a)** $x^2 + y^2 = 9$
   - **b)** $x^2 + y^2 = 81$
   - **c)** $x^2 + y^2 = 15$
   - **d)** $x^2 + y^2 = 27$
   - **e)** $x^2 + y^2 = 6.25$
   - **f)** $x^2 + y^2 = 17.64$

3. A point on the circle $x^2 + y^2 = 169$ has an x-coordinate of 12. What are the possible values of the y-coordinate?

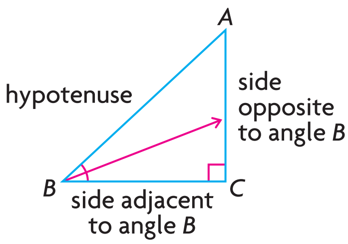
4. Plot the circle with the given equation.
   - **a)** $x^2 + y^2 = 16$
   - **b)** $x^2 + y^2 = 49$
   - **c)** $x^2 + y^2 = 100$
   - **d)** $x^2 + y^2 = 12.25$

A–16 Trigonometry of Right Triangles

By the Pythagorean relationship, $a^2 + b^2 = c^2$ for any right triangle, where c is the length of the hypotenuse and a and b are the lengths of the other two sides.



In any right triangle, there are three primary trigonometric ratios that associate the measure of an angle with the ratio of two sides. For example, for $\angle ABC$, in Figure 1,



For $\angle B$

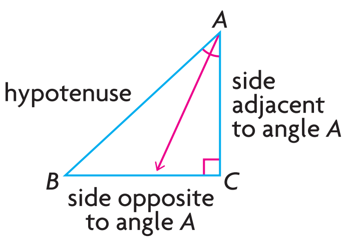
$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos B = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan B = \frac{\text{opposite}}{\text{adjacent}}$$

Figure 1

Similarly, in Figure 2,



For $\angle A$

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

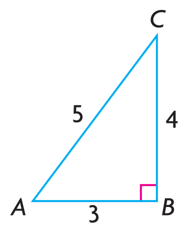
$$\tan A = \frac{\text{opposite}}{\text{adjacent}}$$

Figure 2

Note how the opposite and adjacent sides change in Figures 1 and 2 with angles A and B.

EXAMPLE 1

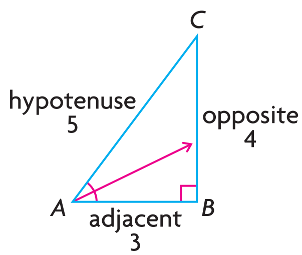
State the primary trigonometric ratios of $\angle A$.



Solution

Sketch the triangle. Then label the opposite side, the adjacent side, and the hypotenuse.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$= \frac{4}{5}$$



$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$= \frac{3}{5}$$

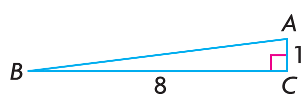
$$\tan A = \frac{\text{opposite}}{\text{adjacent}}$$
$$= \frac{4}{3}$$

EXAMPLE 2

A ramp must have a rise of one unit for every eight units of run. What is the angle of inclination of the ramp?

Solution

The slope of the ramp is $\frac{\text{rise}}{\text{run}} = \frac{1}{8}$. Draw a labelled sketch.



Calculate the measure of $\angle B$ to determine the angle of inclination.

The trigonometric ratio that associates $\angle B$ with the opposite and adjacent sides is the tangent. Therefore,

$$\tan B = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan B = \frac{1}{8}$$

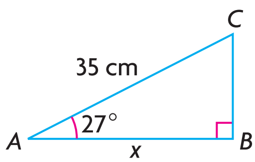
$$B = \tan^{-1}\left(\frac{1}{8}\right)$$

$$B \doteq 7°$$

The angle of inclination is about 7°.

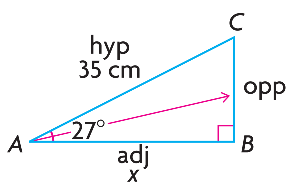
EXAMPLE 3

Determine x to the nearest centimetre.



Solution

Label the sketch. The cosine ratio associates $\angle A$ with the adjacent side and the hypotenuse.



Then,

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 27° = \frac{x}{35}$$

$$x = 35 \cos 27° \doteq 31$$

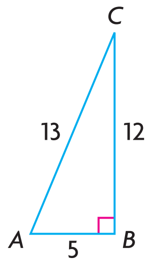
So x is about 31 cm.

Practising

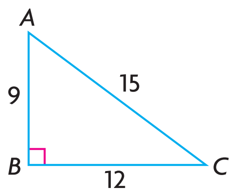
1. A rectangular lot is 15 m by 22 m. How long is the diagonal, to the nearest metre?

2. State the primary trigonometric ratios for $\angle A$.

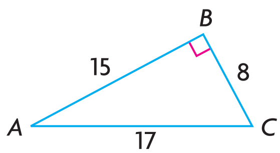
   a)



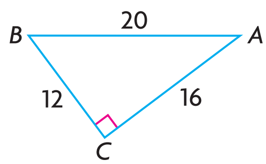
   b)



   c)



   d)



3. Solve for x to one decimal place.

   a) $\sin 39° = \frac{x}{7}$

   b) $\cos 65° = \frac{x}{16}$

   c) $\tan 15° = \frac{x}{22}$

   d) $\tan 49° = \frac{31}{x}$

4. Solve for $\angle A$ to the nearest degree.

   a) $\sin A = \frac{5}{8}$

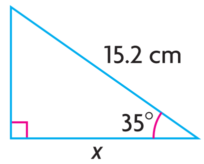
   b) $\cos A = \frac{13}{22}$

   c) $\tan B = \frac{19}{22}$

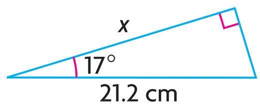
   d) $\cos B = \frac{3}{7}$

5. Determine x to one decimal place.

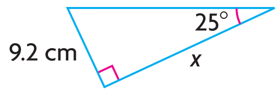
   a)



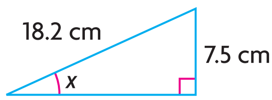
   b)



   c)



   d)



6. In $\triangle ABC$, $\angle B = 90°$ and $AC = 13$ cm. Determine

   a) BC if $\angle C = 17°$

   b) AB if $\angle C = 26°$

   c) $\angle A$ if $BC = 6$ cm

   d) $\angle C$ if $BC = 9$ cm

7. A tree casts a shadow 9.3 m long when the angle of the sun is 43°. How tall is the tree?

8. Janine stands 30.0 m from the base of a communications tower. The angle of elevation from her eyes to the top of the tower is 70°. How high is the tower if her eyes are 1.8 m above the ground?

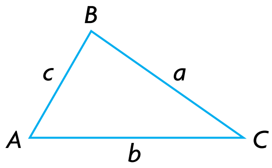
9. A surveillance camera is mounted on the top of a building that is 80 m tall. The angle of elevation from the camera to the top of another building is 42°. The angle of depression from the camera to the same building is 32°. How tall is the other building?

A–17 Trigonometry of Acute Triangles: The Sine Law and the Cosine Law

An acute triangle contains three angles less than 90°.

Sine Law

The sine law states that, for $\triangle ABC$,

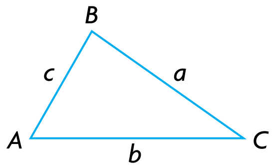


- $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ or
- $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

To use the sine law, two angles and one side (AAS) or two sides and an opposite angle (SSA) must be given.

Cosine Law

The cosine law states that, for $\triangle ABC$,



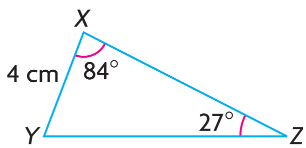
- $c^2 = a^2 + b^2 - 2ab \cos C$ or
- $a^2 = b^2 + c^2 - 2bc \cos A$ or
- $b^2 = a^2 + c^2 - 2ac \cos B$

To use the cosine law, two sides and the contained angle (SAS) or three sides (SSS) must be given.

EXAMPLE 1

Determine the length of XZ to one decimal place.

Solution



In the triangle shown, $\angle X = 84°$, $\angle Z = 27°$, and $z = 4$ cm.

$$\begin{aligned}\angle Y &= 180° - (84° + 27°)\\ &= 180° - 111°\\ &= 69°\end{aligned}$$

This is not a right triangle, so the primary trigonometric ratios do not apply. Two angles and one side are known, so the sine law can be used.

$$\begin{aligned}\frac{\sin Y}{y} &= \frac{\sin Z}{z}\\ \frac{\sin 69°}{y} &= \frac{\sin 27°}{4}\\ y &= \frac{4 \sin 69°}{\sin 27°}\\ &\doteq 8.2\end{aligned}$$

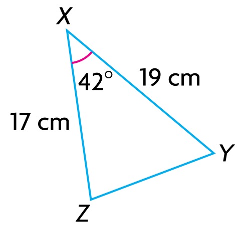
Then XZ is about 8.2 cm.

EXAMPLE 2

Determine the length of ZY to one decimal place.

Solution

In this triangle, $\angle X = 42°$, $y = 17$ cm, and $z = 19$ cm.



There is not enough information to use the primary trigonometric ratios or the sine law. However, two sides and the contained angle are known, so the cosine law can be used.

$$x^2 = y^2 + z^2 - 2yz \cos X$$

$$x^2 = 17^2 + 19^2 - 2(17)(19) \cos 42°$$

$$x = \sqrt{17^2 + 19^2 - 2(17)(19) \cos 42°}$$

$$x \doteq 13$$

Therefore, ZY is about 13 cm.

Practising

1. Solve to one decimal place.

a) $\dfrac{\sin 35°}{c} = \dfrac{\sin 42°}{12}$

b) $\dfrac{15}{\sin 43°} = \dfrac{13}{\sin B}$

c) $19^2 = 15^2 + 13^2 - 2(15)(13)\cos A$

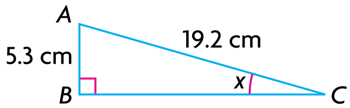
d) $c^2 = 12^2 + 17^2 - 2(12)(17)\cos 47°$

e) $\dfrac{\sin A}{12.3} = \dfrac{\sin 58°}{14.2}$

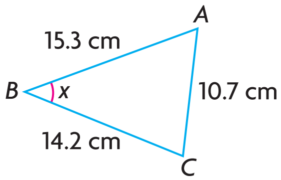
f) $\dfrac{\sin 14°}{3.1} = \dfrac{\sin 27°}{b}$

2. Determine x to one decimal place.

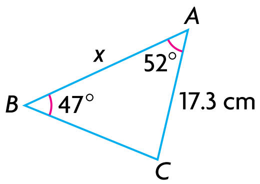
a)



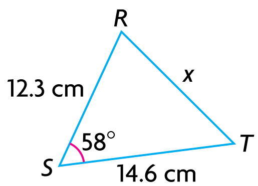
b)



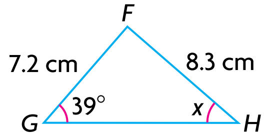
c)



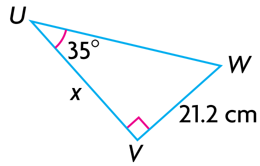
d)



e)



f)



3. Solve each triangle for all missing sides and angles.

a) $\triangle CAT$, with $c = 5.2$ cm, $a = 6.8$ cm, and $\angle T = 59°$.

b) $\triangle ABC$, with $a = 4.3$ cm, $b = 5.2$ cm, and $c = 7.5$.

c) $\triangle DEF$, with $DE = 14.3$ cm, $EF = 17.2$ cm, and $\angle D = 39°$.

4. A swamp separates points L and R. To determine the distance between them, Ciana stands at L and looks toward R. She turns about 45° and walks 52 paces from L to point P. From P, she looks at R and estimates that $\angle LPR$ is about 60°. How many paces is it from L to R?

5. An observation helicopter using a laser device determines that the helicopter is 1800 m from a boat in distress. The helicopter is 1200 m from a rescue boat. The angle formed between the helicopter and the two boats is 35°. How far apart are the boats?

6. Neil designs a cottage that is 15 m wide. The roof rafters are the same length and meet at an angle of 80°. The rafters hang over the supporting wall by 0.5 m. How long are the rafters?

Appendix B

Review of Technical Skills

PART 1 USING THE TI-83 PLUS AND TI-84 GRAPHING CALCULATORS

B–1 Preparing the Calculator

Before you graph any function, be sure to clear any information left on the calculator from the last time it was used. You should always do the following:

1. **Clear all data in the lists.**

   Press [2nd] [+] [4] [ENTER].

2. **Turn off all stat plots.**

   Press [2nd] [Y=] [4] [ENTER].

3. **Clear all equations in the equation editor.**

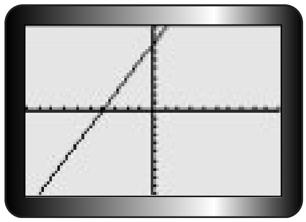
   Press [Y=], then press [CLEAR] for each equation.

4. **Set the window so that the axes range from −10 to 10.**

   Press [ZOOM] [6]. Press [WINDOW] to verify.

B–2 Entering and Graphing Functions

Enter the equation of the function into the equation editor. The calculator will display the graph.

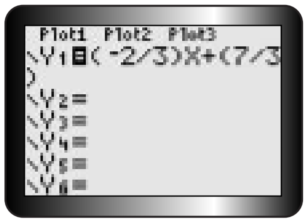


1. **Graph.**

   To graph $y = 2x + 8$, press [Y=] [2] [X, T, Θ, *n*] [+] [8] [GRAPH]. The graph will be displayed as shown.

2. **Enter all linear equations in the form $y = mx + b$.**

   If m or b are fractions, enter them between brackets. For example, enter $2x + 3y = 7$ in the form $y = -\frac{2}{3}x + \frac{7}{3}$, as shown.

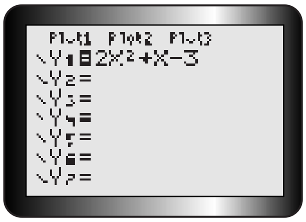


3. **Press [GRAPH] to view the graph.**

4. **Press [TRACE] to find the coordinates of any point on a graph.**

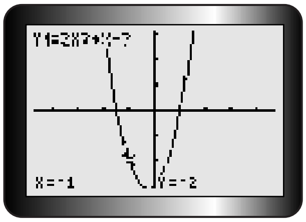
   Use the left and right arrow keys to cursor along the graph.
   Press [ZOOM] [8] [ENTER] [TRACE] to trace using integer intervals. If you are working with several graphs at the same time, use the up and down arrow to scroll between graphs.

B–3 Evaluating a Function



1. **Enter the function into the equation editor.**

   To enter $y = 2x^2 + x - 3$, press [Y=] [2] [X, T, Θ, *n*] [x^2] [+] [X, T, Θ, *n*] [−] [3].



2. **Use the value operation to evaluate the function.**

   To find the value of the function at $x = -1$, press [2nd] [TRACE] [ENTER], enter [(−)] [1] at the cursor, then press [ENTER].

3. **Use function notation and the Y-VARS operation to evaluate the function.**

   There is another way to evaluate the function, say at $x = 37.5$.

   Press [CLEAR], then [VARS], then cursor right to **Y-VARS** and press [ENTER]. Press [1] to select **Y1**. Finally, press [(] [3] [7] [.] [5] [)], then [ENTER].

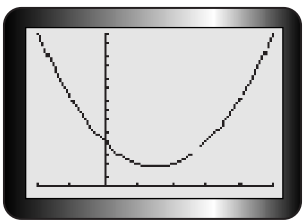
B–4 Changing Window Settings

The window settings can be changed to show a graph for a given domain and range.

1. **Enter the function $y = x^2 - 3x + 4$ in the equation editor.**

2. **Use the WINDOW function to set the domain and range.**

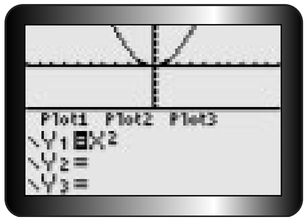
   To display the function over the domain $\{x \mid -2 \le x \le 5\}$ and range $\{y \mid 0 \le y \le 14\}$, press [WINDOW] [(−)] [2] [ENTER], then [5] [ENTER], then [1] [ENTER], then [0] [ENTER], then [1] [4] [ENTER], then [1] [ENTER], and [1] [ENTER].



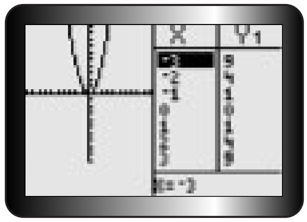
3. **Press [GRAPH] to show the function with this domain and range.**

B–5 Using the Split Screen

To see a graph and the equation editor at the same time, press MODE and cursor to **Horiz**. Press ENTER to select this, then press 2nd MODE to return to the home screen. Enter $y = x^2$ in **Y1** of the equation editor, then press GRAPH.

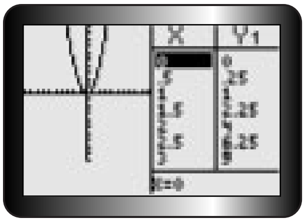


To see a graph and a table at the same time: press MODE and cursor to **G–T** (Graph-Table). Press ENTER to select this, then press GRAPH.



It is possible to view the table with different increments. For example, to see the table start at $x = 0$ and increase in increments of 0.5, press 2nd WINDOW and adjust the settings as shown.

Press GRAPH.



B–6 Using the TABLE Feature

A function such as $y = -0.1x^3 + 2x + 3$ can be displayed in a table of values.

1. **Enter the function into the equation editor.**

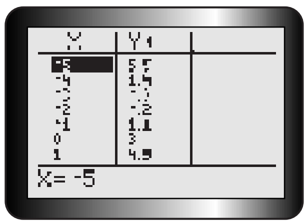
   To enter $y = -0.1x^3 + 2x + 3$, press Y= (–) . 1 X, T, Θ, *n* ^ 3 + 2 X, T, Θ, *n* + 3.

2. **Set the start point and step size for the table.**

   Press 2nd WINDOW. The cursor is alongside "TblStart=." To start at $x = -5$, press (–) 5 ENTER. The cursor is now alongside "ΔTbl=" (Δ, the Greek capital letter delta, stands for "change in.") To increase the x-value in steps of 1, press 1 ENTER.

3. **To view the table, press 2nd GRAPH.**

   Use ▲ and ▼ to move up and down the table. Notice that you can look at higher or lower x-values than the original range.

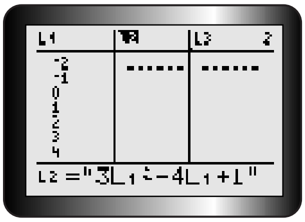


B–7 Making a Table of Differences

To create a table with the first and second differences for a function, use the STAT lists.

1. **Press** STAT 1 **and enter the *x*-values into L1.**

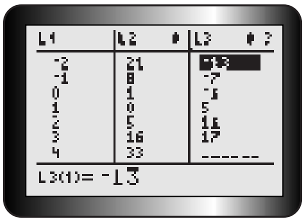
   For the function $f(x) = 3x^2 - 4x + 1$, use x-values from -2 to 4.



2. **Enter the function.**

   Scroll right and up to select **L2**. Enter the function $f(x)$, using **L1** as the variable x. Press ALPHA + 3 2nd 1 x^2 − 4 2nd 1 + 1 ALPHA +.

3. **Press** ENTER **to display the values of the function in L2.**



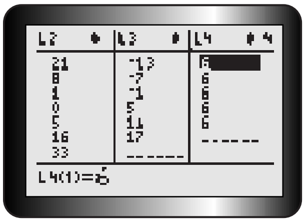
4. **Find the first differences.**

   Scroll right and up to select **L3**. Then press 2nd STAT.

   Scroll right to **OPS** and press 7 to choose **ΔList(**.

   Enter **L2** by pressing 2nd 2).

   Press ENTER to see the first differences displayed in **L3**.



5. **Find the second differences.**

   Scroll right and up to select **L4**. Repeat step 4, using **L3** in place of **L2**. Press ENTER to see the second differences displayed in **L4**.

B–8 Finding the Zeros of a Function

To find the zeros of a function, use the **zero** operation.

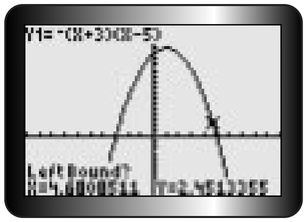
1. **Start by entering $y = -(x + 3)(x - 5)$ in the equation editor, then press** GRAPH ZOOM 6.

2. **Access the zero operation.**

   Press 2nd TRACE 2.

3. **Use the left and right arrow keys to cursor along the curve to any point to the left of the zero.**

   Press ENTER to set the left bound.

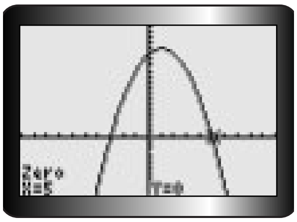


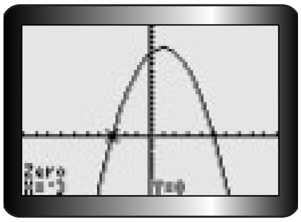
4. **Cursor along the curve to any point to the right of the zero.**

   Press ENTER to set the right bound.

5. **Press ENTER again to display the coordinates of the zero (the *x*-intercept).**

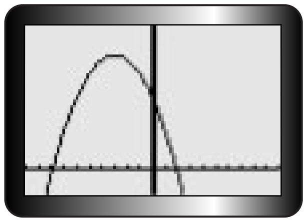
6. **Repeat to find the second zero.**





B–9 Finding the Maximum or Minimum Values of a Function

The least or greatest value can be found using the **minimum** operation or the **maximum** operation.



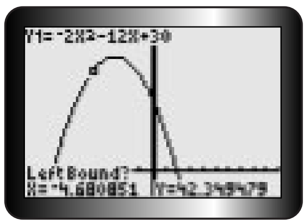
1. **Enter $y = -2x^2 - 12x + 30$.**
   Graph it and adjust the window as shown. This graph opens downward, so it has a maximum.

2. **Use the maximum operation.**

   Press 2nd TRACE 4. For parabolas that open upward, press 2nd TRACE 3 to use the **minimum** operation.

3. **Use the left and right arrow keys to cursor along the curve to any point to the left of the maximum value.**

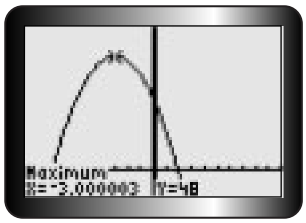
   Press ENTER to set the left bound.



4. **Cursor along the curve to any point right of the maximum value.**

   Press ENTER to set the right bound.

5. **Press ENTER again to display the coordinates of the optimal value.**

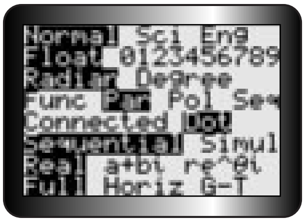


B–10 Graphing the Inverse of a Function

Parametric equations allow you to graph any function and its inverse. For example, the function $y = 2 - x^2$ with domain $x \geq 0$ can be graphed using parametric mode. For a parametric equation, both x and y must be expressed in terms of a parameter, t. Replace x with t. Then $x = t$ and $y = 2 - t^2$. The inverse of this function can now be graphed.

1. **Clear the calculator and press MODE.**

   Change the setting to the parametric mode by scrolling down to the fourth line and to the right to **Par**, as shown on the screen. Press ENTER.

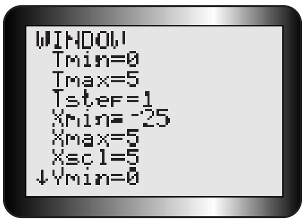


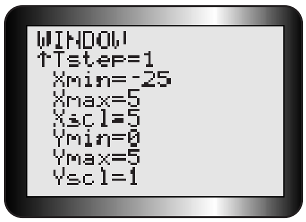
2. **Enter the inverse function by swapping the parametric equations $x = t$, $y = 2 - t^2$ to $x = 2 - t^2$, $y = t$.**

   Press Y=. At X1T=, enter 2 − X, T, Θ, *n* x^2 ENTER. At Y1T=, enter X, T, Θ, *n*.

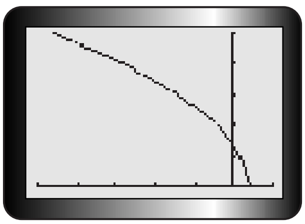
3. **Press WINDOW.**

   The original domain, $x \geq 0$, is also the domain of t. Use window settings such as the ones shown to display the graph.





4. **Press GRAPH to display the inverse function.**

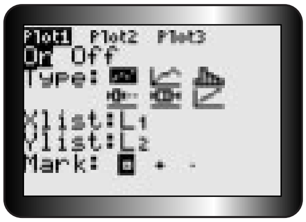


B–11 Creating Scatter Plots and Determining Lines and Curves of Best Fit Using Regression

This table gives the height of a baseball above ground, from the time it was hit to the time it touched the ground.

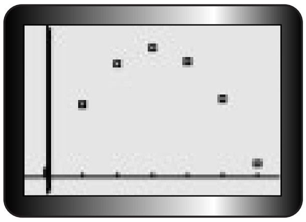
Time (s)	0	1	2	3	4	5	6
Height (m)	2	27	42	48	43	29	5

Create a scatter plot of the data.

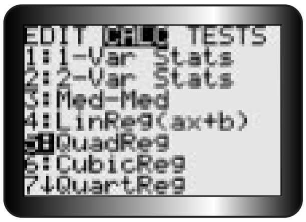


1. **Enter the data into lists.** To start press STAT ENTER. Move the cursor over to the first position in **L1** and enter the values for time. Press ENTER after each value. Repeat this for height in **L2**.

2. **Create a scatter plot.** Press 2nd Y= and 1 ENTER. Turn on Plot 1 by making sure the cursor is over **On**, the **Type** is set to the graph type you prefer, and **L1** and **L2** appear after **Xlist** and **Ylist**.



3. **Display the graph.** Press ZOOM 9 to activate **ZoomStat**.



4. **Apply the appropriate regression analysis.** To determine the equation of the line or curve of best fit press STAT and scroll over to **CALC**. Press:

   4 to enable **LinReg(ax+b)**

   5 to enable **QuadReg**.

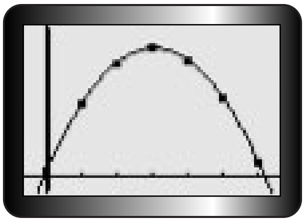
   0 to enable **ExpReg**.

   ALPHA C to enable **SinReg**.

   Press 2nd 1 , 2nd 2 , VARS. Scroll over to **Y-VARS**. Press 1 twice. This action stores the equation of the line or curve of best fit into **Y1** of the equation editor.

5. **Display and analyze the results.**

   Press ENTER. In this case, the letters a, b, and c are the coefficients of the general quadratic equation $y = ax^2 + bx + c$ for the curve of best fit. R^2 is the percent of data variation represented by the model. In this case, the equation is about $y = -4.90x^2 + 29.93x + 1.98$.



Note: In the case of linear regression, if r is not displayed, turn on the diagnostics function. Press 2nd 0 and scroll down to **DiagnosticOn**. Press ENTER twice. Repeat steps 4 to 6.

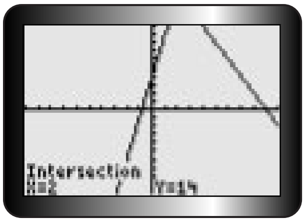
6. **Plot the curve.**

   Press GRAPH

B–12 Finding the Points of Intersection of Two Functions

1. **Enter both functions into the equation editor.** In this case we will use $y = 5x + 4$ and $y = -2x + 18$.
2. **Graph both functions**. Press GRAPH. Adjust the window settings until the point(s) of intersection are displayed.
3. **Use the intersect operation.**

   Press 2nd TRACE 5.

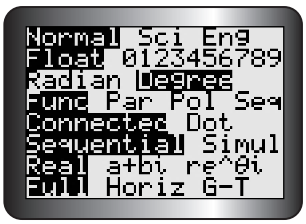


4. **Determine a point of intersection**. You will be asked to verify the two curves and enter a guess (optional) for the point of intersection. Press ENTER after each screen appears.

   The point of intersection is exactly (2, 14).

5. **Determine any additional points of intersection.** Press TRACE and move the cursor close to the other point you wish to identify. Repeat step 4.

B–13 Evaluating Trigonometric Ratios and Finding Angles

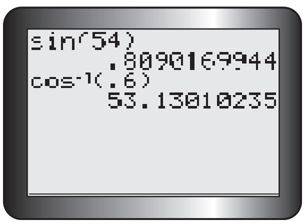


1. **Put the calculator in degree mode.**

   Press MODE. Scroll down and across to Degree. Press ENTER.

2. **Use the SIN, COS, or TAN key to calculate trigonometric ratios.**

   To find the value of sin 54°, press SIN 5 4) ENTER.

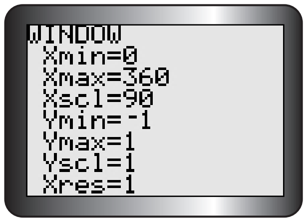


3. **Use SIN^{-1}, COS^{-1}, or TAN^{-1} to calculate angles.**

   To find the angle whose cosine is 0.6, press 2nd COS . 6) ENTER.

B–14 Graphing Trigonometric Functions

You can graph trigonometric functions in degree measure using the TI-83 Plus or TI-84 calculators. Graph the function $y = \sin x$ for $0° \leq x \leq 360°$.



step 3

1. **Put the calculator in degree mode.**

   Press MODE . Scroll down and across to **Degree**. Press ENTER .

2. **Enter $y = \sin x$ into the equation editor.**

   Press Y= SIN X, T, Θ, *n*) .

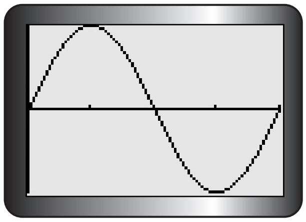
3. **Adjust the window to correspond to the given domain.**

   Press WINDOW . Set **Xmin = 0**, **Xmax = 360**, and **Xscl = 90**. These settings display the graph from 0° to 360°, using an interval of 90° on the *x*-axis. In this case, set **Ymin = −1** and **Ymax = 1**, since the sine function lies between these values. However, if this fact is not known, this step can be omitted.

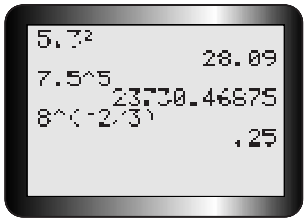
4. **Graph the function using ZoomFit.**

   Press ZOOM 0 . The graph is displayed over the domain and the calculator determines the best values to use for **Ymax** and **Ymin** in the display window.

   Note: You can use **ZoomTrig** (press ZOOM 7) to graph the function in step 4. **ZoomTrig** will always display the graph in a window where **Xmin = −360°**, **Xmax = 360°**, **Ymin = −4**, and **Ymax = 4**.


step 4

B–15 Evaluating Powers and Roots



1. **Evaluate the power $(5.3)^2$.**

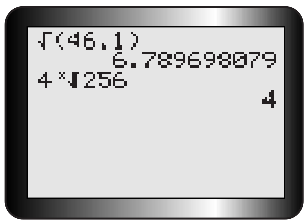
   Press 5 . 3 x^2 ENTER .

2. **Evaluate the power 7^5.**

   Press 7 ^ 5 ENTER .

3. **Evaluate the power $8^{-\frac{2}{3}}$.**

   Press 8 ^ (− 2 ÷ 3) ENTER .



4. **Evaluate the square root of 46.1.**

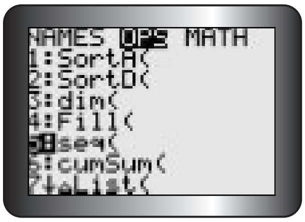
   Press 2nd x^2 4 6 . 1) ENTER .

5. **Evaluate $\sqrt[4]{256}$.**

   Press 4 MATH 5 2 5 6 ENTER .

B–16 Generating the Terms of a Sequence

Generate or list the first five terms of the sequence defined by $t_n = n^2$.



step 1

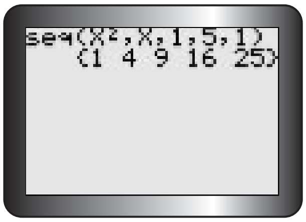
1. **Select sequence from the List OPS menu.**

   Press [2nd] [STAT] [▶]. Scroll down to sequence and press [ENTER].

step 2

2. **Enter the information for sequence.**
   You will need to enter the following:
   - the expression of the general term
   - the variable n — let [X, T, Θ, n] represent n
   - the first position number
   - the last position number
   - the increment — the increment is 1, because the difference between each pair of consecutive natural numbers is always 1

   Press [X, T, Θ, n] [x^2] [,] [X, T, Θ, n] [,] [1] [,] [5] [,] [1] [)].



step 3

3. **Generate the first five terms of the sequence.**

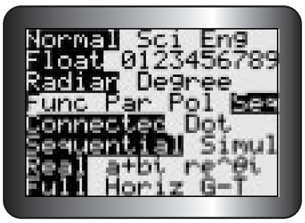
   Press [ENTER].

B–17 Graphing Sequences

Part 1: Creating a Table and Graphing — The General Term

Using the TI-83 or TI-84 Plus calculator, you can first create a table for a sequence and then a graph for the sequence. You will need the general term.

Create and graph the sequence defined by $t_n = 3n - 1$.



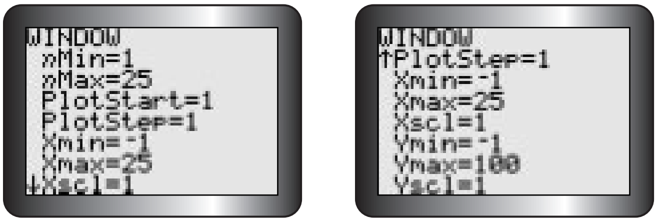
step 1

1. **Change the graphing mode from function to sequence.**
   The graphing modes are listed on the fourth line of the MODE menu. Press [MODE] and scroll down and across to **Seq**. Press [ENTER].

step 2

2. **Enter the general term into the sequence editor.**
   Press [Y=]. In this editor, $\boldsymbol{u(n)}$, $\boldsymbol{v(n)}$, and $\boldsymbol{w(n)}$ represent the general terms of sequences. You can change the minimum value of n (**nMin**). In most cases, you will not need to change the value 1, because, when $n = 1$, the first term is generated. Scroll down to $\boldsymbol{u(n)}$ and position the cursor to the right of the equal sign. Press [3] [X, T, Θ, n] [−] [1].

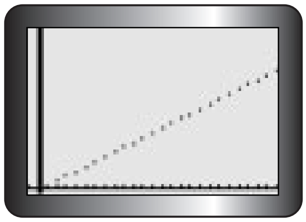
3. **Adjust the window to display the required number of terms.**
Press WINDOW. The setting ***n*Min** indicates the smallest *n*-value for the calculator to evaluate, while ***n*Max** indicates the largest *n*-value for the calculator to evaluate. **PlotStart=1** means that the graph starts at the first term. **PlotStep=1** means that each consecutive term will be plotted. You can change these settings, but use these window settings for this example.



step 3

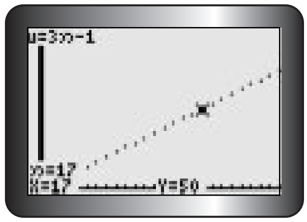
4. **Graph the sequence.**

Press GRAPH.



step 4

5. **Trace along the graph to identify specific terms of the sequence.**
Press TRACE. Use ◀ and ▶ to move from point to point. The *n*-value, or position, and the *x*- and *y*-coordinates of each term are displayed at each point. The *y*-coordinate represents the value of the term.

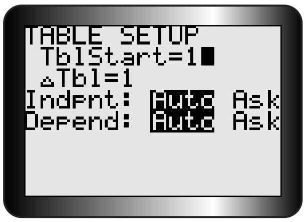


step 5

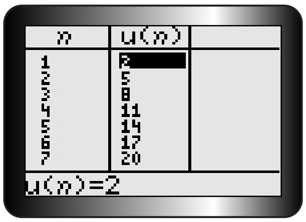
6. **View the terms of the sequence in a table.**

a) Press 2nd WINDOW. Set **TblStart** to 1 and **ΔTbl** to 1.

b) Press 2nd GRAPH to display the table. Use the cursor keys to scroll through the table.



step 6 a)



step 6 b)

Note: To see the graph and the table at the same time, use split-screen mode.

Press MODE, then scroll down and across to G–T (on the last line of the MODE menu). Press ENTER GRAPH.

Part 2: Creating a Table and Graphing — The Recursive Formula

Using the TI-83 Plus or TI-84 calculator, you can graph recursive sequences in the same way, with one exception: you must specify an initial value or values for **u(*n*Min)** in the sequence editor.

Graph the sequence $t_1 = 500$, $t_n = 500 + 0.26t_{n-1}$.

1. **Enter the recursive formula in the sequence editor and set the initial value.**

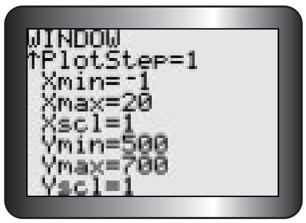
   Press Y= . Then, for the sequence **u(*n*)**, press 5 0 0 + 0 . 2 6 2nd 7 (X, T, Θ, *n* − 1) .

   Set the initial value to 500. Position the cursor to the right of the equal sign for **u(*n*Min)** and press 2nd (5 0 0 2nd) ENTER .

   Note: You do not have to enter the braces (2nd (and 2nd)) around the initial value, or the first term. However, if you were to enter, for example, $t^1 = 0$ and $t^2 = 1$, then you would press 2nd (1 , 0 2nd) ENTER .

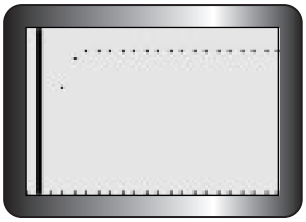
2. **Set the window.**

   Press WINDOW and enter the values shown.



3. Draw the graph.

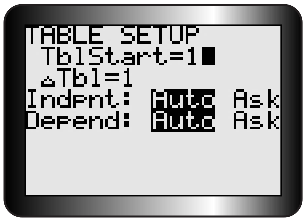
Press GRAPH.

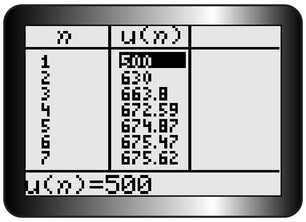


4. View the terms of the sequence in a table.

Press 2nd WINDOW. Set **TblStart** to 1 and **ΔTbl** to 1.

Press 2nd GRAPH to display the table. Use the cursor keys to scroll through the table.





B–18 Finding the Sum and Cumulative Sum of a Series

Find the sum and the cumulative sum of the first five terms of the sequence defined by $t_n = n^2$.

Part 1: Finding the Sum of the Terms of a Series

step 1

1. **Generate the first five terms of the sequence.**

   a) Select sequence from the List OPS menu. Press 2nd STAT ▶.

   Scroll down to **5:seq** and press ENTER.

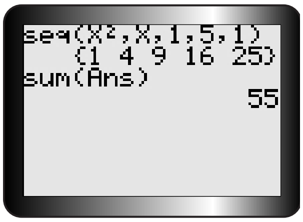
   b) Enter the expression of the general term, the variable, the starting value of the variable, the ending value of the variable, and the increment, 1. Press ENTER to generate the first five terms, as shown.

step 2

2. **Select sum from the List MATH menu.**

   Press 2nd STAT ▶ ▶ and scroll down to **5:sum.**

   Press ENTER.



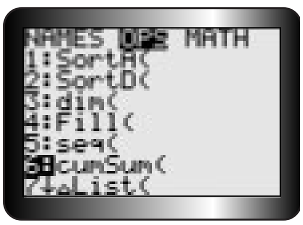
step 3

3. **Find the sum of the series.**
   Use **Ans**, last answer, to insert the terms in **sum**.
   Press 2nd –) ENTER. The sum is displayed.

Part 2: Finding the Cumulative Sum of the Terms of a Series

The cumulative sum displays the progression of sums of the terms of a series.

1. **Generate the terms of the sequence again.**
   Follow step 1 of Part 1.



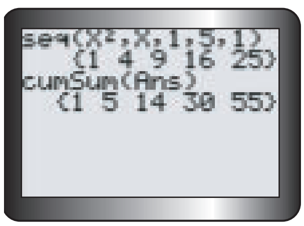
step 2

2. **Select cumulative sum from the LIST OPS menu.**

   Press 2nd STAT ▶ and scroll down to **6:cumSum**. Press ENTER.

3. **Find the cumulative sums of the terms of the sequence.**

   Press 2nd –) ENTER. The cumulative sums are displayed.



$1 = 1$
$5 = 1 + 4$
$14 = 1 + 4 + 9$
$30 = 1 + 4 + 9 + 16$
$55 = 1 + 4 + 9 + 16 + 25$

step 3

B–19 Analyzing Financial Situations Using the TVM Solver

Part 1: Introducing the TVM Solver

Press MODE and change the fixed decimal mode to 2, because most of the values that you are working with here represent dollars and cents. Scroll down to **Float**, across to **2**, and press ENTER .

Press APPS and then select **1:Finance**. From the Finance CALC menu, select **1:TVM Solver**. The screen that appears should be similar to the second one shown, but the values may be different.

You will notice eight variables on the screen.

N	total number of payment periods, or the number of interest conversion periods for simple annuities
I%	annual interest rate as a percent, not as a decimal
PV	present or discounted value
PMT	regular payment amount
FV	future or accumulated value
P/Y	number of payment periods per year
C/Y	number of interest conversion periods per year
PMT	Choose **BEGIN** if the payments are made at the beginning of the payment intervals. Choose **END** if the payments are made at the end of the payment intervals.

You may enter different values for the variables. Enter the value for money that is *paid* as a negative number, since the investment is a cash outflow; enter the value of money that is *received* as a positive number, since the money is a cash inflow. When you enter a whole number, you will see that the calculator adds the decimal and two zeros.

To solve for a variable, move the cursor to that variable and press ALPHA ENTER , and the calculator will calculate this value. A small shaded box to the left of the line containing the calculated value will appear.

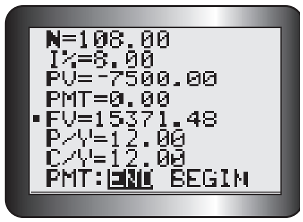
Part 2: Determining Future Value and Present Value

EXAMPLE 1

Find the future value or amount of $7500 invested for nine years at 8%/a, compounded monthly.

Solution

The number of interest conversion periods, **N**, is $9 \times 12 = 108$, **I%** $= 8$, and **PV** $= -7500$. The value for present value, **PV**, is negative, because the investment represents a cash outflow. **PMT** $= 0$ and **FV** $= 0$. The payments per year, **P/Y**, and the compounding periods per year, **C/Y**, are both 12. Open the **TVM Solver** and enter these values. Scroll to the line containing **FV**, the future value, and press ALPHA ENTER.



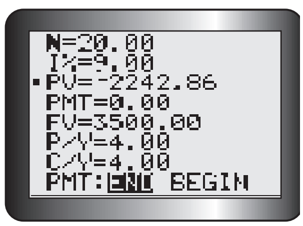
The investment will be worth $15 371.48 after nine years.

EXAMPLE 2

Maeve would like to have $3500 at the end of five years, so she can visit Europe. How much money should she deposit now in a savings account that pays 9%/a, compounded quarterly, to finance her trip?

Solution

Open the **TVM Solver** and enter the values shown in the screen, except the value for **PV**. The value for **FV** is positive, because the future value of the investment will be “paid” to Maeve, representing a cash inflow. Scroll to the line containing **PV** and press ALPHA ENTER to get $-\$2242.86$. The solution for **PV** is negative, because Maeve must pay this money and the payment is a cash outflow.



Part 3: Determining the Future or Accumulated Value of an Ordinary Simple Annuity

EXAMPLE 3

Celia deposits \$1500 at the end of each year in a savings account that pays 4.5%/a, compounded annually. What will be the balance in the account after five years?

Solution

N = 5 and **I%** = 4.5. Because there is no money in the account at the beginning of the term, **PV** = 0. **PMT** = −1500. The payment, **PMT**, is negative, because Celia makes a payment, which is a cash outflow. **P/Y** = 1 and **C/Y** = 1. Open the **TVM Solver** and enter these values.
Scroll to the line containing **FV** and press ALPHA ENTER.

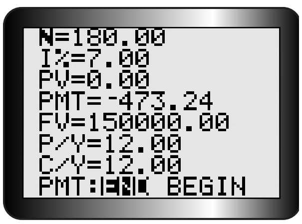
The balance in Celia's account at the end of the year will be \$8206.06.

EXAMPLE 4

Mr. Bartolluci would like to have \$150 000 in his account when he retires in 15 years. How much should he deposit at the end of each month in an account that pays 7%/a, compounded monthly?

Solution

Open the **TVM Solver** and enter the values shown, except for **PMT**. Note that **N** = 12 × 15 = 180, and the future value, **FV**, is positive, since he will receive the money at some future time. Scroll to the line containing **PMT** and press ALPHA ENTER.



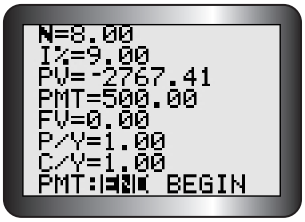
Mr. Bartolluci must deposit \$473.24 at the end of each month for 15 years. The payment appears as −473.24, because it is a cash outflow.

Part 4: Determining Present or Discounted Value of an Ordinary Simple Annuity

EXAMPLE 5

Northern Lights High School wishes to establish a scholarship fund. A $500 scholarship will be awarded at the end of each school year for the next eight years. If the fund earns 9%/a, compounded annually, what does the school need to invest now to pay for the fund?

Solution



Open the **TVM Solver** and enter 8 for **N**, 9 for **I%**, and 500 for **PMT**. The value for **PMT** is positive, because someone will receive $500 each year. Enter 0 for **FV**, since the fund will be depleted at the end of the term. Enter 1 for both **P/Y** and **C/Y**. Scroll to the line containing **PV** and press ALPHA ENTER.

The school must invest $2767.41 now for the scholarship fund. The present value appears as −2767.41, because the school must pay this money to establish the fund. The payment is a cash outflow.

EXAMPLE 6

Monica buys a snowboard for $150 down and pays $35 at the end of each month for 1.5 years. If the finance charge is 16%/a, compounded monthly, find the selling price of the snowboard.

Solution

Open the **TVM Solver** and enter the values as shown in the screen, except the value for **PV**. The payment, **PMT**, is positive, because the payments are a cash inflow for the snowboard's seller. Scroll to the line containing **PV** and press ALPHA ENTER.

The present value is $556.82. The present value appears as a negative value on the screen, because it represents what Monica would have to pay now if she were to pay cash.

The selling price is the sum of the positive present value and the down payment. Since the down payment is also a payment, add both numbers. The total cash price is $PV + \$150 = \$706.82$. Under this finance plan, Monica will pay $\$35 \times 18 + \$150 = \$780$.

B–20 Creating Repayment Schedules Using the TVM Solver

In this section, you will create a repayment schedule, by applying some of the financial functions from the Finance CALC menu.

Part 1: Introducing Other Finance CALC Menu Functions

You have used the **TVM Solver** to find, for example, future value and present value. The calculator can use the information that you have entered into the **TVM Solver** to perform other functions. Here are three other functions:

ΣInt(*A*, *B*, *roundvalue*)	calculates the sum of the interest paid from period *A* to period *B*
ΣPrn(*A*, *B*, *roundvalue*)	calculates the sum of the principal paid from period *A* to period *B*
bal(*x*, *roundvalue*)	calculates the balance owing after period *x*

The calculator rounds as it calculates. You will need to tell the calculator the value for rounding, *roundvalue*. The greater *roundvalue* is, the greater the accuracy of the calculations. In this section, the *roundvalue* is 6, which is also the value that banks use.

Part 2: Using the TVM Solver and Other Finance CALC Menu Functions

Press MODE and change the fixed decimal mode to 2, because most of the values in this section represent dollars and cents.

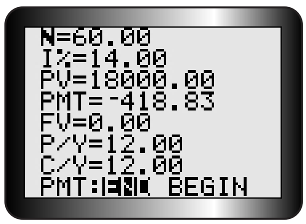
EXAMPLE 1

Eleanor finances the purchase of a new pickup truck by borrowing \$18 000. She will repay the loan with monthly payments. The term of the loan is five years. The interest rate is 14%/a, compounded monthly.

a) How much is the monthly payment?
b) How much will she pay in interest?
c) How much will she still owe on the loan after the 30th payment, that is, at the halfway point in repaying the loan?
d) What portion of the 30th payment reduces the principal?

Solution

a) Press APPS and select **1:Finance**. Then press ENTER to select **1:TVM Solver** from the Finance CALC menu. Enter **N** = 60, because 12 × 5 = 60. Enter **I%** = 14, **PV** = 18 000, **FV** = 0, **P/Y** = 12, and **C/Y** = 12. Notice that the present value, **PV**, is a positive number because Eleanor receives (a cash inflow) \$18 000 from the bank.



Scroll to the line containing **PMT** and press ALPHA ENTER. The monthly payment is \$418.83.

The payment appears as a negative value, because Eleanor pays this amount each month. The actual value is −418.828 515 3, which the calculator rounded to −418.83.

b) Use **ΣInt**(A, B, *roundvalue*) to calculate the total interest that Eleanor will pay.

Press [2nd] [MODE] to return to the home screen. Press [APPS] and select **1:Finance** from the Finance CALC menu. Select **ΣInt** by scrolling down or by pressing [ALPHA] [MATH].

Press [ENTER].

Press [1] [,] [6] [0] [,] [6] [)] [ENTER].

The sum of the interest paid from the first period to the 60th period is calculated.

By the end of the loan, Eleanor will have paid \$7129.71 in total interest. Eleanor will have paid \$7129.71 + \$18 000 = \$25 129.71 in interest and principal for the truck. Note that the product of the payment, \$418.83, and the total number of payments, 60, is \$25 129.80. The difference of \$0.09 is due to rounding, because 418.828 515 3 was rounded to 418.83.

c) Find the balance on the loan after the 30th payment. *roundvalue* must be consistent, that is, 6.

From the Finance CALC menu, select **bal** by scrolling or by pressing [9]. Press [3] [0] [,] [6] [)] [ENTER].

Eleanor still owes \$10 550.27 after the 30th payment. (Why is this amount not \$9000?)

d) Find the portion of the 30th payment that reduces the principal by calculating the sum of the principal paid from the 30th payment to the 30th payment. In the words, you are calculating the sum of only one item, the 30th payment. *roundvalue* is again 6. From the Finance CALC menu,

select **ΣPrn** by scrolling down or by pressing [0]. Press [3] [0] [,] [3] [0] [,] [6] [)] [ENTER].

The portion of the 30th payment that reduces the principal is \$292.33. The other portion of this payment, \$126.50, is interest.

Part 3: Using the Finance Functions to Create Repayment Schedules

Use the functions described in parts 1 and 2 to create repayment schedules or amortization tables.

EXAMPLE 2

Recall that Eleanor borrows $18 000 to purchase a pickup truck. She will repay the loan with monthly payments. The term of the loan is five years. The interest rate is 14%/a compounded monthly.

a) What will be the monthly outstanding balance on the loan after each of the first seven months?

b) Create a repayment schedule for the first seven months of the loan.

c) Use the repayment schedule to verify that the loan is completely paid after five years or 60 payments.

Solution

a) Find the outstanding balance after each payment for the first seven months. You will combine **sequence** (List OPS menu) and **bal**.

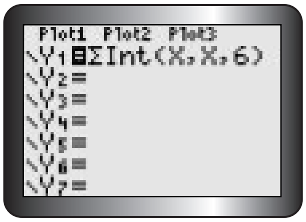
From the home screen, press [2nd] [STAT] [▶] [5] to select sequence.

Then press [APPS] [ENTER] [9] to select **bal**.

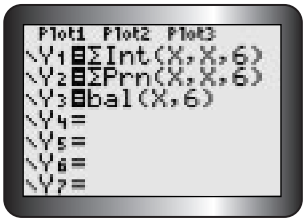
Press [X, T, Θ, *n*] [,] [6] [)] [,] [X, T, Θ, *n*] [,] [1] [,] [7] [,] [1] [)].

Press [ENTER] to calculate the sequence of balances, beginning with the first month, 1, and ending with the last month, 7. The increment for this sequence is 1, which is the last value entered. Recall that the increment is the change from payment number to payment number. Scroll right ([▶]) to see the other balances.

b) Create a repayment, or an amortization, schedule for the first seven months by comparing the interest, principal, and balance in a table.



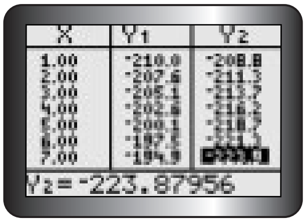
Begin by opening the equation editor. Press Y= . Clear **Y1** to **Y3**, if necessary. Store the interest portion of each payment in **Y1**. Move the cursor to the right of **Y1=**. Press APPS ENTER ALPHA MATH to select **ΣInt**. Press X, T, Θ, *n* , X, T, Θ, *n* , 6) ENTER .



To store the principal portion of each payment in Y2, press APPS ENTER 0 to select **ΣPrn**. Press X, T, Θ, *n* , X, T, Θ, *n* , 6) ENTER .

To store the outsanding balance after each payment in **Y3**, press APPS ENTER 9 to select **bal**. Press X, T, Θ, *n* , 6) ENTER .

Before viewing the table, press 2nd WINDOW . Set **TblStart** to 1 and **ΔTbl** to 1. The table will start with payment 0 and the payment number will increase by 1 at each step.



Press 2nd GRAPH to see the amortization table. Notice that the interest portion and the principal portion of each payment appear as negative values. Each payment, which is a combination of interest and principal, is a cash outflow for Eleanor.

Scroll right to see the values for **Y3**, the outstanding balance.

The outstanding balance after seven payments is $16 486.03.

c) Scroll or reset the tables's start value to see other entries in the amortization table. Scroll up to the beginning of the table. Notice that a substantial portion of the $418.83 payment is interest. Scroll down the table. At the end of the amortization, none of the payment is applied to the principal. The final outstanding balance is **2.5E⁻5**. As a decimal, this value is $0.000 25. Therefore, the amortization period of 60 payments is correct. The loan will be paid completely after five years or 60 payments.

PART 2 USING A SPREADSHEET

B–21 Introduction to Spreadsheets

A spreadsheet is a computer program that can be used to create a table of values and then graph the values. It is made up of cells that are identified by column letter and row number, such as A2 or B5. A cell can hold a label, a number, or a formula.

Creating a Table

Use spreadsheets to solve problems like this:
How long will it take to double your money if you invest $1000 at 5%/a compounded quarterly?

To create a spreadsheet, label cell A1 as Number of Quarters, Cell B1 as Time (years), and cell C1 as Amount ($). Enter the initial values of 0 in A2, 0 in B2, and 1000 in C2. Enter the formulas =A2+1 in A3, =A3/4 in B3, and =1000*(1.0125)^A3 in C3 to generate the next values in the table.

	A	B	C
1	**Number of Quarters**	**Time (years)**	**Amount ($)**
2	0	0	1000
3	=A2+1	=A3/4	-1000*(1.0125^A3)
4			

Notice that an equal sign is in front of each formula, an asterisk (*) is used for multiplication, and a caret (^) is used for exponents. Next, use the cursor to select cell A3 to C3 and several rows of cells below them. Use the **Fill Down** command. This command inserts the appropriate formula into each selected cell.

	A	B	C
1	**Number of Quarters**	**Time (years)**	**Amount ($)**
2	0	0	1000
3	1	0.25	
4			

When the **Fill Down** command is used, the computer automatically calculates and enters the values in each cell, as shown below in the screen on the left.

Continue to select the cells in the last row of the table and use the **Fill Down** command to generate more values until the solution appears, as shown below in the screen on the right.

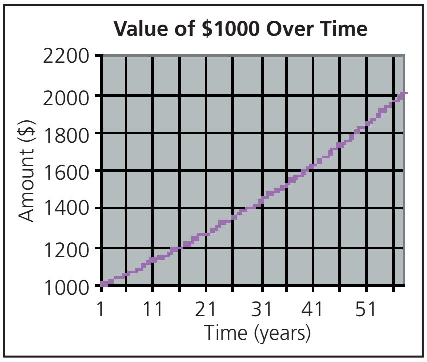
	A	B	C
1	**Number of Quarters**	**Time (years)**	**Amount ($)**
2	0	0	1000
3	1	0.25	1012.50
4	2	0.5	1025.16
5	3	0.75	1037.97
6	4	1	1050.94

	A	B	C
1	**Number of Quarters**	**Time (years)**	**Amount ($)**
2	0	0	1000
3	1	0.25	1012.50
4	2	0.5	1025.1563
⋮	⋮	⋮	⋮
56	54	13.5	1955.8328
57	55	13.75	1980.2807
58	56	14	2005.0342

Creating a Graph

Use the spreadsheet's graphing command to graph the results. Use the cursor to highlight the portion of the table you would like to graph. In this case, Time versus Amount.

	A	B	C
1	**Number of Quarters**	**Time (years)**	**Amount ($)**
2	0	0	1000
3	1	0.25	1012.50
4	2	0.5	1025.1563
⋮	⋮	⋮	⋮
56	54	13.5	1955.8328
57	55	13.75	1980.2807
58	56	14	2005.0342



Different spreadsheets have different graphing commands. Check your spreadsheet's instructions to find the proper command. This graph appears above on the right.

Determining the Equation of the Curve of Best Fit

Different spreadsheets have different commands for finding the equation of the curve of best fit using regression. Check your spreadsheet's instructions to find the proper command for the type of regression that suits the data.

B–22 Creating an Amortization Table

You have decided to purchase $4000 in furniture for your apartment. Your monthly payment is $520 and the interest rate 12%/a, compounded monthly. A spreadsheet can create a table of the outstanding balance and the graph.

Create a Table of the Outstanding Balance

You may find that the spreadsheet software that you are using may have different commands or techniques for achieving the same results.

1. Create a spreadsheet with five columns, labelled Payment Number, Payment, Interest Paid, Principal Paid and Outstanding Balance in cells A1, B1, C1, D1 and E1, respectively.
2. In cells A2 and E2, enter 0 and 4000, respectively. Leave cell B2, C2, and D2 empty.
3. In cell A3, enter the expression =A2+1 and press ENTER. In cell B3, enter 520. In cell C3, enter the expression for interest, =E2*0.01 and press ENTER. In cell D3, enter the expression =B3−C3 and press ENTER. In cell E3 enter = E2−D3 and press ENTER.

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Outstanding Balance
2	0				4000
3	=A2 + 1	520	=E2*0.01	=B3 − C3	=E2 − D3

4. Select cells A3 across to E3 and down to E11. Use the Fill Down command to complete the table. You can display financial data in different ways. Select cells B3 to E11 and click the **$** button or icon. Each number will appear with a dollar sign, a comma, and will be rounded to two decimal places.

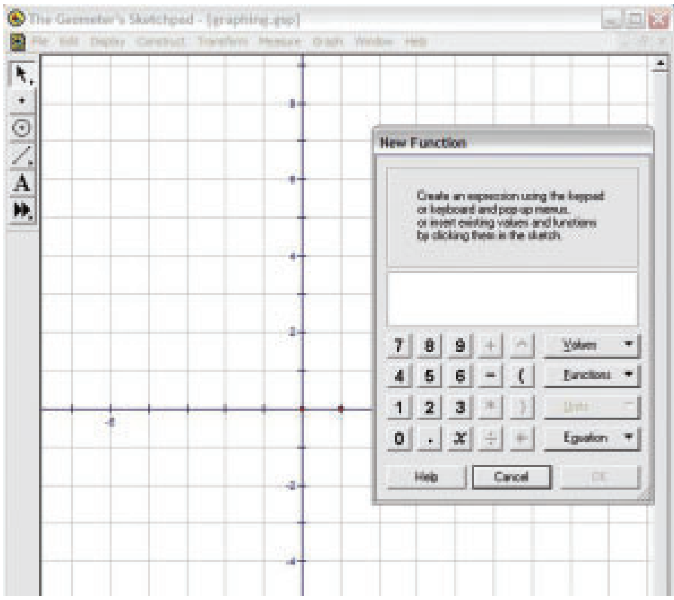
	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Outstanding Balance
2	0				$4,000.00
3	1	$520.00	$40.00	$480.00	$3,520.00
4	2	$520.00	$35.20	$484.80	$3,035.20
5	3	$520.00	$30.35	$489.65	$2,545.55
6	4	$520.00	$25.46	$494.54	$2,051.01
7	5	$520.00	$20.51	$499.49	$1,551.52
8	6	$520.00	$15.52	$504.48	$1047.03
9	7	$520.00	$10.47	$509.53	$537.50
10	8	$520.00	$5.38	$514.62	$22.88
11	9	$520.00	$0.23	$519.77	$(496.89)

PART 3 USING THE GEOMETER'S SKETCHPAD

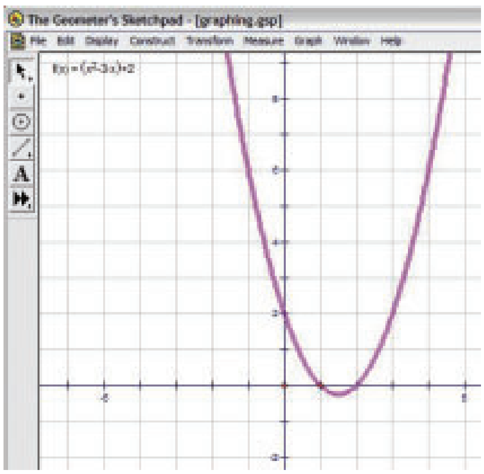
B–23 Graphing Functions

1. **Turn on the grid.**
   From the **Graph** menu, choose **Show Grid**.

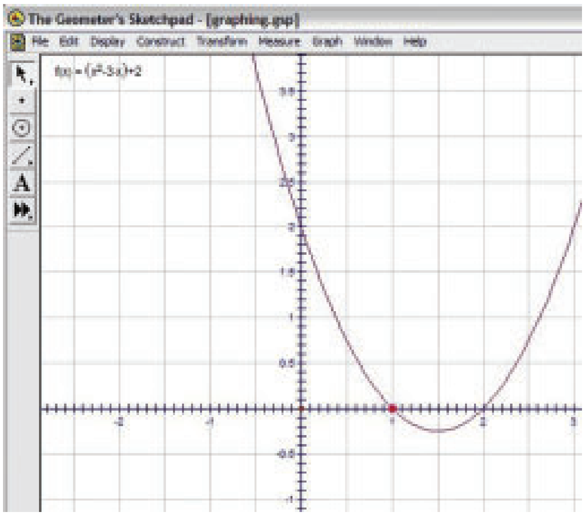
2. **Enter the function.** From the **Graph** menu, choose **Plot New Function**.
   The function calculator should appear.



3. **Graph the function $y = x^2 - 3x + 2$.**
   Use either the calculator keypad or the keyboard to enter "x ^ 2 - 3 * x + 2".
   Then press the "OK" button on the calculator keypad. The graph of
   $y = x^2 - 3x + 2$ should appear on the grid.



4. Adjust the origin and/or scale.
To adjust the origin, left-click on the point at the origin to select it. Then left-click and drag the origin as desired. To adjust the scale, left-click in blank space to deselect, then left-click on the point at (1, 0) to select it. Left-click and drag this point to change the scale.

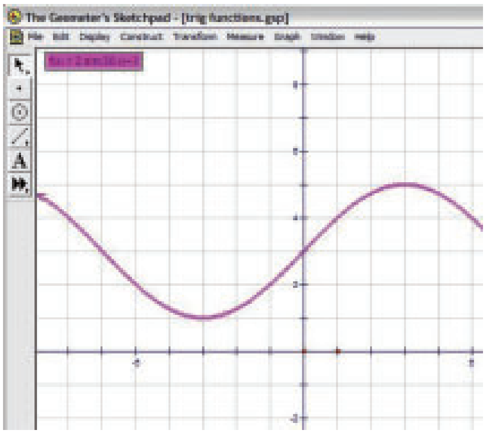


Appendix B

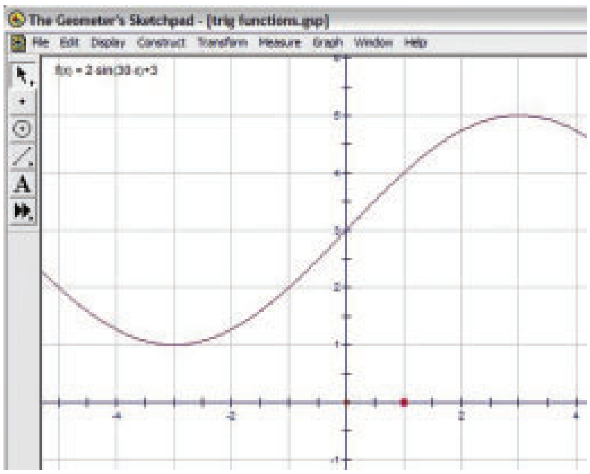
B–24 Graphing Trigonometric Functions

1. **Turn on the grid.**
   From the **Graph** menu, choose **Show Grid**.

2. **Graph the function $y = 2 \sin (30x) + 3$.**
   - From the **Graph** menu, choose **Plot New Function**. The function calculator should appear.
   - Use either the calculator keypad or the keyboard to enter "2 * sin (30 * x) + 3". To enter "sin," use the pull-down "Functions" menu on the calculator keypad.
   - Click on the "OK" button on the calculator keypad.
   - Click on "No" in the pop-up panel to keep degrees as the angle unit. The graph of $y = 2 \sin (30x) + 3$ should appear on the grid.



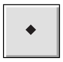
3. **Adjust the origin and/or scale.**
   Left-click on and drag either the origin or the point (1, 0).

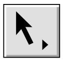


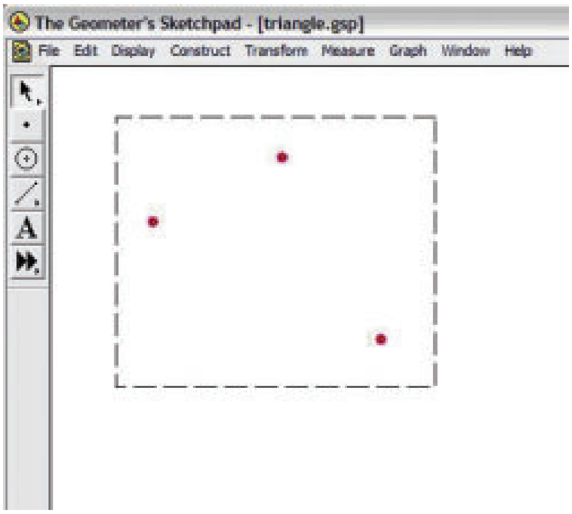
B–25 Creating Geometric Figures

To draw a triangle, follow these steps.

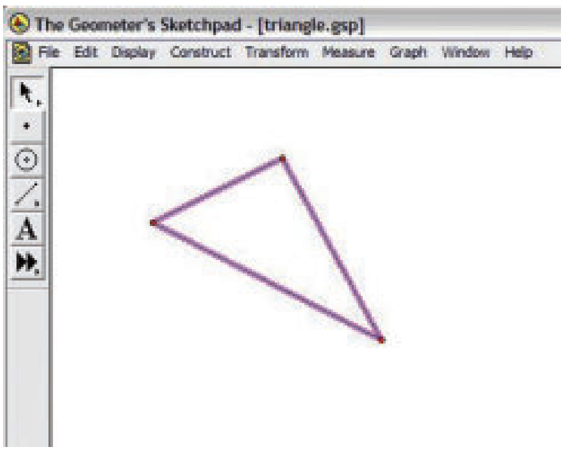
1. **Plot the vertices of the figure.**

   Choose the **Point** tool .

2. **Select the vertices.**
   Choose the **Arrow Selection** tool . Left-click and drag around the three vertices to select them.



3. **Draw the sides of the figure.**
   From the **Construct** menu, choose **Segments**. The sides of the triangle should appear.

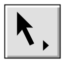


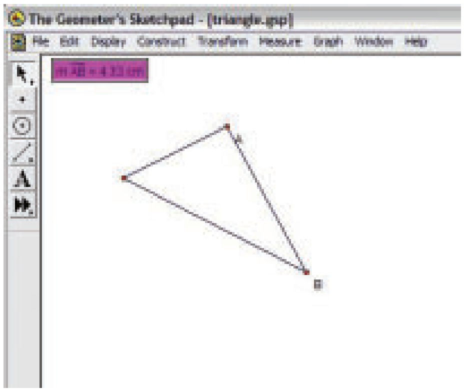
B–26 Measuring Sides and Angles, and Using the Calculator

As an example, draw a triangle and verify that the sine law holds for it.

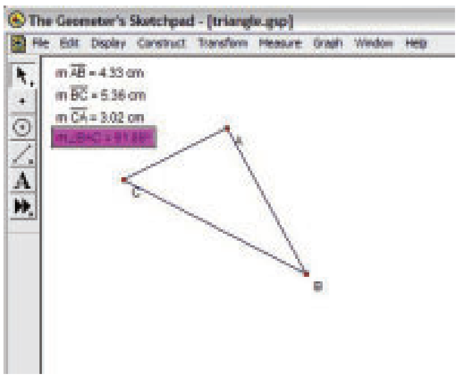
1. **Draw a triangle.**
   See "Creating Geometric Figures" (B-25).

2. **Find the measure of each side.**

   - Choose the **Arrow Selection**  tool.
   - Left-click on any blank area to deselect.
   - Left-click on a side to select it.
   - From the **Measure** menu, choose **Length**.
   - Repeat for the other two sides.



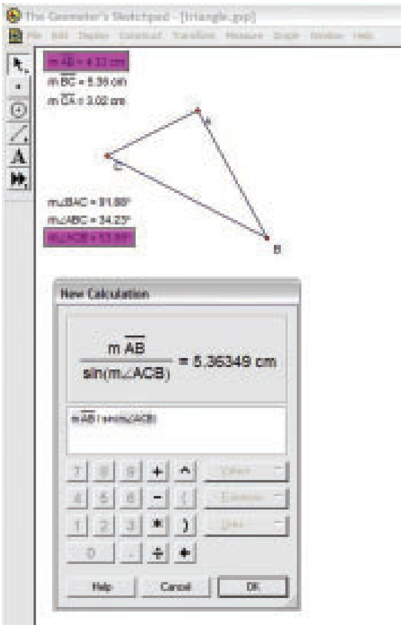
3. **Find the measure of each angle.**
   - Left-click on any blank area to deselect.
   - To measure *BAC*, select the vertices *in that order*: first *B*, then *A*, then *C*.
   - From the **Measure** menu, choose **Angle**.
   - Repeat for the other two angles.



4. For side *AB* and the opposite *ACB*, calculate the ratio $\frac{AB}{\sin ACB}$.

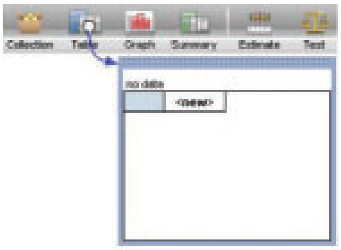
- From the Measure menu, choose Calculate…. The calculator keypad should appear.
- Left-click on the measure "m$\overline{AB}$" to select it.
- Use the calculator keypad to enter "÷".
- Use the "Functions" menu on the keypad to select "sin".
- Select the measure "m∠ACB".
- Press the "OK" button on the calculator keypad. The ratio should appear as a new measure.

5. Repeat step 4 for the other two pairs of corresponding sides and angles.



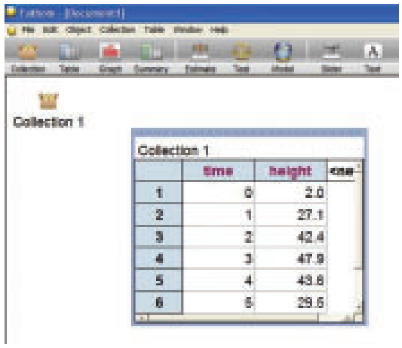
PART 4 USING FATHOM

B–27 Creating a Scatter Plot and Determining the Equation of a Line or Curve of Good Fit

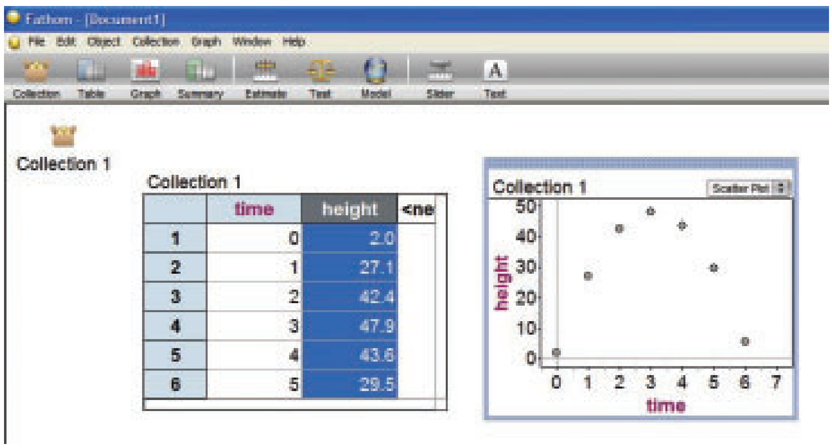


1. **Create a case table.** Drag a case table from the object shelf, and drop it in the document.

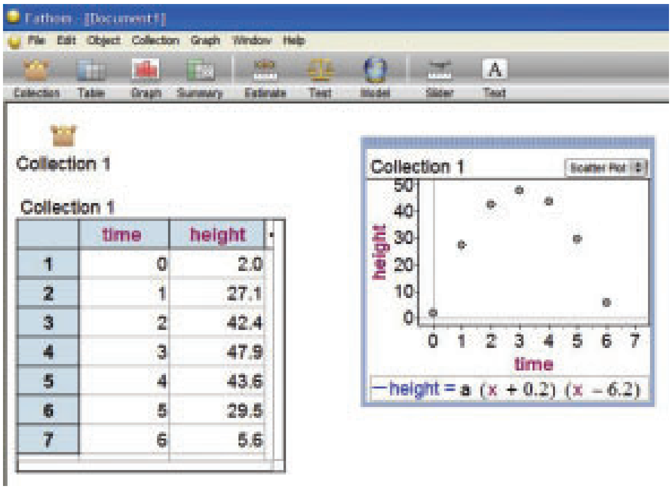
2. **Enter the Variables and Data.** Click <new>, type a name for the new variable or attribute and press Enter. (If necessary, repeat this step to add more attributes; pressing Tab instead of Enter moves you to the next column.) When you name your first attribute, Fathom creates an empty collection to hold your data (a little, empty box). The collection is where your data are actually stored. Deleting the collection deletes your data. When you add cases by typing values, the collection icon fills with gold balls. To enter the data click in the blank cell under the attribute name and begin typing values. (Press Tab to move from cell to cell.)



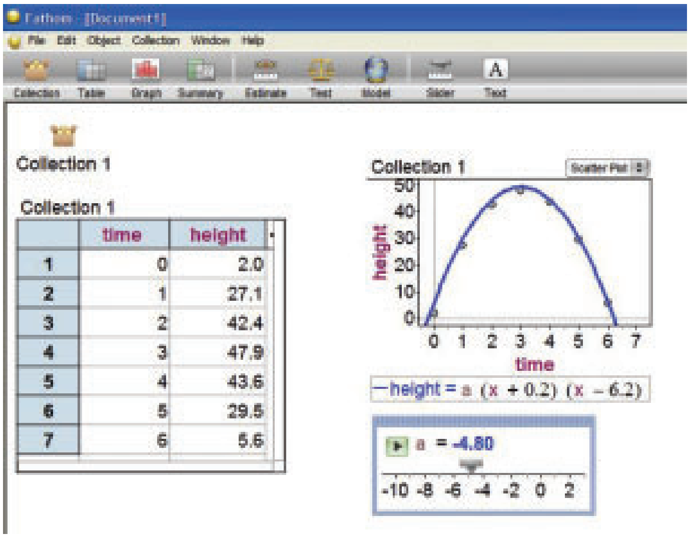
3. **Graph the data.** Drag a new graph from the object shelf at the top of the Fathom window, and drop it in a blank space in your document. Drag an attribute from the case table, and drop it on the prompt below and/or to the left of the appropriate axis in the graph.



4. **Create a function.** Right click the graph and select **Plot Function.** Enter your function using a parameter that can be adjusted to fit the curve to the scatter plot. (a was used in this case).



5. **Create a slider for the parameter(s) in your equation.** Drag a new slider from the object shelf at the top of the Fathom window, and drop it in a blank space below your graph. Type in the letter of the parameter used in your function in step 4 over V1. Click on the number then adjust the value of the slider until you are satisfied with the fit.



The equation of a curve of good fit is $y = -4.8(x + 0.2)(x - 6.2)$.

Glossary

Instructional Words

C

calculate: Figure out the number that answers a question; compute

clarify: Make a statement easier to understand; provide an example

classify: Put things into groups according to a rule and label the groups; organize into categories

compare: Look at two or more objects or numbers and identify how they are the same and how they are different (e.g., Compare the numbers 6.5 and 5.6. Compare the size of the students' feet. Compare two shapes.)

conclude: Judge or decide after reflection or after considering data

construct: Make or build a model; draw an accurate geometric shape (e.g., Use a ruler and a protractor to construct an angle.)

create: Make your own example

D

describe: Tell, draw, or write about what something is or what something looks like; tell about a process in a step-by-step way

determine: Decide with certainty as a result of calculation, experiment, or exploration

draw: 1. Show something in picture form (e.g., Draw a diagram.)
2. Pull or select an object (e.g., Draw a card from the deck. Draw a tile from the bag.)

E

estimate: Use your knowledge to make a sensible decision about an amount; make a reasonable guess (e.g., Estimate how long it takes to cycle from your home to school. Estimate how many leaves are on a tree. What is your estimate of 3210 + 789?)

evaluate: 1. Determine if something makes sense; judge
2. Calculate the value as a number

explain: Tell what you did; show your mathematical thinking at every stage; show how you know

explore: Investigate a problem by questioning, brainstorming, and trying new ideas

extend: 1. In patterning, continue the pattern
2. In problem solving, create a new problem that takes the idea of the original problem further

J

justify: Give convincing reasons for a prediction, an estimate, or a solution; tell why you think your answer is correct

M

measure: Use a tool to describe an object or determine an amount (e.g., Use a ruler to measure the height or distance around something. Use a protractor to measure an angle. Use balance scales to measure mass. Use a measuring cup to measure capacity. Use a stopwatch to measure the time in seconds or minutes.)

model: Show or demonstrate an idea using objects and/or pictures (e.g., Model addition of integers using red and blue counters.)

P

predict: Use what you know to work out what is going to happen (e.g., Predict the next number in the pattern 1, 2, 4, 7, ...)

R

reason: Develop ideas and relate them to the purpose of the task and to each other; analyze relevant information to show understanding

relate: Describe how two or more objects, drawings, ideas, or numbers are similar

represent: Show information or an idea in a different way that makes it easier to understand (e.g., Draw a graph. Make a model.)

S

show (your work): Record all calculations, drawings, numbers, words, or symbols that make up the solution

sketch: Make a rough drawing (e.g., Sketch a picture of the field with dimensions.)

solve: Develop and carry out a process for finding a solution to a problem

sort: Separate a set of objects, drawings, ideas, or numbers according to an attribute (e.g., Sort 2-D shapes by the number of sides.)

V

validate: Check an idea by showing that it works

verify: Work out an answer or solution again, usually in another way; show evidence of

visualize: Form a picture in your head of what something is like; imagine

Mathematical Words

A

absolute value: Written as $|x|$; describes the distance of x from 0; equals x when $x \geq 0$ or $-x$ when $x < 0$; for example, $|3| = 3$ and $|-3| = -(-3) = 3$

the ambiguous case of the sine law: A situation in which 0, 1, or 2 triangles can be drawn given the information in a problem. This occurs when you know two side lengths and an angle *opposite* one of the sides rather than *between* them (an SSA triangle). If the given angle is acute, 0, 1, or 2 triangles are possible. If the given angle is obtuse, 0 or 1 triangle is possible

amortization schedule: A record of payments showing the interest paid, the principal, and the current balance on a loan or investment

amount: The total value of an investment or loan. The amount is given by $A = P + I$, where A is the amount, P is the principal, and I is the interest

amplitude: Half the difference between the maximum and minimum values; it is also the vertical distance from the function's axis to the maximum or minimum value

angle of depression (declination): The angle between a line below the horizontal and the horizontal

angle of elevation: The angle formed by the horizontal and the line of sight (to an object above the horizontal)

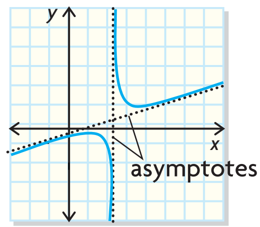
annuity: A series of payments or investments made at regular intervals, A **simple** annuity is an annuity in which the payments coincide with the compounding period, or *conversion* period; an **ordinary** annuity is an annuity in which the payments are made at the end of each interval

arithmetic sequence: A sequence that has the same difference, the **common difference**, between any pair of consecutive terms

arithmetic series: The sum of the terms of an arithmetic sequence

associative property: With addition and multiplication, you can add or multiply in any order: $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$

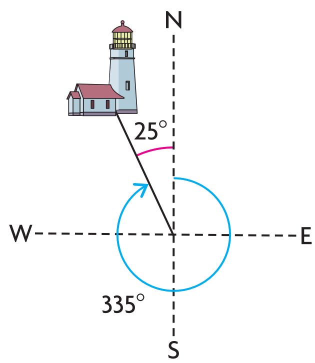
asymptote: A line that the graph of a relation or function gets closer and closer to, but never meets, on some portion of its domain



axis of symmetry: A line in a two-dimensional figure such that, if a perpendicular is constructed, any two points lying on the perpendicular and the figure are at equal distances from this line

B

bearing: The direction in which you have to move in order to reach an object. A bearing is a clockwise angle from magnetic north. For example, the bearing of the lighthouse shown is 335°



Glossary

C

common difference: The constant difference between two consecutive terms in an arithmetic sequence or series

common ratio: The constant ratio (quotient) between two consecutive terms in a geometric sequence or series

commutative property: The order in which you add or multiply numbers does not matter. The result is the same; $a + b = b + a$ and $a \times b = b \times a$

completing the square: The process of adding a constant to a given quadratic expression to form a perfect trinomial square. For example, $x^2 + 6x + 2$ is not a perfect square, but if 7 is added it becomes $x^2 + 6x + 9$, which is the square of $x + 3$

compound interest: Interest that is added to the principal *before* new interest earned is calculated. So interest is calculated on the principal *and* on the interest already earned. Interest is paid at regular time intervals called the **compounding period**

compounding period: The intervals at which interest is calculated, for example,

annually $\Rightarrow$ 1 time per year

semi-annually $\Rightarrow$ 2 times per year

quarterly $\Rightarrow$ 4 times per year

monthly $\Rightarrow$ 12 times per year

cosine law: The relationship, for any triangle, involving the cosine of one of the angles and the lengths of the three sides; used to determine unknown sides and angles in triangles. If a triangle has sides a, b, and c, and if the angle A is opposite side a, then $a^2 = b^2 + c^2 - 2bc \cos A$

curve of best fit: The curve that best describes the distribution of points in a scatter plot. Typically found using regression analysis

curve of good fit: A curve that reasonably describes the distribution of points in a scatter plot. Typically found using an informal process

D

direction of opening: The direction in which a parabola opens; up or down

discriminant: The expression $b^2 - 4ac$ in the quadratic formula

distributive property: The principal that says that, when a polynomial is expanded, each of its terms is multiplied or divided by the number or term outside of the brackets

domain: The set of all values of the independent variable of a relation

down payment: The partial amount of a purchase paid at the time of purchase

E

entire radical: A radical with coefficient 1, for example $\sqrt{12}$

equation of the axis: The equation of the horizontal line halfway between the maximum and the minimum; it is determined by

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$

exponential function: A function of the form $y = a(b^x)$

F

factored form: A quadratic relation in the form $f(x) = a(x - r)(x - s)$

family: A collection of functions (or lines or curves) sharing common characteristics

family of parabolas: A group of parabolas that all share a common characteristic

function: A relation where each value of the independent variable corresponds with only one value of the dependent variable

function notation: Notation, such as $f(x)$, used to represent the value of the dependent variable—the output—for a given value of the independent variable, x—the input

future value: The total amount, A, of an investment after a certain length of time

G

general term: A formula, labelled t_n, that expresses each term of a sequence as a function of its position. For example, if the general term of a sequence is $t_n = 2n$, then to calculate the 12th term (t_{12}), substitute $n = 12$

$$\begin{aligned} t_{12} &= 2(12) \\ &= 24 \end{aligned}$$

geometric sequence: A sequence that has the same ratio, the **common ratio**, between any pair of consecutive terms

geometric series: The sum of the terms of a geometric sequence

H

half life: The time required for a quantity to decay to half of its initial value

harmonic mean: A type of average, denoted by H, involving the reciprocals of a list of numbers

$$H = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}, \text{ where}$$

$a_1, \ldots, a_n$ are positive real numbers

hypotenuse: The longest side of a right triangle; the side that is opposite the right angle

I

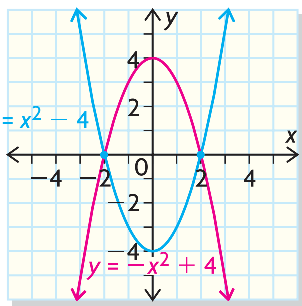
identity: A mathematical statement that is true for all values of the given variables. If the identity involves fractions, the denominators cannot be zero. Any restrictions on a variable must be stated

independent variable: In an algebraic relation, a variable whose values may be freely chosen and upon which the values of the other variables depend. Often represented by x

index (plural **indices**)**:** The number at the left of the radical sign. It tells which root is indicated: 3 for cube root, 4 for fourth root, etc. If there is no number, the square root is intended

interest: The cost of borrowing money or the money earned from an investment

invariant point: A point on a graph (or figure) that is unchanged by a transformation; for example, $(-2, 0)$ and $(2, 0)$ for this graph and transformation



inverse of a function: The reverse of the original function; undoes what the original function has done

L

like radicals: Radicals that have the same number under the radical symbol, such as $3\sqrt{6}$ and $-2\sqrt{6}$

linear relation: A relation between two variables that appears as a straight line when graphed on a coordinate system. May also be referred to as a *linear function*

line of best fit: The straight line that best describes the distribution of points in a scatter plot. Typically found using linear regression analysis

line of good fit: The straight line that reasonably describes the distribution of points in a scatter plot. Typically found using an informal process

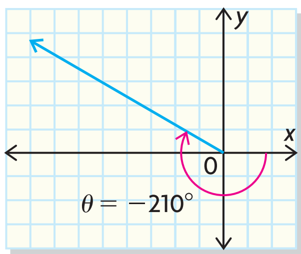
lowest common denominator (LCD): The smallest multiple shared by two or more denominators

M

mixed radical: A radical with coefficient other than 1; for example, $2\sqrt{3}$

N

negative angle: An angle measured *clockwise* from the positive x-axis



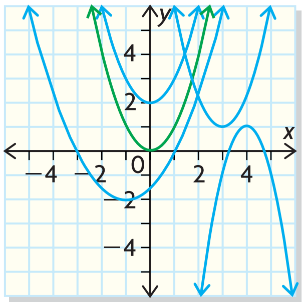
negative correlation: This indicates that as one variable in a linear relationship increases, the other decreases or vice versa

nonperiodic function: Any function that does not repeat at regular intervals

P

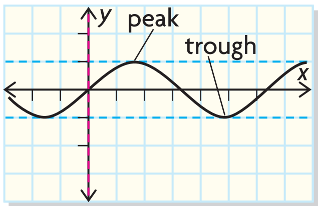
parabola: The graph of a quadratic relation of the form $y = ax^2 + bx + c\ (a \neq 0)$. The graph, which resembles the letter "U," is symmetrical

parent function: The simplest, or base, function in a family

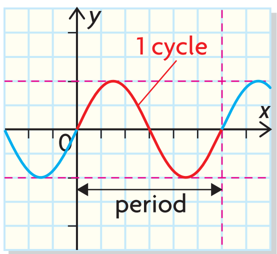


partial sum: The sum, S_n, of the first n terms of a sequence

peak: The maximum point on a graph



period: The change in the independent variable (typically x) corresponding to one cycle; a cycle of a periodic function is a portion of the graph that repeats



periodic function: A function whose graph repeats at regular intervals; the y-values in the table of values show a repetitive pattern when the x-values change by the same increment

phase shift: The horizontal translation of a sinusoidal function is also called a phase shift

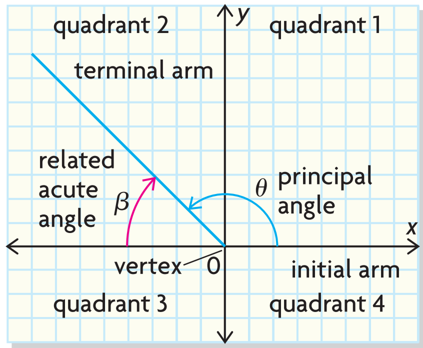
positive correlation: This indicates that both variables in a linear relationship increase or decrease together

present value: The principal that would have to be invested now to get a specific future value in a certain amount of time; PV is used for present value instead of P, since P is used for principal

prime number: A number with only two factors, 1 and itself (e.g., 17 is a prime number since its only factors are 1 and 17.)

principal: A sum of money that is borrowed or invested

principal angle: The counterclockwise angle between the initial arm and the terminal arm of an angle in standard position. Its value is between 0° and 360°



Pythagorean theorem: The conclusion that, in a right triangle, the square of the length of the longest side is equal to the sum of the squares of the lengths of the other two sides

Q

quadratic formula: A formula for determining the roots of a quadratic equation of the form $ax^2 + bx + c = 0$. The formula is phrased in terms of the coefficients of the quadratic equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

quadratic relation: A relation whose equation is in quadratic form; for example, $y = x^2 + 7x + 10$

R

radical: A square, cube, or higher root, such as $\sqrt{4} = 2$ or $\sqrt[3]{27} = 3$; $\sqrt{}$ is called the radical symbol

range: The set of all values of the dependent variable of a relation

rational form: A number written as an integer or a fraction, such as -3 or $-\frac{2}{3}$

rational expression: A quotient of polynomials; for example, $\frac{2x - 1}{3x}$, $x \neq 0$

rational function: Any function whose output can be given by an expression that is the ratio of two polynomials.

A rational function can be expressed as $f(x) = \frac{R(x)}{S(x)}$, where R and S are polynomials and $S \neq 0$—for example,

$$f(x) = \frac{x^2 - 2x + 3}{4x - 1}, \; x \neq \frac{1}{4}$$

real numbers: Numbers that are either rational or irrational; these include positive and negative integers, zero, fractions, and irrational numbers such as $\sqrt{2}$ and π

reciprocal trigonometric ratios: The reciprocal ratios are defined by dividing 1 by each of the primary trigonometric ratios

$$\csc \theta = \frac{1}{\sin \theta} = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

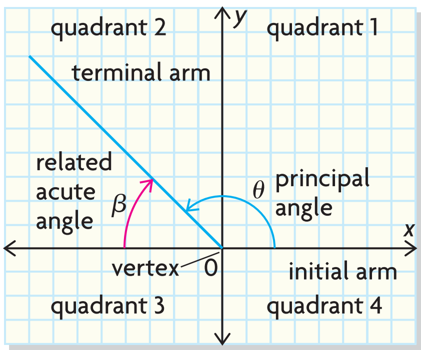
$$\cot \theta = \frac{1}{\tan \theta} = \frac{\text{adjacent}}{\text{opposite}}$$

$\cot \theta$ is the short form for the cotangent of angle θ, $\sec \theta$ is the short form for the secant of angle θ, and $\csc \theta$ is the short form for the cosecant of angle θ

recursive formula: A formula relating the general term of a sequence to the previous term or terms

recursive sequence: A sequence for which one term (or more) is given and each successive term is determined from the previous term(s)

related acute angle: The acute angle between the terminal arm of an angle in standard position and the x-axis when the terminal arm lies in quadrants 2, 3, or 4



relation: A set of ordered pairs; values of the independent variable are paired with values of the dependent variable

restrictions: The values of the variable(s) in a rational function or rational expression that cause the function to be undefined. These are the zeros of the denominator or, equivalently, the numbers that are not in the domain of the function

S

scatter plot: A graph that attempts to show a relationship between two variables by means of points plotted on a coordinate grid

sequence: An ordered list of numbers

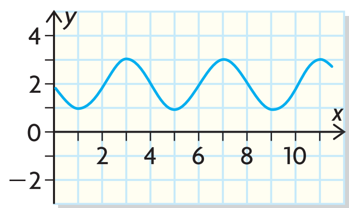
series: The sum of the terms of a sequence

simple interest: Interest earned or paid only on the original sum of money that was invested or borrowed

sine law: The relationships, for any triangle, involving the sines of two of the angles and the lengths of the opposite sides; used to determine unknown sides and angles in triangles. If a triangle has sides a, b, and c, and if the angles opposite each side are A, B, and C, respectively, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

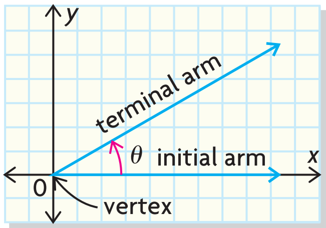
sinusoidal function: A periodic function whose graph looks like smooth symmetrical waves, where any portion of the wave can be horizontally translated onto another portion of the curve; graphs of sinusoidal functions can be created by transforming the graph of the function $y = \sin x$ or $y = \cos x$



standard form: A quadratic relation in the form $f(x) = ax^2 + bx + c$

Glossary

standard position: An angle in the Cartesian plane whose vertex lies at the origin and whose initial arm (the arm that is fixed) lies on the positive x-axis. Angle θ is measured from initial arm to terminal arm (the arm that rotates)



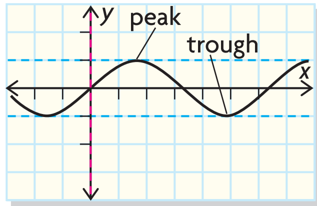
T

term: A number in a sequence. Subscripts are usually used to identify the position of the terms

transformation: A geometric operation such as a translation, rotation, dilation, or reflection

trend: A relationship between two variables for which the independent variable is time

trough: The minimum point on a graph



V

vertex (plural **vertices**): The point at the corner of an angle or shape (e.g., A cube has eight vertices. A triangle has three vertices. An angle has one vertex.)

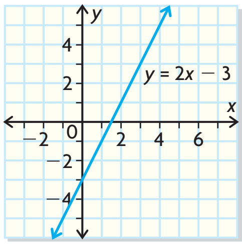
vertex form: A quadratic function written in the form $f(x) = a(x - h)^2 + k$ is in vertex form; the vertex is (h, k)

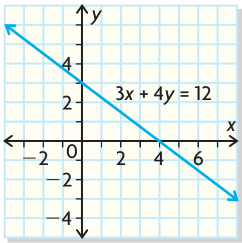
vertical line test: If any vertical line intersects the graph of a relation more than once, then the relation is not a function

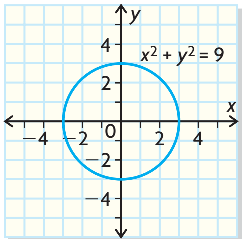
Answers

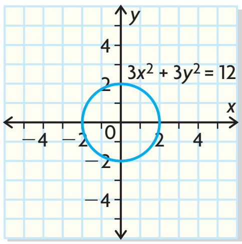
Chapter 1

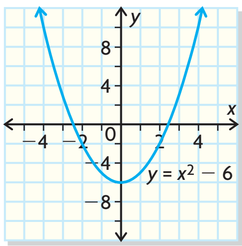
Getting Started, p. 2

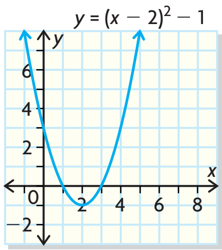
1. a) $-2x + 8y$ b) $16x^2 - y^2$ c) $-x^2 + 2$ d) $2x^2 + 16x$
2. a) -46 b) 119 c) -7 d) 66
3. a) $x = 3$ b) $x = -2$ c) $y = -36$ d) $x = -2$
4. a) 

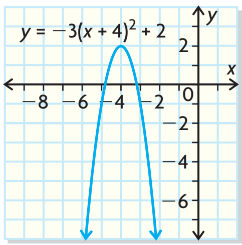
b) 

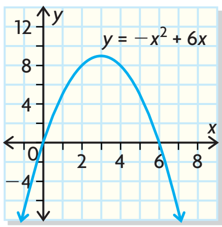
5. a) 

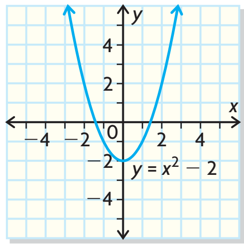
b) 

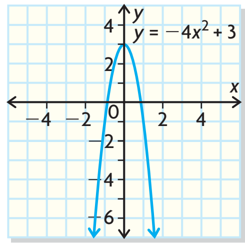
6. a) 

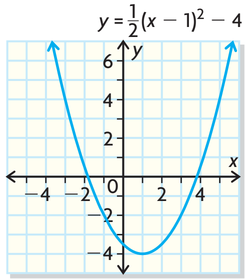
b) 

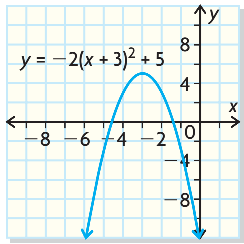
c) 

d) 

7. a) Translate down 2 units


b) Reflect in x-axis, then vertical stretch, scale factor 4, then translate up 3 units


c) Vertical compression, scale factor, then translate right 1 unit and down 4 units


d) Reflect in x-axis, then vertical stretch, scale factor 2, then translate left 3 units and up 5 units


8. a) $x = 2$ or 3 b) $x = \pm 5$
9. Similarities: none of the equations for the relations involve powers higher than 2. Differences: linear and quadratic relations assign one y-value to each x-value, but circles may assign 0, 1, or 2 y-values; the relations have different shapes and model different types of problems; linear relations may only enter 2 or 3 quadrants.

Property	Linear Relations	Circles	Quadratic Relations
Equation(s)	$y = mx + c$ or $Ax + By = C$	$(x - h)^2 + (y - k)^2 = r^2$	$y = ax^2 + bx + c$ or $y = a(x - h)^2 + k$
Shape of graph	Straight line	Circle	Parabola
Number of quadrants graph enters	2 or 3	1, 2, 3, or 4	1, 2, 3, or 4
Descriptive features of graph	Slope is constant; crosses each axis at most once	Graph has upper and lower parts	Graph has a single lowest or highest point (vertex); crosses y-axis once, x-axis 0, 1, or 2 times
Types of problems modelled by the relation	Direct and partial variation	Constant distance from a point	Some economic functions; motion of a projectile; area

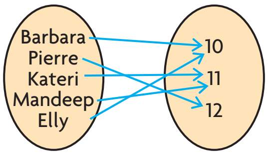
Lesson 1.1, pp. 10–12

1. **a)** Function; each x-value has only one y-value
b) Not a function; for $x = 1$, $y = -3$ and 0
c) Not a Function; for $x = 0$, $y = 4$ and 1
d) Function; each x-value has only one y-value

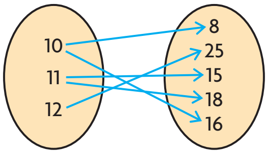
2. **a)** Not a function **c)** Function **e)** Function
b) Not a function **d)** Not a function **f)** Not a function

3. For $y = x^2 - 5x$, each x-value gives a single y-value. For $x = y^2 - 5y$, each x-value gives a quadratic equation in y, which may have two solutions.

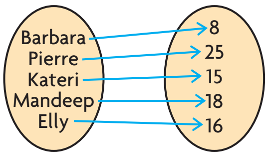
4. **a)** {(Barbara, 10), (Pierre, 12), (Kateri, 11), (Mandeep, 11), (Elly, 10)}



{(10, 8), (12, 25), (11, 15), (11, 18), (10, 16)}

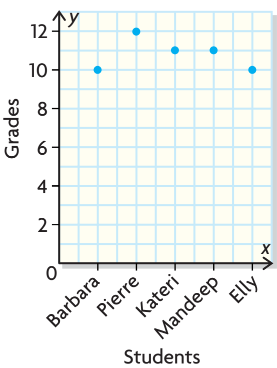


{(Barbara, 8), (Pierre, 25), (Kateri, 15), (Mandeep, 18), (Elly, 16)}

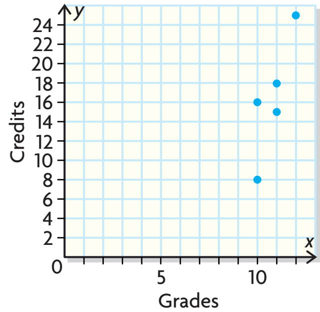


b) students, grades: domain = {Barbara, Pierre, Kateri, Mandeep, Elly}, range = {10, 11, 12} grades, credits: domain = {10, 11, 12}, range = {8, 15, 16, 18, 25} students, credits: domain = {Barbara, Pierre, Kateri, Mandeep, Elly}, range = {8, 15, 16, 18, 25}
c) Only grades-credits relation is not a function; it has repeated range values for single domain values.

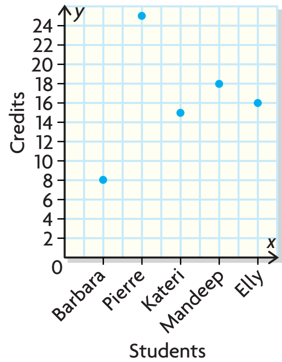
5. students, grades:



grades, credits:

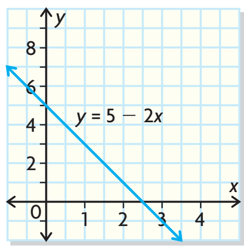


students, credits:

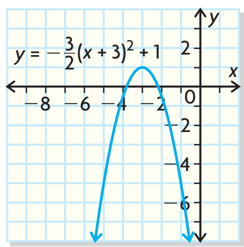


6. $y = 3$: horizontal line; function (passes vertical-line test).
$x = 3$: vertical line; not a function (fails vertical-line test)

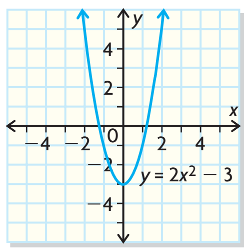
7. **a)** Linear, function



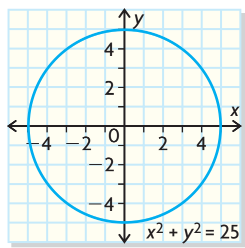
c) Quadratic, function



b) Quadratic, function



d) Circle, not a function



8. **a)** **i)** 1.25; 2.75 **ii)** ± 2; 0 **iii)** 2; -2 **iv)** 0; $\pm\sqrt{2}$
b) Functions: (i), (iii)
c) Graph relation and apply vertical-line test, or solve equation for y and check for multiple values

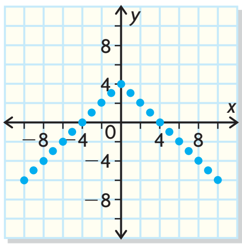
9. Functions: (a), (b), (d)

10. Not a function; for example, when $x = 6$, $y = 2$ or -2; graph fails vertical-line test

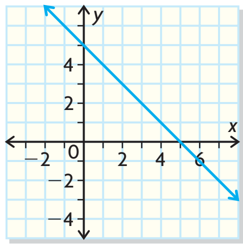
11. Functions: (a), (b)

12. **a)** domain $= \{x \in \mathbf{R} \mid x \geq 0\}$, range $= \{y \in \mathbf{R} \mid y \geq 44\}$
b) Distance cannot be negative, cost cannot be lower than daily rental charge.
c) Yes, it passes the vertical line test.

13. **a)** Answers may vary; for example:



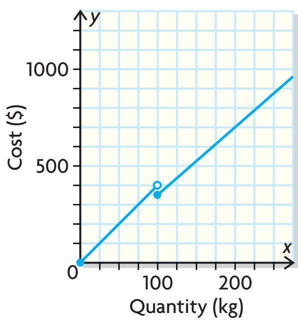
b) Answers may vary; for example:



14. Answers may vary; for example:

Definition:	**Characteristics:**
A relation with only one y-value for each x-value	A vertical line crosses the graph in at most one place
Function	
Examples:	**Non-examples:**
$3x + y = 2$ $y = -2x^2 + 7$	$x^2 + y^2 = 16$ $y = \pm\sqrt{x - 7}$

15. **a)** Each order quantity determines a single cost.
b) domain $= \{x \in \mathbf{R} \mid x \geq 0\}$, range $= \{y \in \mathbf{R} \mid y \geq 0\}$
c)



d) Answers may vary. For example, the company currently charges more for an order of 100 kg ($350) than for an order of 99 kg ($396). A better system would be for the company to charge $50 plus $3.50 per kilogram for orders of 100 kg or more. This would make the prices strictly increasing as the weight of the order increases.

Lesson 1.2, pp. 22–24

1. **a)** -4 **b)** 2 **c)** 14 **d)** $\frac{1}{2}$ **e)** $2 - 3a$ **f)** $2 - 9b$

2. **a)** 2 **b)** 4 **c)** -5 **d)** -3 or -4

3. **a)** $f(x) = 1200 - 3x$
b) 840 mL **c)** 3:10 pm

4. **a)** 8, 0, -0.75 **b)** $-5, -25, -2.5$

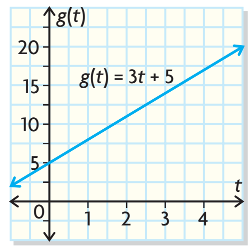
5. **a)** $-\frac{1}{6}$ **b)** undefined **c)** $\frac{1}{3}$ **d)** $2\frac{2}{3}$

6. **a)** domain $= \{-2, 0, 2, 3, 5, 7\}$, range $= \{1, 2, 3, 4, 5\}$
b) **i)** 4 **ii)** 2 **iii)** 5 **iv)** -2

7. **a)** $2a - 5$ **b)** $2b - 3$ **c)** $6c - 7$ **d)** $-10x - 1$

8. **a)**

t	$g(t)$
0	5
1	8
2	11
3	14
4	17
5	20



b) **i)** 5 **ii)** 14 **iii)** 3 **iv)** 3 **v)** 3 **vi)** 3

9. **a)**

s	$f(s)$
0	9
1	4
2	1
3	0

b) **i)** 9 **ii)** 4 **iii)** 1 **iv)** 0 **v)** 2 **vi)** 2
c) They are the same; they represent the second differences, which are constant for a quadratic function.

10. **a)** 49
b) The y-coordinate of the point on the graph with x-coordinate -2
c) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq -1\}$
d) It passes the vertical-line test.

11. **a)** 2 **b)** 0.4 **c)** 0.8 **d)** $\frac{17}{25}$

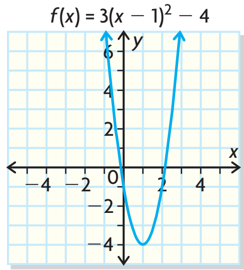
12. **a)** $f(x) = 0.15x + 50$ **b)** $120.80 **c)** 200 km

13. **a)** $f(x) = (24 - 3x)x$ **b)** 45, -195, -60 **c)** 48

14. $f(x) \doteq 0.0034x(281 - x)$

15. **a)**

$f(x) = 3(x - 1)^2 - 4$



b) The y-coordinate of the point on the graph with x-coordinate -1; start from -1 on x-axis, move up to curve, then across to y-axis
c) **i)** 3 **ii)** 9 **iii)** $3x^2 - 4$

16. **a)** 3, -5 **b)** 1, -3 **c)** -1

17. **a)** $\frac{1}{4}$ **b)** $\frac{1}{3}$, -1

Answers

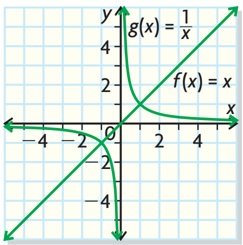
18. Answers may vary; for example: $f(x)$ is defined as equal to an expression involving x, for each x-value in the functions domain; the graph of $f(x)$ is the set of all points $(x, f(x))$ for which x is in the domain. Student examples will vary.

19. **a)** $f(x) = \frac{2}{3}x + 10$ **b)** $73\frac{1}{3}, 126\frac{2}{3}, 153\frac{1}{3}, 180$

20. **a)** 8, 27, 64, 125, 216 **b)** cube of x or x^3

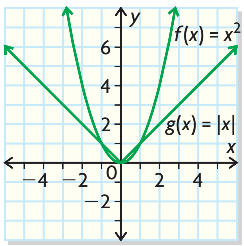
Lesson 1.3, p. 28

1.



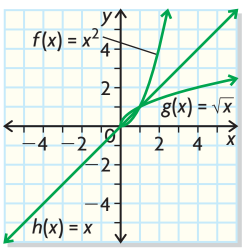
Both graphs lie in quadrants 1 and 3; graph of $g(x)$ is in two curved parts, and does not intersect axes. Vertical asymptote of $g(x)$: $x = 0$; horizontal asymptote of $g(x)$: $g(x) = 0$.

2.



Both graphs lie in quadrants 1 and 2; graph of $f(x)$ curves, but graph of $g(x)$ is formed by two straight half-lines.

3.



Graph of $g(x)$ is reflection of graph of $f(x)$ in graph of $h(x)$.

Lesson 1.4, pp. 35–37

1. **a)** domain $= \{1900, 1920, 1940, 1960, 1980, 2000\}$, range $= \{47.3, 54.1, 62.9, 69.7, 73.7, 77.0\}$
b) domain $= \{-5, -1, 0, 3\}$, range $= \{9, 15, 17, 23\}$
c) domain $= \{-4, 0, 3, 5\}$, range $= \{-1, 0, 3, 5, 7\}$

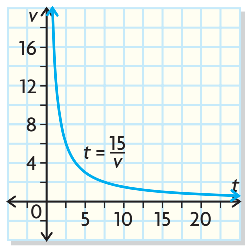
2. **a)** domain $= \{0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10\}$, range $= \{-8, -7, -6, -5, -4, -2, 0, 4, 8\}$
b) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R}\}$
c) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq -8\}$
d) domain $= \{x \in \mathbf{R} \mid -6 \leq x \leq 6\}$, range $= \{y \in \mathbf{R} \mid -6 \leq y \leq 6\}$
e) domain $= \{x \in \mathbf{R} \mid x \leq 6\}$, range $= \{y \in \mathbf{R} \mid y \geq -2\}$
f) domain $= \{x \in \mathbf{R} \mid x \geq -10\}$, range $= \{y \in \mathbf{R} \mid y = -6, -2 \leq y < 2, y > 4\}$

3. **1.** (a), (b); **2.** (b), (c), (e), (f)

4. domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq -3\}$

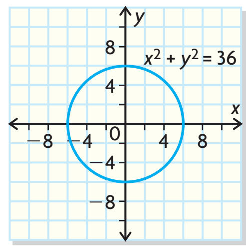
5. **a)** Even at masses when the price changes, a single price (the lower one) is assigned. It would not make sense to assign two or more prices to the same mass.
b) domain $= \{x \in \mathbf{R} \mid 0 < x \leq 500\}$, range $= \{0.52, 0.93, 1.20, 1.86, 2.55\}$

6.



Graph passes vertical line test; domain $= \{v \in \mathbf{R} \mid v > 0\}$, range $= \{t \in \mathbf{R} \mid t > 0\}$

7. **a)**

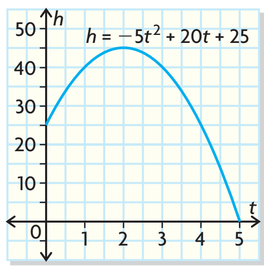


b) domain $= \{x \in \mathbf{R} \mid -6 \leq x \leq 6\}$, range $= \{y \in \mathbf{R} \mid -6 \leq y \leq 6\}$
c) No; fails vertical line test

8. $V(t) = t$; domain $= \{t \in \mathbf{R} \mid 0 \leq t \leq 2500\}$, range $= \{V \in \mathbf{R} \mid 0 \leq V \leq 2500\}$

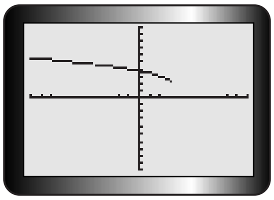
9. **a)** domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R}\}$
b) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \leq 4\}$
c) domain $= \{x \in \mathbf{R} \mid x \geq 1\}$, range $= \{y \in \mathbf{R} \mid y \geq 0\}$
d) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq -5\}$
e) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R}\}$
f) domain $= \{x \in \mathbf{R} \mid x \leq 5\}$, range $= \{y \in \mathbf{R} \mid y \geq 0\}$

10. **a)**

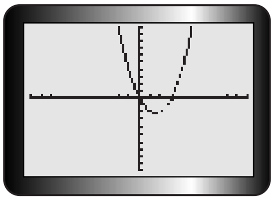


b) domain $= \{t \in \mathbf{R} \mid 0 \leq t \leq 5\}$, range $= \{h \in \mathbf{R} \mid 0 \leq h \leq 45\}$
c) $h = -5t^2 + 20t + 25$

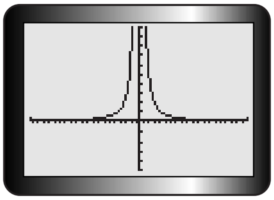
11. **a)** domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R}\}$
b) domain $= \{x \in \mathbf{R} \mid x \geq 2\}$, range $= \{y \in \mathbf{R} \mid y \geq 0\}$
c) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq -4\}$
d) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \leq -5\}$

12. a) 

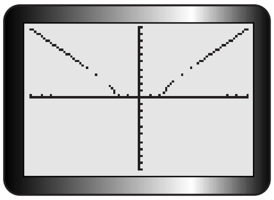
domain $= \{x \in \mathbf{R} \mid x \leq 3\}$, range $= \{y \in \mathbf{R} \mid y \geq 2\}$

b) 

domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq -2.25\}$

c) 

domain $= \{x \in \mathbf{R} \mid x \neq 0\}$, range $= \{y \in \mathbf{R} \mid y > 0\}$

d) 

domain $= \{x \in \mathbf{R} \mid x \leq -\sqrt{5}, x \geq \sqrt{5}\}$,
range $= \{y \in \mathbf{R} \mid y \geq 0\}$

13. a) $A = \left(\frac{450 - 3w}{2}\right)w$

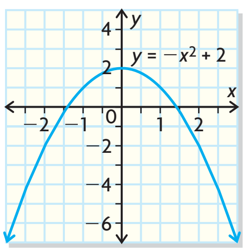
b) domain $= \{w \in \mathbf{R} \mid 0 < w < 150\}$,
range $= \{A \in \mathbf{R} \mid 0 < A \leq 8437.5\}$

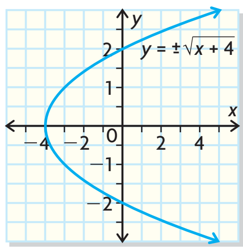
c) $l = 112.5$ m, $w = 75$ m

14. a) $\{-14, -3.5, 4, 7, 13\}$ b) $\{1, 6, 28, 55\}$

15. The domain is the set of x-values for a relation or function; the range is the set of y-values corresponding to these x-values. Domain and range are determined by values in x-column and y-column; x-coordinates and y-coordinates of graph; x-values for which relation or function is defined, and all possible corresponding y-values. Students' examples will vary.

16. a) Answers may vary; for example: b) Answers may vary; for example:





17. a) $A = x^2 + (10 - x)^2$ or $2x^2 - 20x + 100$

b) domain $= \{x \in \mathbf{R} \mid 0 \leq x \leq 10\}$,
range $= \{A \in \mathbf{R} \mid 50 \leq A \leq 100\}$

c) $P = 4\sqrt{x^2 + (10 - x)^2}$ or $\sqrt{2x^2 - 20x + 100}$

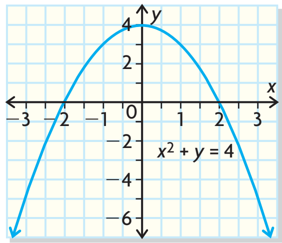
d) domain $= \{x \in \mathbf{R} \mid 0 \leq x \leq 10\}$,
range $= \{P \in \mathbf{R} \mid 50\sqrt{2} \leq P \leq 400\}$

Mid-Chapter Review, p. 40

1. a) Not a function
b) Function; each x-value goes to a single y-value
c) Function; passes vertical line test
d) Not a function
e) Function; each x-value determines a single y-value
f) Function; each x-value determines a single y-value

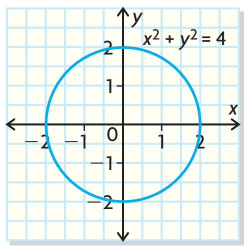
2. $x^2 + y = 4$:

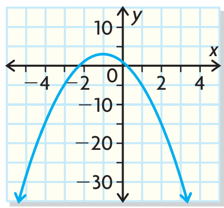
x	y
−3	−5
−2	0
−1	3
0	4
1	3
2	0
3	−5



$x^2 + y^2 = 4$:

x	y
−2	0
0	±2
2	0



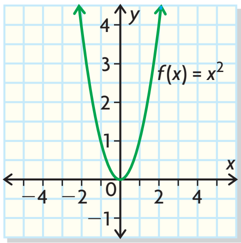
3. a) 

$f(x) = -2(x + 1)^2 + 3$

b) -5
c) y-coordinate of the point on the graph with x-coordinate -3
d) i) -6 ii) -50 iii) $-2(3 - x)^2 + 3$

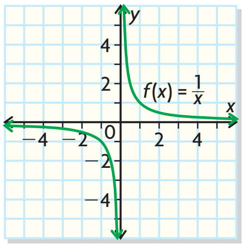
4. a) $f(x) = (20 - 5x)x$ b) $15, -25, -105$ c) 20

5. a)



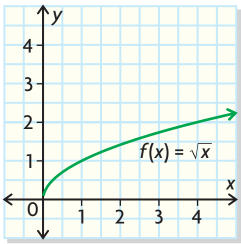
domain $= \{x \in \mathbf{R}\}$,
range $= \{y \in \mathbf{R} \mid y \geq 0\}$

b)



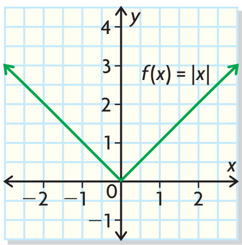
domain $= \{x \in \mathbf{R} \mid x \neq 0\}$,
range $= \{y \in \mathbf{R} \mid y \neq 0\}$

c)



domain $= \{x \in \mathbf{R} \mid x \geq 0\}$,
range $= \{y \in \mathbf{R} \mid y \geq 0\}$

d)



domain $= \{x \in \mathbf{R}\}$,
range $= \{y \in \mathbf{R} \mid y \geq 0\}$

6. a) domain $= \{1, 2, 4\}$, range $= \{2, 3, 4, 5\}$
b) domain $= \{-2, 0, 3, 7\}$, range $= \{-1, 1, 3, 4\}$
c) domain $= \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$,
range $= \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
d) domain $= \{x \in \mathbf{R} \mid x \geq -3\}$, range $= \{y \in \mathbf{R}\}$
e) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \leq 5\}$
f) domain $= \{x \in \mathbf{R} \mid x \geq 4\}$, range $= \{y \in \mathbf{R} \mid y \geq 0\}$

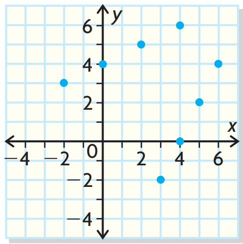
7. a) $A = \left(\frac{600 - 4w}{2}\right)w$

b) domain $= \{w \in \mathbf{R} \mid 0 < w < 150\}$,
range $= \{A \in \mathbf{R} \mid 0 < A \leq 11\ 250\}$
c) $l = 150$ m, $w = 75$ m

8. a) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \leq 5\}$
b) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq 4\}$
c) domain $= \{x \in \mathbf{R} \mid -7 \leq x \leq 7\}$,
range $= \{y \in \mathbf{R} \mid -7 \leq y \leq 7\}$
d) domain $= \{x \in \mathbf{R} \mid -2 \leq x \leq 6\}$,
range $= \{y \in \mathbf{R} \mid 1 \leq y \leq 9\}$

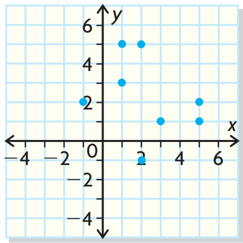
Lesson 1.5, pp. 46–49

1. a) $\{(3, -2), (4, 0), (5, 2), (6, 4)\}$



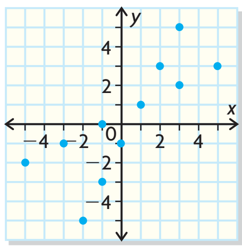
Both relation and inverse relation are functions.

b) $\{(5, 2), (-1, 2), (1, 3), (1, 5)\}$



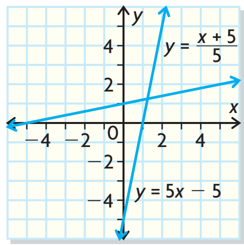
Both relation and inverse relation are not functions.

2. a)



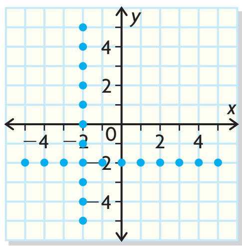
Function, point (1, 1) is common

d)



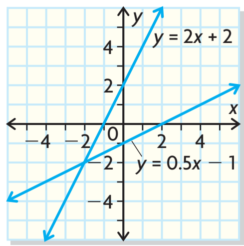
Function, point (1.25, 1.25) is common

b)



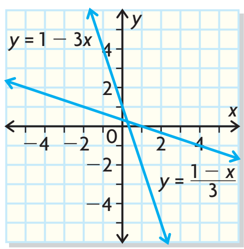
Not a function, point $(-2, -2)$ is common

e)



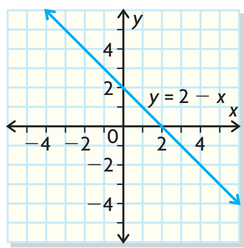
Function, point $(-2, -2)$ is common

c)



Function, point (0.25, 0.25) is common

f)



Function, all points are common. The function and its inverse are the same graph.

3. **a)** No **b)** Yes

4. **a)** $y = \dfrac{x+3}{4}$ **c)** $y = \dfrac{6-4x}{3}$

b) $y = 2(2 - x)$ **d)** $y = \dfrac{2x - 10}{5}$

5. **a)** $f^{-1}(x) = x + 4$ **d)** $f^{-1}(x) = 2(x + 1)$

b) $f^{-1}(x) = \dfrac{x-1}{3}$ **e)** $f^{-1}(x) = \dfrac{6-x}{5}$

c) $f^{-1}(x) = \dfrac{1}{5}x$ **f)** $f^{-1}(x) = \dfrac{4}{3}(x - 2)$

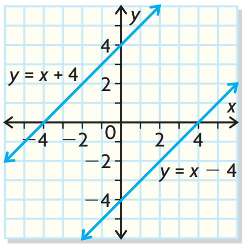
6. **a)** $f^{1}(x) = x - 7$ **d)** $f^{1}(x) = -5(x + 2)$

b) $f^{1}(x) = 2 - x$ **e)** $f^{1}(x) = x$

c) $x = 5$ **f)** $f^{-1}(x) = 4x + 3$

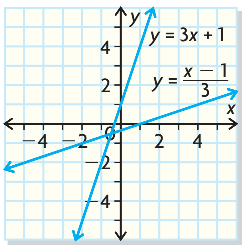
7. Question 5.

a)



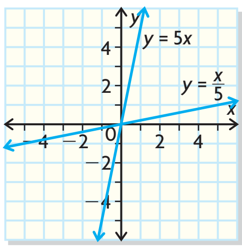
Function, linear

b)



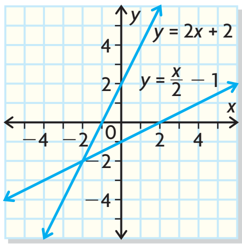
Function, linear

c)



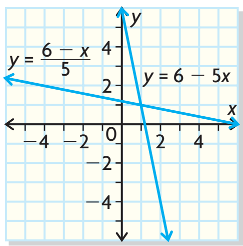
Function, linear

d)



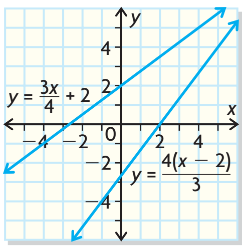
Function, linear

e)



Function, linear

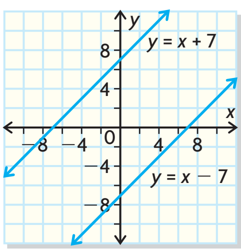
f)



Function, linear

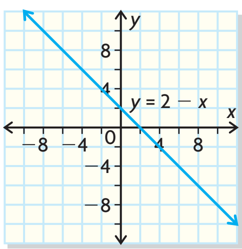
Question 6.

a)



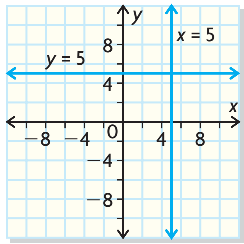
Function, linear

b)



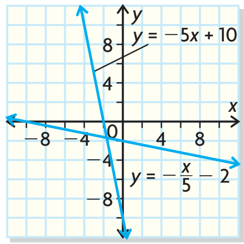
Function, linear
The function and its inverse are the same graph.

c)



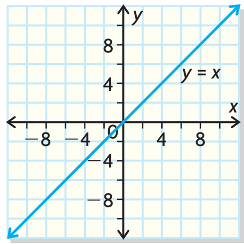
Not a function, linear

d)



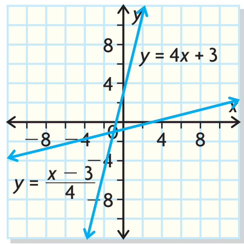
Function, linear

e)



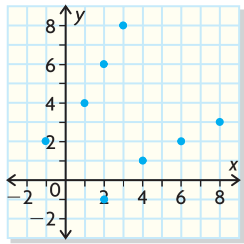
Function, linear
The function and its inverse are the same graph.

f)



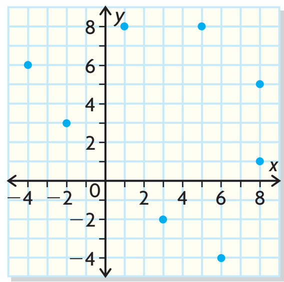
Function, linear

8. **a)** $\{(2, -1), (4, 1), (6, 2), (8, 3)\}$



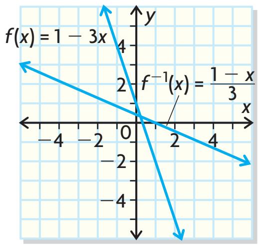
Function: domain $= \{-1, 1, 2, 3\}$, range $= \{2, 4, 6, 8\}$;
inverse: domain $= \{2, 4, 6, 8\}$, range $= \{-1, 1, 2, 3\}$;
domain, range are interchanged

b) $\{(6, -4), (3, -2), (8, 1), (8, 5)\}$



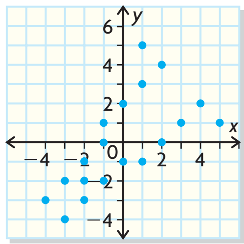
Function: domain $= \{-4, -2, 1, 5\}$, range $= \{3, 6, 8\}$;
inverse: domain $= \{3, 6, 8\}$, range $= \{-4, -2, 1, 5\}$;
domain, range are interchanged

c) $f^{-1}(x) = \dfrac{1 - x}{3}$



Function: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R}\}$;
inverse: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R}\}$;
domain, range are identical for both

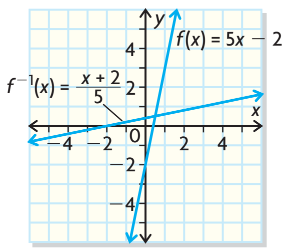
d) $\{(-3, -4), (-2, -3), (-2, -2), (-2, -1), (-1, 0), (-1, 1), (0, 2), (1, 3), (2, 4), (1, 5)\}$



Function: domain $= \{0, \pm1, \pm2, \pm3, \pm4, 5\}$, range $= \{0, \pm1, \pm2, -3\}$; inverse: domain $= \{0, \pm1, \pm2, -3\}$, range $= \{0, \pm1, \pm2, \pm3, \pm4, 5\}$; domain, range are interchanged

9. a) $f^{-1}(x) = \dfrac{x + 2}{5}$

b)

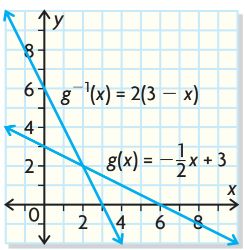


c) Function equation is linear in x; or, graph is a straight line.

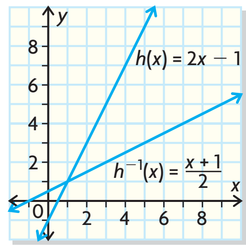
d) $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$

e) Slopes are reciprocals of each other.

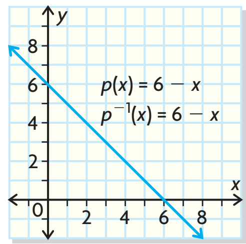
f) $g^{-1}(x) = 2(3 - x)$



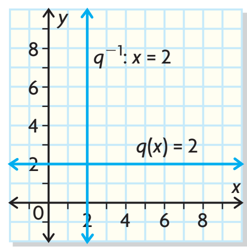
Function equation is linear in x, or, graph is a straight line; $(2, 2)$; slopes are reciprocals of each other. $h^{-1}(x) = \dfrac{x + 1}{2}$



Function equation is linear in x, or, graph is a straight line; $(1, 1)$; slopes are reciprocals of each other. $p^{-1}(x) = 6 - x$



Function equation is linear in x, or, graph is a straight line; all points on graphs; slopes are equal. q^{-1} is the relation $x = 2$



In this case, q^{-1} is not a function, but its graph is a straight line; $(2, 2)$; slopes are 0 and undefined.

10. a) 37 b) 19 c) 3 d) 5 e) 3 f) $\dfrac{1}{3}$

11. c) is slope of $g(t)$; f) is slope of $g^{-1}(t)$

12. a) $f(x) = 2x + 30$

b) Subtract 30, then halve; a Canadian visiting the United States

c) $f^{1}(x) = \dfrac{1}{2}(x - 30)$

d) $f(14) = 2(14) + 30 = 58\ °F$

e) $f^{1}(70) = \dfrac{1}{2}(70 - 30) = 20\ °C$

13. a) Multiply by 10, then divide by 4

b) A Canadian visiting the U.S.

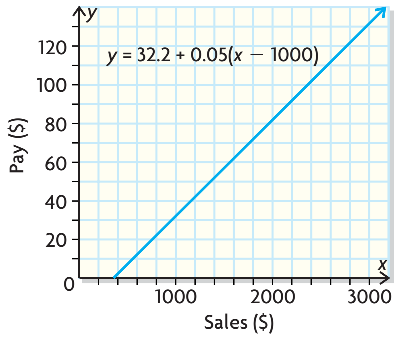
c) $g(x) = \dfrac{4x}{10}$; $g^{-1}(x) = \dfrac{10x}{4}$

d) $g(15) = \dfrac{4(15)}{10} = 6$ in.

e) $g^{-1}(70) = \dfrac{10(70)}{4} = 175$ cm

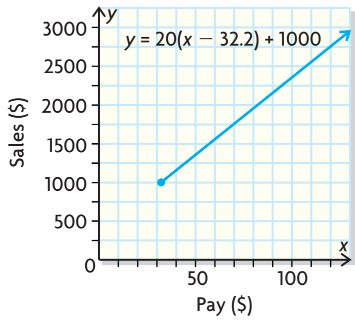
14. $y = 0.38x + 0.50$

15. **a)**



b) $f(x) = 32.2 + 0.05(x - 1000)$

c)



d) $f^{1}(x) = 1000 + 20(x - 32.2)$

e) $f^{-1}(420) = 1000 + 20(420 - 32.2) = \$8756$

16. Because (1, 2) cannot belong to *f*, as well as (1, 5).

17. $k = 2$

18.

Definition:	**Methods:**
Inverse of a function of form $f(x) = mx + c$	Switch x and y and solve for y Take reciprocal of slope, switch x- and y-intercepts
Examples: $f(x) = 3x + 2, f^{-1}(x) = \frac{x-2}{3}$ $g(x) = \frac{5}{7}(2 - 5x), g^{-1}(x) = \frac{2}{5} - \frac{7}{25}x$	**Properties:** Has form $f^{-1}(x) = mx + c$ or $x = c$ Graph is straight line

Inverse of a Linear Function

19. Answers may vary; for example: $y = x$; $y = -x$; $y = 1 - x$

20. $(f^{-1})^{-1}(x) = 3x + 4$

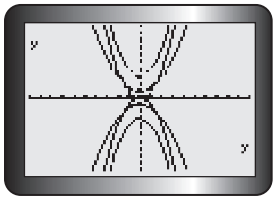
Lesson 1.6, p. 51

1. **a)** Upper half-parabola, opening right, vertex at (1, 2)
b) V-shape, opening up, vertex at (1, 2)
c) Hyperbola, asymptotes $x = 1$ and $y = 2$, graph lying to upper right and lower left of asymptotes

2. **a)** Graph of $y = \sqrt{x}$ is upper half, graph of $y = -\sqrt{x}$ lower half of parabola opening right.
b) Graph of $y = |x|$ opens up, graph of $y = -|x|$ opens down.
c) Graph of $y = \frac{1}{x}$ lies to upper right and lower left of asymptotes, graph of $y = -\frac{1}{x}$ lies to lower right and upper left.

3. **a)** Graph of $y = 2\sqrt{x}$ is narrower (steeper) than graph of $y = \sqrt{x}$.
b) Graph of $y = 2|x|$ is narrower (steeper) than graph of $y = |x|$.
c) Graph of $y = \frac{2}{x}$ is narrower (steeper) than graph of $y = \frac{1}{x}$.

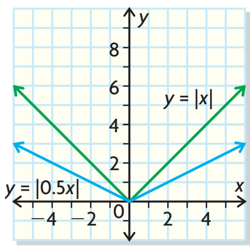
4. Answers will vary. For example,



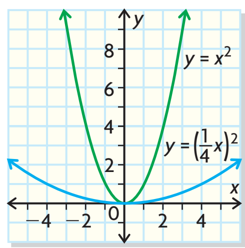
Lesson 1.7, pp. 58–60

1. **a)** $y = (3x)^2$ **b)** $y = \sqrt{-\frac{1}{2}x}$

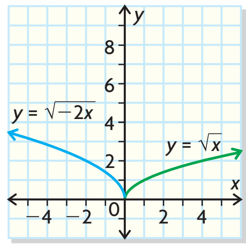
2. **a)** $y = |x|$; horizontal stretch factor 2



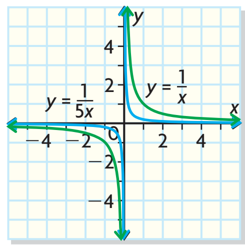
b) $y = x^2$; horizontal stretch factor 4



c) $y = \sqrt{x}$; horizontal compression factor $\frac{1}{2}$ and reflection in y-axis

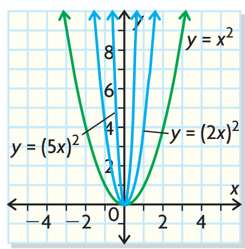


d) $y = \frac{1}{x}$; horizontal compression factor $\frac{1}{5}$



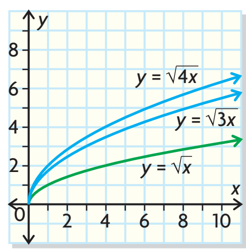
3. **a)** (1.5, 4) **b)** (6, 4) **c)** (9, 4) **d)** (−0.75, 4)

4. **a)**



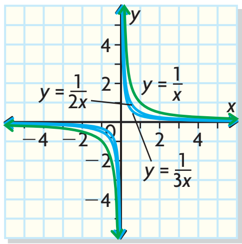
Horizontal compression factor $\frac{1}{2}$, (0, 0); horizontal compression factor $\frac{1}{5}$, (0, 0)

b)

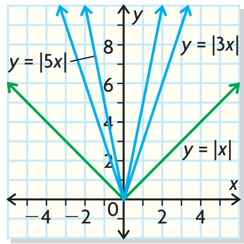


Horizontal compression factor $\frac{1}{3}$, (0, 0); horizontal compression factor $\frac{1}{4}$, (0, 0)

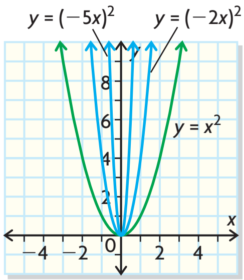
Answers

c) 

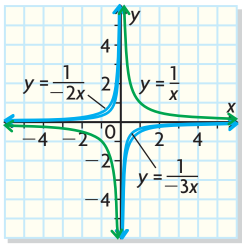
Horizontal compression factor $\frac{1}{2}$, no invariant points; horizontal compression factor $\frac{1}{3}$, no invariant points

d) 

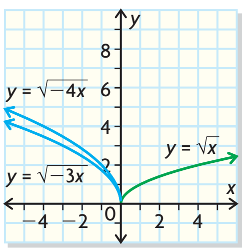
Horizontal compression factor $\frac{1}{3}$, (0, 0); horizontal compression factor $\frac{1}{5}$, (0, 0)

5. a) 

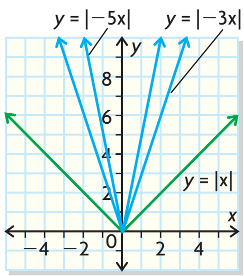
Horizontal compression factor $\frac{1}{2}$ and (optional) reflection in y-axis, (0, 0); horizontal compression factor $\frac{1}{5}$ and (optional) reflection in y-axis, (0, 0)

c) 

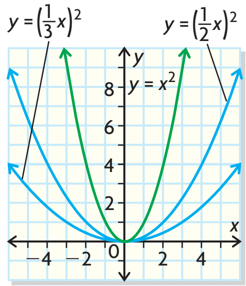
Horizontal compression factor $\frac{1}{2}$ and reflection in y-axis, no invariant points; horizontal compression factor $\frac{1}{3}$ and reflection in y-axis, no invariant points

b) 

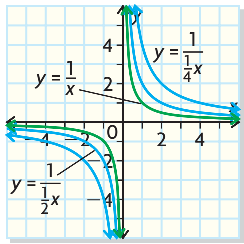
Horizontal compression factor $\frac{1}{3}$ and reflection in y-axis, (0, 0); horizontal compression factor $\frac{1}{4}$ and reflection in y-axis, (0, 0)

d) 

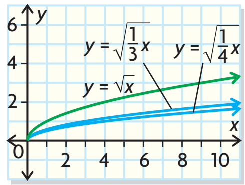
Horizontal compression factor $\frac{1}{3}$ and (optional) reflection in y-axis, (0, 0); horizontal compression factor $\frac{1}{5}$ and (optional) reflection in y-axis, (0, 0)

6. a) 

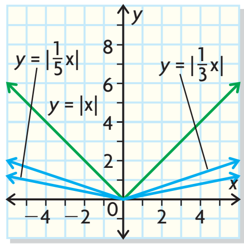
Horizontal stretch factor 2, (0, 0); horizontal stretch factor 3, (0, 0)

c) 

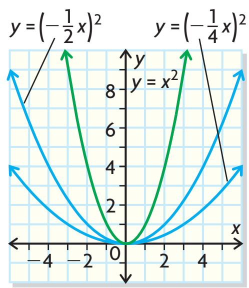
Horizontal stretch factor 2, no invariant points; horizontal stretch factor 4, no invariant points

b) 

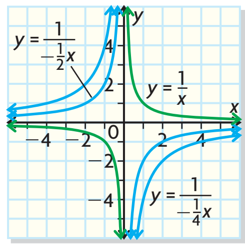
Horizontal stretch factor 2, (0, 0); horizontal stretch factor 3, (0, 0)

d) 

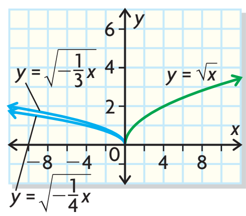
Horizontal stretch factor 3, (0, 0); horizontal stretch factor 5, (0, 0)

7. a) 

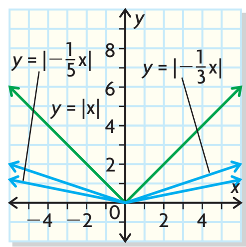
Horizontal stretch factor 2 and (optional) reflection in y-axis, (0, 0); horizontal stretch factor 3 and (optional) reflection in y-axis, (0, 0)

c) 

Horizontal stretch factor 2 and reflection in y-axis, no invariant points; horizontal stretch factor 4 and reflection in y-axis, no invariant points

b) 

Horizontal stretch factor 2 and reflection in y-axis, (0, 0); horizontal stretch factor 3 and reflection in y-axis, (0, 0)

d) 

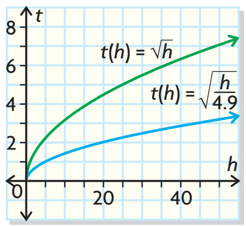
Horizontal stretch factor 3 and (optional) reflection in y-axis, (0, 0); horizontal stretch factor 5 and (optional) reflection in y-axis, (0, 0)

8. **a)** $g(x) = |2x|$ **b)** $g(x) = \dfrac{1}{-2x}$ **c)** $g(x) = \left(\dfrac{1}{4}x\right)^2$

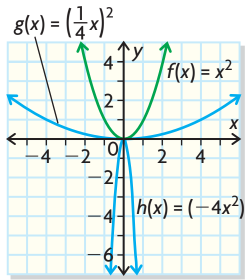
d) $g(x) = \sqrt{-3x}$

9. **a)** domain: $\{h \in \mathbf{R} \mid h \geq 0\}$; range: $\{t \in \mathbf{R} \mid t \geq 0\}$

b)

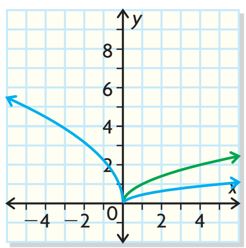


10. **a)**



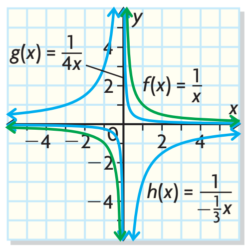
$g(x)$: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq 0\}$;
$h(x)$: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \leq 0\}$

b)



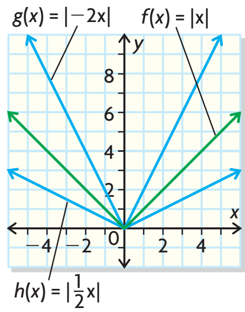
$g(x)$: domain $= \{x \in \mathbf{R} \mid x \geq 0\}$, range $= \{y \in \mathbf{R} \mid y \geq 0\}$;
$h(x)$: domain $= \{x \in \mathbf{R} \mid x \leq 0\}$, range $= \{y \in \mathbf{R} \mid y \geq 0\}$

c)



$g(x)$: domain $= \{x \in \mathbf{R} \mid x \neq 0\}$, range $= \{y \in \mathbf{R} \mid y \neq 0\}$;
$h(x)$: domain $= \{x \in \mathbf{R} \mid x \neq 0\}$, range $= \{y \in \mathbf{R} \mid y \neq 0\}$

d)



$g(x)$: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq 0\}$;
$h(x)$: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq 0\}$

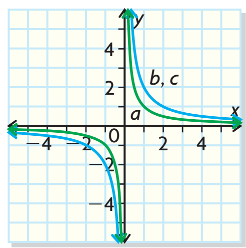
11. **a)** $\dfrac{1}{4}$ **b)** 2 **c)** -1 **d)** -5

12. **a)** $1.5, -1$ **b)** $9, -6$ **c)** $-1, \dfrac{2}{3}$

13. **a)** For $k > 1$, effect is a horizontal compression with scale factor $\dfrac{1}{k}$; for $0 < k < 1$, a horizontal stretch with scale factor $\dfrac{1}{k}$; for $k < 0$, reflection in the y-axis, then a horizontal compression or stretch with scale factor $\dfrac{1}{|k|}$. Apply these transformations to the graph of $y = f(x)$ to sketch the graph of $y = f(kx)$.

b) Answers may vary; for example: A horizontal compression or stretch is equivalent to a vertical stretch or compression, respectively; scale factors are reciprocals of each other for some functions, such as $f(x) = ax$, $f(x) = a|x|$, and $f(x) = \dfrac{a}{x}$, but not for others, such as $f(x) = ax^2 + bx + c$ and $f(x) = a\sqrt{x - d} + c$.

14. **a)–c)**



c) The horizontal and vertical stretches give the same graph.

d) $y = \dfrac{1}{\frac{1}{2}x}$, $y = 2\left(\dfrac{1}{x}\right)$; both equations simplify to $y = \dfrac{2}{x}$

15. Translation 4 units left, then horizontal compression factor $\dfrac{1}{2}$; yes; check students' parent function investigations

Lesson 1.8, pp. 70–73

1. A: vertical stretch, factor 5; B: reflection in y-axis; C: horizontal compression, factor 5; D: translation 2 units right; E: translation 4 units up

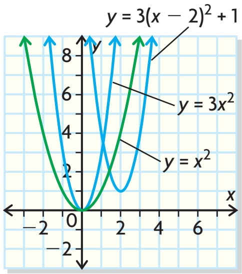
2. Divide the x-coordinates by 3: C; Multiply the y-coordinates by 5: A; Multiply the x-coordinates by -1: B; Add 4 to the y-coordinate: E; Add 2 to the x-coordinate: D

3.

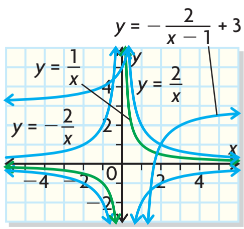
$f(x)$	$f(3x)$	$f(-3x)$	$5f(-3x)$	$5f[-3(x-2)]+4$
$(1, 1)$	$\left(\frac{1}{3}, 1\right)$	$\left(-\frac{1}{3}, 1\right)$	$\left(-\frac{1}{3}, 5\right)$	$\left(1\frac{2}{3}, 9\right)$

4. **a)** Vertical stretch, factor 3, then translation 1 unit down
b) Translation 2 units right and 3 units up
c) Horizontal compression, factor $\frac{1}{2}$, then translation 5 units down
d) Reflection in x-axis, horizontal stretch with factor 2, and then translation 2 units down
e) Vertical compression, factor $\frac{2}{3}$, then translation 3 units left and 1 unit up
f) Vertical stretch with factor 4, reflection in y-axis, and then translation 4 units down

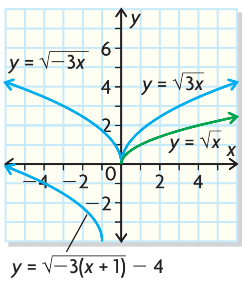
5. **a)**



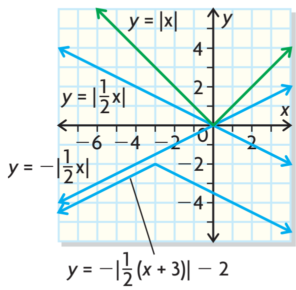
c)



b)

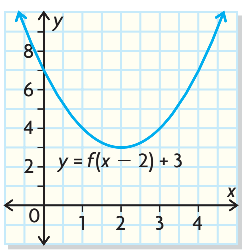


d)



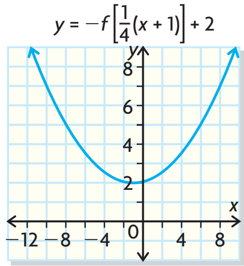
6. **a)** Horizontal stretch, factor 3, then translation 4 units left
b) Vertical stretch with factor 2, reflection in y-axis, and then translation 3 units right and 1 unit up
c) Reflection in x-axis, vertical stretch with factor 3, horizontal compression with factor $\frac{1}{2}$, then translation 1 unit right and 3 units down

7. **a)**



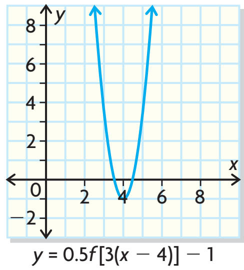
domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq 3\}$

b)



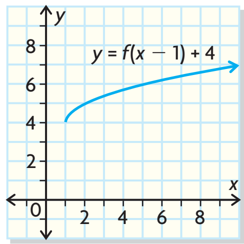
domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq 2\}$

c)



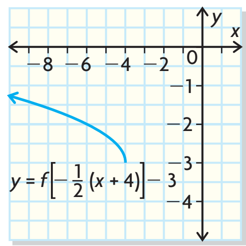
domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \leq -3\}$

8. **a)**



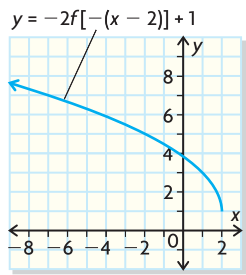
domain $= \{x \in \mathbf{R} \mid x \geq 1\}$, range $= \{y \in \mathbf{R} \mid y \geq 4\}$

b)



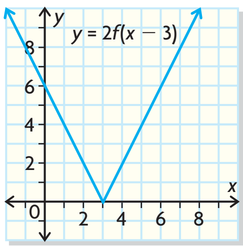
domain $= \{x \in \mathbf{R} \mid x \leq -4\}$, range $= \{y \in \mathbf{R} \mid y \geq -3\}$

c)



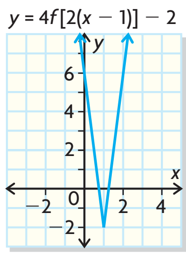
domain $= \{x \in \mathbf{R} \mid x \leq 2\}$, range $= \{y \in \mathbf{R} \mid y \leq 1\}$

9. **a)**



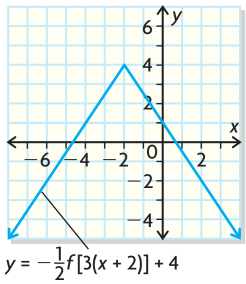
domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq 0\}$

b)



domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq -2\}$

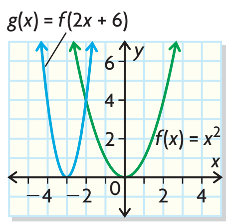
c)



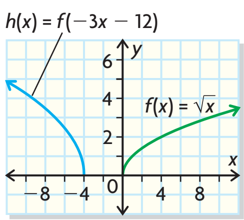
domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq 4\}$

10. **a)** Translation right 2
b) Translation up 2
c) Vertical compression, factor 0.5
d) Vertical stretch, factor 2
e) Horizontal compression, factor 0.5
f) Reflection in *x*-axis

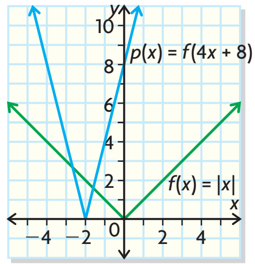
11.



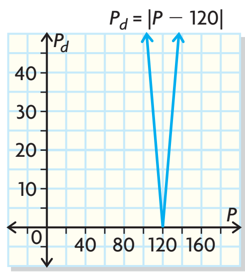
12.



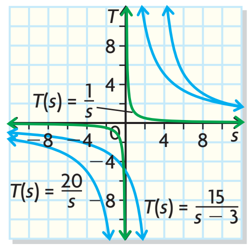
13.



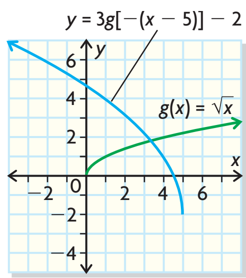
14.



15.



16.

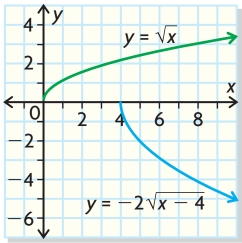


$y = 3\sqrt{-(x - 5)} - 2$

17. $g(x) = 3f[-(x + 1)] + 2$

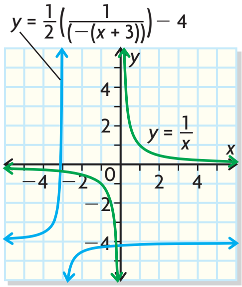
18. **a)** C; parent graph is $y = \frac{1}{x}$, asymptotes are translated 2 units right and 1 unit up, and graph has been reflected in one of the axes
b) E; parent graph is $y = |x|$, and vertex is translated 3 units right and 2 units down
c) A: parent graph is $y = \sqrt{x}$, graph has been reflected in *y*-axis, and vertex is translated 3 units left and 2 units down
d) G: parent graph is $y = x^2$, and vertex is translated 2 units right and 3 units down
e) F; parent graph is $y = \frac{1}{x}$, asymptotes are translated 3 units down, and graph has been reflected in one of the axes
f) D; parent graph is $y = |x|$, graph has been reflected in *y*-axis, and vertex is translated 4 units left and 2 units up
g) H: parent graph is $y = \sqrt{x}$, graph has been reflected in *x*- and *y*-axes, and vertex is translated 1 unit right and 1 unit up
h) B: parent graph is $y = x^2$, graph has been reflected in *y*-axis, and vertex is translated 4 units left and 1 unit up

19. **a)** $a = -2, k = 1, c = 0, d = 4$



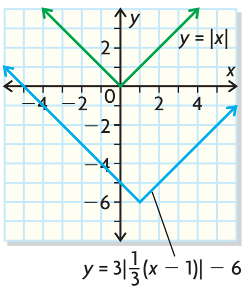
domain $= \{x \in \mathbf{R} \mid x \geq 4\}$, range $= \{y \in \mathbf{R} \mid y \leq 0\}$

b) $a = \frac{1}{2}, k = -1, c = -3, d = -4$



domain $= \{x \in \mathbf{R} \mid x \neq -3\}$, range $= \{y \in \mathbf{R} \mid y \neq -4\}$

c) $a = 3, k = \frac{1}{3}, c = -6, d = 1$



domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq -6\}$

20. **a)** $2, -5$ **b)** $2, -5$ **c)** $-\frac{2}{3}, 1\frac{2}{3}$ **d)** $-4, 3$

21. A. Sketch parent function; B. Apply reflections in x-axis if $a < 0$ and in y-axis if $k < 0$; apply vertical stretch or compression with factor $|a|$, and stretch or compression with factor $\frac{1}{|k|}$; D. Translate c units right (or $-c$ units left if $c < 0$) and d units up (or $-d$ units down if $d < 0$). Transformations in steps B and C can be done in any order, but must precede translation in step D.

22. **a)** Reflection in x-axis, vertical compression factor $\frac{1}{4}$ [or horizontal stretch factor 2], and then translation 3 units left and 1 unit up

b) $y = -\frac{1}{4}(x + 3)^2 + 2 \left[\text{or } y = -\left[\frac{1}{2}(x + 3)\right]^2 + 2\right]$

23. Graphs are both based on a parabola, but open in different directions, and graph of $g(x)$ is only an upper half-parabola. Reflect right half of graph of $f(x)$ in the line $y = x$.

Chapter Review, pp. 76–77

1. **a)** domain $= \{-3, -1, 0, 4\}$, range $= \{0, 1, 5, 6\}$; not a function, because two y-values are assigned to $x = 0$

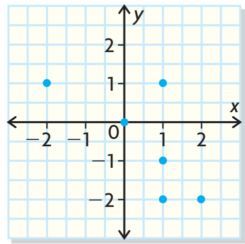
b) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R}\}$; function, because each x-value has only one y-value assigned

c) domain $= \{x \in \mathbf{R} \mid x \geq -4\}$, range $= \{y \in \mathbf{R}\}$; not a function, because each $x > -4$ has two y-values assigned

d) domain $= \{x \in \mathbf{R} \mid -4 \leq x \leq 4\}$, range $= \{y \in \mathbf{R} \mid -4 \leq y \leq 4\}$; not a function, because each x except ± 4 has two y-values assigned

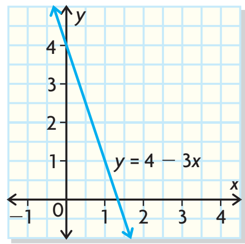
2. Vertical-line test

a)



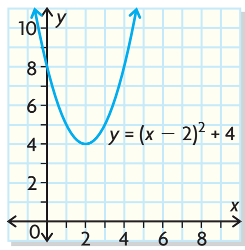
Not a function

b)



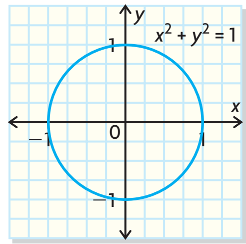
Function

c)



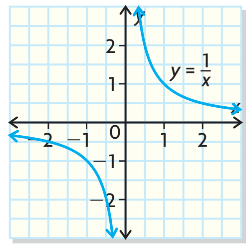
Function

d)



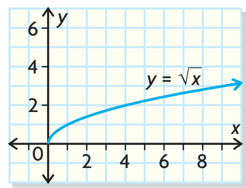
Not a function

e)



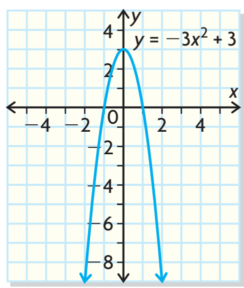
Function

f)



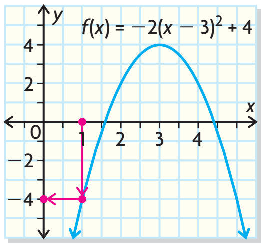
Function

3. Answers may vary; for example:



4. **a)** -7 **b)** -5 **c)** -2 **d)** $4b^2 + 6b - 5$ **e)** $-8a - 1$ **f)** 1 or -2

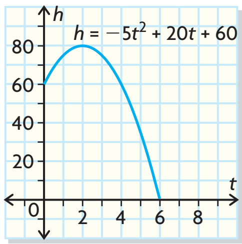
5. **a), b)**



a) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \le 4\}$
b) $f(1)$ represents the y-coordinate corresponding to $x = 1$.
c) **i)** 2 **ii)** -1 **iii)** $-2(-x - 2)^2 + 4$

6. 5, -1

7. **a)**



b) domain $= \{t \in \mathbf{R} \mid 0 \le t \le 6\}$, range $= \{h \in \mathbf{R} \mid 0 \le h \le 80\}$
c) $h = -5t^2 + 20t + 60$

8. **a)** domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \ge 3\}$
b) domain $= \{x \in \mathbf{R} \mid x \ge -2\}$, range $= \{y \in \mathbf{R} \mid y \ge 0\}$

9. **a)** $A(w) = \left(\dfrac{540 - 3w}{2}\right)w$
b) domain $= \{w \in \mathbf{R} \mid 0 < w < 180\}$, range $= \{A \in \mathbf{R} \mid 0 < A \le 12\,150\}$
c) $l = 270$ m, $w = 90$ m

10. **a)** Graph $y = 2x - 5$, and reflect it in the line $y = x$ to get graph of inverse. Use graph to determine the slope-intercept form of inverse; slope is 0.5 and y-intercept is 2.5, so $f^{-1}(x) = 0.5x + 2.5$.
b) Switch x and y, then solve for y:

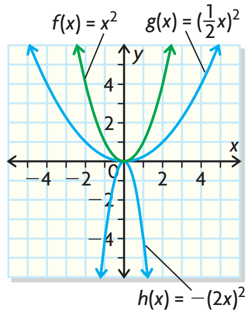
$$y = \frac{x + 3}{7}$$
$$x = \frac{y + 3}{7}$$
$$7x = y + 3$$
$$7x - 3 = y$$
$$f^{-1}(x) = 7x - 3$$

c) Reverse operations: for f, divide by 2 and subtract from 4, so for f^{-1}, subtract from 4 (operation is self-inverse) and multiply by 2. Therefore, $f^{-1}(x) = 2(4 - x)$.

11. **a)** $f(x) = 30x + 15\,000$
b) domain $= \{x \in \mathbf{R} \mid x \ge 0\}$, range $= \{y \in \mathbf{R} \mid y \ge 15\,000\}$; number of people cannot be negative, and income cannot be less than corporate sponsorship
c) $f^{-1}(x) = \dfrac{x - 15\,000}{30}$; domain $= \{x \in \mathbf{R} \mid x \ge 15\,000\}$

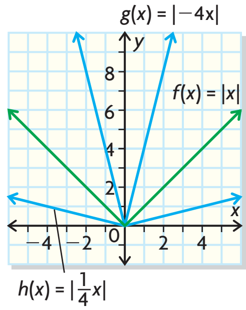
12. **a)** $y = \sqrt{4x}$
b) $y = \dfrac{1}{-\frac{1}{5}x}$

13. **a)**



$f(x)$: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \ge 0\}$;
$g(x)$: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \ge 0\}$;
$h(x)$: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \le 0\}$

b)

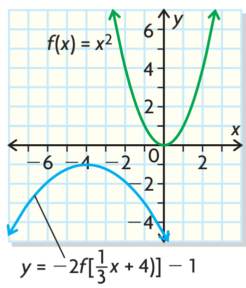


$f(x)$: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \ge 0\}$;
$g(x)$: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \ge 0\}$;
$h(x)$: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \ge 0\}$

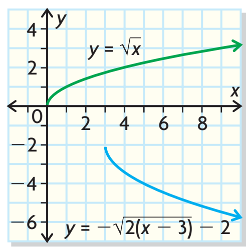
14. **a)** Yes; translations must be done last.
b) Yes: vertical stretch with factor 2, translation 4 units down, and translation 3 units right

15. $(-5, 10)$

16. **a)** Reflection in x-axis, vertical stretch factor 2, horizontal stretch factor 3, then translation left 4 and down 1
b)

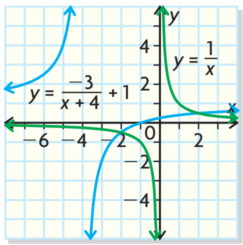


17. **a)** $y = -\sqrt{2(x - 3)} - 2$



domain $= \{x \in \mathbf{R} \mid x \ge 3\}$, range $= \{y \in \mathbf{R} \mid y \le -2\}$

b) $y = \dfrac{-3}{x+4} + 1$



domain $= \{x \in \mathbf{R} \mid x \neq -4\}$, range $= \{y \in \mathbf{R} \mid y \neq 1\}$

18. **a)** $4, -3$ **b)** $4, -3$ **c)** $-8, 6$ **d)** $-5, 2$

19. **a)** domain $= \{x \in \mathbf{R} \mid x > -4\}$, range $= \{y \in \mathbf{R} \mid y < -2\}$

b) domain $= \{x \in \mathbf{R} \mid x < 4\}$, range $= \{y \in \mathbf{R} \mid y < -1\}$

c) domain $= \{x \in \mathbf{R} \mid x > -5\}$, range $= \{y \in \mathbf{R} \mid y < 1\}$

d) domain $= \{x \in \mathbf{R} \mid x < -1\}$, range $= \{y \in \mathbf{R} \mid y < 3\}$

Chapter Self-Test, p. 78

1. **a)** domain $= \{-5, -2, 0, 3\}$, range $= \{-1, 1, 7\}$; function, because each x-value has only one y-value assigned

b) domain $= \{x \in \mathbf{R} \mid x \geq -2\}$, range $= \{y \in \mathbf{R} \mid y \geq 0\}$; function, same reason as part (a)

2. **a)** $f(x) = 0.004x + 0.65$, $g(x) = 0.001x + 3.50$

b) f: domain $= \{x \in \mathbf{R} \mid x \geq 0\}$, range $= \{y \in \mathbf{R} \mid y \geq 0.65\}$; g: domain $= \{x \in \mathbf{R} \mid x \geq 0\}$, range $= \{y \in \mathbf{R} \mid y \geq 3.50\}$

c) 950 h

d) Regular bulb costs \$3.72 more than fluorescent.

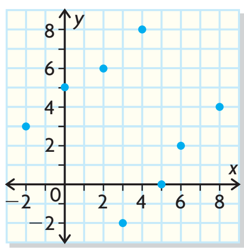
3. **a)** domain $= \{x \in \mathbf{R} \mid x \neq 2\}$, range $= \{y \in \mathbf{R} \mid y \neq 0\}$

b) domain $= \{x \in \mathbf{R} \mid x \leq 3\}$, range $= \{y \in \mathbf{R} \mid y \geq -4\}$

c) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \leq 3\}$

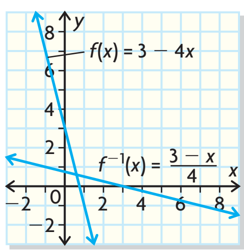
4. The inverse of a linear function is either the linear function obtained by reversing the operations of the original function, or if the original function is $f(x) = c$ constant, the relation $x = c$. Domain and range are exchanged for the inverse.

5. **a)** $\{(3, -2), (5, 0), (6, 2), (8, 4)\}$



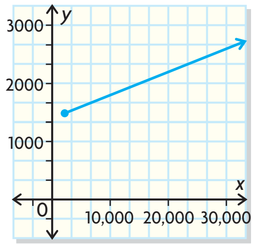
Function: domain $= \{-2, 0, 2, 4\}$, range $= \{3, 5, 6, 8\}$; inverse: domain $= \{3, 5, 6, 8\}$, range $= \{-2, 0, 2, 4\}$

b) $f^{-1}(x) = \dfrac{3-x}{4}$



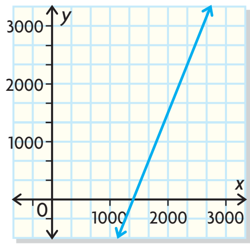
Function: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R}\}$; inverse: domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R}\}$

6. **a)** $f(x) = 0.04(x - 2500) + 1500$



b) $f(x) = 0.04(x - 2500) + 1500$ for $x \geq 2500$

c) $f^{-1}(x) = 25(x - 1500) + 2500$

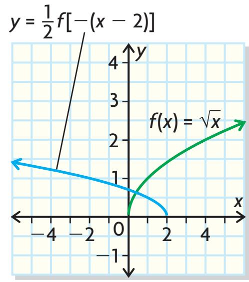


d) $f^{-1}(x) = 25(x - 1500) + 2500$ for $x \geq 1500$

e) $f^{-1}(1740) = 25(1740 - 1500) + 2500 = \$51\,000$

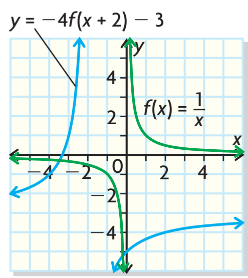
7. **a)** $\dfrac{1}{5}$ **b)** -3

8. **a)** $a = \dfrac{1}{2}$, $k = -1$, $c = 0$, $d = 2$



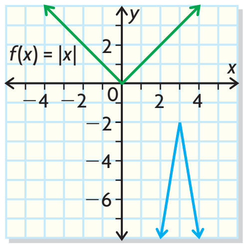
domain $= \{x \in \mathbf{R} \mid x \leq 2\}$, range $= \{y \in \mathbf{R} \mid y \geq 0\}$

b) $a = -4$, $k = 1$, $c = -3$, $d = -2$



domain $= \{x \in \mathbf{R} \mid x \neq -2\}$, range $= \{y \in \mathbf{R} \mid y \neq -3\}$

c) $a = -\frac{3}{2}, k = 4, c = -2, d = 3$



$y = -\frac{3}{2}f[4(x - 3)] - 2$

domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \leq -2\}$

Chapter 2

Getting Started, p. 82

1.

	Type	Degree
a)	binomial	1
b)	monomial	0
c)	binomial	2
d)	monomial	2
e)	trinomial	2

2. a) $9x - 2$ c) $8x^2 - 2x - 15$
b) $2x^2 - 4x - 9$ d) $4x^2 - 4x + 1$

3. Factoring is the opposite of expanding. To expand a polynomial, you multiply using the distributive property. To factor, you try to determine the polynomials that multiply together to give you the given polynomial. e.g., $(x + 2)(3x - 1) = 3x^2 + 5x - 2$

$\xrightarrow{\text{expanding}}$

$\xleftarrow{\text{factoring}}$

4. a) $2xy^3(3 - 4x)$ d) $(3 - 5x)(3 + 5x)$
b) $(a - 2)(a - 5)$ e) not possible
c) $(4n + 5)(3n - 2)$ f) $(y - 9)(y + 4)$

5. a) $\frac{11}{12}$ b) $-\frac{1}{2}$ c) $\frac{8}{15}$ d) -10

6. a) $\frac{5x^5}{6}$ b) $\frac{15}{2x^2}$ c) $8x^3y^5$ d) $5x^3y^2$

7. a) $\{x \in \mathbf{R}\}$ d) $x = 0, \{x \in \mathbf{R} | x \neq 0\}$
b) $\{x \in \mathbf{R}\}$ e) $x = 4, \{x \in \mathbf{R} | x \neq 4\}$
c) $x < 0, \{x \in \mathbf{R} | x \geq 0\}$ f) $x < -10, \{x \in \mathbf{R} | x \geq -10\}$

8.

Definition:	**Characteristics:**
A polynomial is any algebraic expression that contains one or more terms.	• usually contains variables • can contain both like and unlike terms • exponents must be whole numbers
Polynomial	
Examples: $3x^2$ $4x - 3$ $5x^3 - 2xy + 6y$	**Non-examples:** $\frac{5 - x}{3 + x}$ $5x^{-2} + 4x + 3$ $\sqrt{x}$

Lesson 2.1, pp. 88–90

1. a) $4x^2 - 8x + 8$ b) $2x^2 - 4$ c) $2x^2 - x$

2. $f(x) = 7x - 2$
$g(x) = 7x - 2$

3. Answers may vary. For example, $f(1) = -10$; $g(1) = -20$

4. a) $9a - 5c + 5$ d) $3x^2 - 9x - 3$
b) $2x^2 + 3x + 4y + z$ e) $2x^2 - 5xy + 2y^2$
c) $3x - 3y + 1$ f) $5x^2 - y^2 - 1$

5. a) $m - 4n + p + 7$ d) $-2m^2 - 5mn + 15n^2$
b) $-8m - 4q + 1$ e) $-x^2 + 4y^2 + 15$
c) $-a^3 + 4a^2 - 2a$ f) $3x^2 + 50$

6. a) $11x - 7y$ d) $2x + xy - 4y + yz$
b) $4x^2 - 16x - 3$ e) $\frac{3}{10}x + \frac{4}{3}y$
c) $4x^2$ f) $\frac{1}{12}x + \frac{1}{4}y + 1$

7. i) $(3x^2 - x) - (5x^2 - x)$
$= -2x^2$
$\neq -2x^2 - 2x$

ii) Answers will vary. For example, if $x = 1$,
$(3x^2 - x) - (5x^2 - x)$
$= (3 - 1) - (5 - 1)$
$= -2$
but $-2x^2 - 2x$
$= -2 - 2$
$= -4$

8. a) $f(x) = 2x^2 + 4x - 9$ and $g(x) = 2x^2 + 4x - 5 \therefore f(x) \neq g(x)$
b) $s_1(1) = 27$ and $s_1(1) = 9 \therefore s_1(t) \neq s_2(t)$
c) e.g., if $x = -1$, then $y_1 = 2$ and $y_2 = 0$
$\therefore y_1 \neq y_2$
d) $f(n) = 2n^2 + 2n - 9$ and $g(n) = 2n^2 + 2n - 9$
$\therefore f(n) = g(n)$
e) $p = 1, q = 1, y_1 = 9; y_2 = 5 \therefore y_1 \neq y_2$
f) $f(2) = 6$
$g(2) = 14$
$\therefore f(m) \neq g(m)$

9. Answers will vary. For example, $f(x) = 2x$ and $g(x) = x^2$

10. a) $25 - x - y$ b) $5x + y + 25$ c) 95

11. $3x + 3y + 2$

12. a) $P(x) = -50x^2 + 2350x - 9500$ b) \$11 500

13. a) cannot be determined d) cannot be determined
b) cannot be determined e) equivalent
c) not equivalent

14. a) yes b) Replace variables with numbers and simplify.

15. a) $x + (x + 7) + (x + 14) + (x + 15) + (x + 16) = 5(x + 14) - 18$
b) 26
c) $5x - 18$

16. a) $19 + 20 + 21 + 22 + 23$
b) $n = (m - 2) + (m - 1) + m + (m + 1) + (m + 2)$
c) $10 + 11 + 12 + 13 + 14 + 15 + 16$

17. a) both functions are linear; a pair of linear functions intersect at only one point, unless they are equivalent; since the functions are equal at two values, they must be equivalent
b) both functions are quadratic; a pair of quadratic functions intersect at most in two points, unless they are equivalent; since the functions are equal at three values, they must be equivalent

Lesson 2.2, pp. 95–97

1. **a)** $6x^2 - 10x^3 + 8xy$ **c)** $x^2 + 8x + 16$
b) $6x^2 + 7x - 20$ **d)** $x^3 + 3x^2 - x - 3$

2. **a)** no; for $x = 1$, left side is 25, right side is 13
b) $9x^2 + 12x + 4$

3. **a)** $6x^3 + 24x^2 + 14x - 20$ **b)** same as (a)

4. **a)** $25x^3 + 15x^2 - 20x$ **d)** $n^2 - 13n + 72$
b) $2x^2 - 7x - 30$ **e)** $-68x^2 - 52x - 2$
c) $16x^2 - 53x + 33$ **f)** $5a^2 - 26a - 37$

5. **a)** $4x^3 - 100x$ **d)** $-6x^3 + 31x^2 - 23x - 20$
b) $-2a^3 - 16a^2 - 32a$ **e)** $729a^3 - 1215a^2 + 675a - 125$
c) $x^3 - 5x^2 - 4x + 20$ **f)** $a^2 - 2ad - b^2 + 2bc - c^2 + d^2$

6. **a)** yes **c)** no **e)** no
b) yes **d)** yes **f)** yes

7. All real numbers. Expressions are equivalent.

8. **a)** Both methods give $285x^2 + 209x - 266$.
b) Answer may vary. For example, I preferred multiplying the last two factors together first. Multiplying the first two factors together first meant that I had to multiply larger numbers in the second step.

9. **a)** $16x^2 + 8\pi x$ **b)** $8\pi x^3 + 4\pi x^2 - 2\pi x - \pi$

10. **a)** yes
b) no, $x - 3 = -(3 - x)$. A negative number squared is positive (the same); a negative number cubed is negative (different).

11. **a)** $x^4 + 4x^3 + 2x^2 - 4x + 1$ **c)** $x^6 - x^4 - 2x^3 - 3x^2 - 2x - 1$
b) $8 - 12a + 6a^2 - a^3$ **d)** $-16x^2 + 43x - 13$

12. 0

13. **a)** $\frac{1}{2}mv^2 + \frac{1}{2}xv^2$ **b)** $\frac{1}{2}mv^2 + mvy + \frac{1}{2}my^2$

14. **a)** 6 $2 \times 3 = 6$; $(x^7 + x^6)(x^9 + x^4 + 1)$ has 6 terms
b) Multiply the number of terms in each polynomial

15. **a)** **i)** 8 **ii)** 12 **iii)** 6 **iv)** 1
b) **i)** 8 **ii)** 96 **iii)** 384 **iv)** 512
c) **i)** 8 **ii)** $12(n - 2)$ **iii)** $6(n - 2)^2$ **iv)** $(n - 2)^3$
d) same answers

16. **a)** Answers may vary. For example,
115: $11^2 + 11 = 132$
$115^2 = 13\,225$
b) $(10x + 5)^2 = 100x^2 + 100x + 25$ and
$(x^2 + x)100 + 25$ are both the same

Lesson 2.3, pp. 102–104

1. **a)** $(x - 9)(x + 3)$ **c)** $(2x + 5)^2$
b) $(5x + 7)(5x - 7)$ **d)** $(2x + 1)(3x - 2)$

2. **a)** $(c - d)(a + b)$
b) $(x + y + 1)(x - y + 1)$
c) $(x - y - 5)(x + y + 5)$

3. **a)** $(x - 7)(x + 4)$ **c)** $(3x - 7)^2$
b) $(6x - 5)(6x + 5)$ **d)** $(2x + 3)(x - 5)$

4. **a)** $2x(2x - 1)(x - 1)$ **d)** $(x + 1)(7x^2 - x + 6)$
b) $3xy^2(x^2 - 3xy^2 + y)$ **e)** $(x - 4)(3x - 1)$
c) $(a + 1)(4a - 3)$ **f)** $-2t(t - 13)(t - 1)$

5. **a)** $(x - 7)(x + 2)$ **d)** $(2y + 7)(y - 1)$
b) $(x + 5y)(x - y)$ **e)** $(4a - 7b)(2a + 3b)$
c) $6(m - 6)(m - 9)$ **f)** $2(2x + 5)(4x + 9)$

6. **a)** $(x - 3)(x + 3)$ **d)** $(3y - 8)(3y + 2)$
b) $(2n - 7)(2n + 7)$ **e)** $-12(2x - 3)(x - 3)$
c) $(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)$ **f)** $-(pq + 9)(pq - 9)$

7. **a)** $(x + y)(a + b)$ **d)** $(4 - x)(x - 2)$
b) $(b + 1)(2a - 3)$ **e)** $(a - b + 5)(a + b + 5)$
c) $(x + 1)^2(x - 1)$ **f)** $2(m + n)(m - n + 5)$

8. no; $(x - y)(x^2 + y^2) = x^3 - x^2y + xy^2 - y^3$

9. **a)** $(x - 3)(2x - 7)$ **d)** $(y - x + 7)(y + x - 7)$
b) $(x + 5)(y + 6)$ **e)** $3(2x - 7)(x - 2)$
c) $(x - 1)(x - 2)(x + 2)$ **f)** $(2m^2 - 5)(6m - 7)$

10. $f(n) = (n^2 + 3)(2n + 1)$. Since n is a natural number, $2n + 1$ is always odd and greater than 1. Because $(2n + 1)$ is a factor of $f(n)$, the condition is always true.

11. **a)** $a^2 = (c - b)(c + b)$ **b)** $a = \sqrt{33}$ m, $b = 4$ m, $c = 7$ m

12. Saturn
a) **i)** $\pi(r_2 - r_1)(r_2 + r_1)$
ii) $\pi(r_3 - r_1)(r_3 + r_1)$
iii) $\pi(r_3 - r_2)(r_3 + r_2)$
b) The area of the region between the inner ring and outer ring

13. **1.** Always do common factor first.
2. Do difference of squares for 2 square terms separated by a minus sign.
3. Do simple trinomials for 3 terms with $a = 1$ or a prime.
4. Do complex trinomials for 3 terms with $a \neq 1$ or a prime.
5. Do grouping for a difference of squares for 4 or 6 terms with 3 or 4 squares.
6. Do incomplete squares for 3 terms when you can add a square to allow factoring; e.g.,
1. $5x + 10 = 5(x + 2)$
2. $4x^2 - 25y^2 = (2x + 5y)(2x - 5y)$
3. $x^2 - x - 20 = (x - 5)(x + 4)$
4. $12x^2 - x - 20 = (4x + 5)(3x - 4)$
5. $x^2 + 6x + 9 - y^2 = (x + 3 + y)(x + 3 - y)$
6. $x^4 + 5x^2 + 9 = (x^2 + 3 + x)(x^2 + 3 - x)$

14. **a)** $(x^2 - 3x + 6)(x^2 + 3x + 6)$ **b)** $(x^2 - 3x - 7)(x^2 + 3x - 7)$

15. **a)** $x^4 - 1 = (x - 1)(x^3 + x^2 + x + 1)$
b) $x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$
c) $x^n - 1 = (x - 1)\,(x^{(n-1)}y^0 + x^{(n-2)} + \ldots + x^0)$
d) $x^n - y^n = (x - y)(x^{(n-1)})y^0 + x^{(n-2)} + x^{(n-3)}y^2 + \ldots + x^0y^{(n-1)}$

16. **a)** $2^6 + 2^4 + 2^2 - 2^4 - 2^2 - 2^0 = 2^6 - 1$;
$2^6 + 2^3 - 2^3 - 2^0 = 2^6 - 1$
b) $35 = 5 \times 7$
$\therefore\ 2^{35} - 1 = (2^5 - 1)(2^{30} + 2^{25} + 2^{20} + 2^{15} + 2^{10} + 2^5 + 2^0)$ or
$2^{35} - 1 = (2^7 - 1)(2^{28} + 2^{21} + 2^{14} + 2^7 + 2^0)$
c) Yes. If m is composite, then let $m = a \times b$, where a and/or b cannot equal 1.
$2^{m-1} = 2^{ab-1} = \frac{2^{ab}}{2^1} = \frac{(2^a)^b}{2^1}$
This result will always have two factors: $(2^{a-1})\,(2^a)^{b-1}$
Neither of these will ever equal 1, so 2^{m-1} is composite.

Mid–Chapter Review, p. 107

1. **a)** $6a^2 - 7$ **d)** $18x^2 - 12x + 3xy$
b) $x^2 - 11xy + 9y^2$ **e)** $6a^2 + 20a - 10ab + 6b - 18$
c) $-6c^2 + cd - 8d^2 - d$ **f)** $14x^3 + 6x^2 - 48x + 9xy$

2. **a)** No, e.g., $g(0) = -32$ and $h(0) = 32$
b) Yes, $f(x) = 2x^2 - 7x + 5$ and $g(x) = 2x^2 - 7x + 5$
c) Yes
d) No, e.g., $b(0) = 1$ and $c(0) = -1$

3. The resulting polynomial will be a cubic because you add the exponents of the highest terms (linear = 1 and quadratic = 2).

4. $3x - 94$

5. **a)** $6x^2 - 38x + 40$
b) $27x^3 - 27x^2 + 9x - 1$
c) $-2x^4 + 12x^3 - 34x^2 + 48x - 32$
d) $11x^2 - 25x + 11$
e) $-30x + 30$
f) $-x^3 + 3x^2y - 3xy^2 + y^3$

6. a) 2 b) $2w - l - 2$

7. a) $(x - 2)(x - 3)$ b) $(x - 7)(x - 4)$ c) $(3a + 2)(a - 4)$ d) $3(5x + 1)(2x - 1)$ e) $(4 - 5x)(4 + 5x)$ f) $-(5 + 2a)(13 - 2a)$

8. a) $(n - 3m)(2 + 5n)$ b) $(y - 3 - x)(y - 3 + x)$ c) $(y - b)(1 - y + b)$ d) $2(x + 2 - 2y)(x + 2 + 2y)$ e) $(w - a)(w + b)$ f) $(b + 6)(a + b)$

9. $20x + 8$

10. Many answers are possible; for example, $k = -60, -42, -34, -20, -14, -4, 0, 6, 8, 10$. All answers for k are of the form $k = 11b - 3b^2, b \in \mathbf{I}$.

Lesson 2.4, pp. 112–114

1. a) $3 - 2t$ b) $\frac{3}{2x}, x \neq 0$ c) $\frac{b^2}{3a^2}, a \neq 0, b \neq 0$

2. a) $\frac{5}{x - 3}, x \neq -3, 3$ b) $3, x \neq \frac{3}{2}$ c) $\frac{2ab}{2a - b}, a \neq \frac{1}{2}b$

3. a) $\frac{x - 3}{x + 2}, x \neq -2, 1$

   b) $\frac{(x + 1)}{5x - 4)}, x \neq \frac{4}{5}$

   c) $\frac{x - 5y}{x + 3y}, x \neq 2y, -3y$

4. a) $2x^2 - x + 3, x \neq 0$ b) $-\frac{x^2}{2y}, x \neq 0, y \neq 0$ c) $-\frac{2}{5t}, t \neq 0, 5$ d) $\frac{1}{a}(3a^2 - 2b), a \neq 0, \sqrt{\frac{2}{3}b}, b \neq 0$ e) $-\frac{2x}{3}, x \neq -5$ f) $-\frac{2}{3}, a \neq 0, b \neq 3$

5. a) $\frac{1}{a} - 1, a \neq -4, 1$ b) $-x - \frac{3}{5}, x \neq 3$ c) $\frac{x - 3}{x + 5}, x \neq -5, 2$ d) $\frac{2 + p}{5 + p}, p \neq -5, 5$ e) $\frac{t - 4}{t(t - 3)}, t \neq 0, 3$ f) $\frac{3t - 2}{t - 1}, t \neq -\frac{1}{2}, 1$

6. a) the denominator equals 0; $\mathbf{R}, x \neq 0$ b) $\mathbf{R}, x \neq 0, 2$ c) $\mathbf{R}, x \neq -5, 5$ d) $\mathbf{R}, x \neq -1, 1$ e) $\mathbf{R}$ f) $\mathbf{R}, x \neq -1, 1$

7. a) yes b) no, not the same domain

8. a) $\frac{2}{5}, x > -\frac{1}{3}$

   b) Because $x \leq -\frac{1}{3}$ would imply sides of length 0 or less, therefore this would not be a triangle.

9. $\frac{25}{24}$

10. a) $(t + 1)(4t - 1), t \neq 0$ b) $\frac{5}{4(2x - 1)}, x \neq \frac{1}{2}$ c) $\frac{5 - x}{4 + x}, x \neq -4, 4$ d) $\frac{2x + y}{x - y}, x \neq y$

11. $\frac{3w}{7}$

12. Answers will vary. For example, $\frac{(3x - 2)(x - 4)}{(x - 4)}; \frac{5(3x - 2)(x - 4)}{5(x - 4)}$

13. Answers will vary. For example, $\frac{5}{(x - 1)(x - 2)(x - 3)}$

14. a) Answers will vary. For example,

    i) $\frac{(2x + 1)(x + 1)}{(x - 4)(x + 1)}$ ii) $\frac{x(2x + 1)}{x(x - 4)}$ iii) $\frac{(2x + 1)(3x - 2)}{(x - 4)(x + 1)}$ iv) $\frac{(2x + 1)(2x + 1)}{(x - 4)(2x + 1)}$

    b) yes; $\frac{(2x + 1)(x - 4)}{(x - 4)^2}$

15. yes; $\frac{(x + 1)(x + 2)}{(x + 1)(x + 3)}$ and $\frac{(x + 4)(x + 2)}{(x + 4)(x + 3)}$

16. a) $\lim_{x \to \infty} f(x) = 0$ b) $\lim_{x \to \infty} g(x) = \frac{4}{5}$ c) $\lim_{x \to \infty} h(x) = -\infty$

17. a) $\frac{2(3t^2 - 1)}{(1 + t^2)^3}$; no restrictions b) $\frac{(2x + 1)(6x - 11)}{(3x - 2)^4}; x \quad \frac{2}{3}$

Lesson 2.5, p. 116

1. Answers will vary. For example, $y = \frac{2(x - 2)(x + 3)}{(x - 2)(x + 3)}$

2. Answers will vary. For example, $y = \frac{1}{x^2}$

3. Answers will vary. For example, $y = \frac{2}{x(x - 2)} + 2$

Lesson 2.6, pp. 121–123

1. a) $\frac{5}{12}$ b) $\frac{3x^3}{20y}, y \neq 0$ c) $\frac{(x + 1)}{2}, x \neq -4, 5$ d) $\frac{(3x)}{5}, x \neq -\frac{1}{2}, 0$

2. a) $\frac{10}{(3x)}, x \neq 0$ b) $\frac{5}{4}, x \neq 7$ c) $\frac{(3x)}{(x - 7)}, x \neq -2, 6, 7$ d) $-6(x - 1), x \neq -1, 2$

3. a) $\frac{(x^2 - 1)}{(x + 3)^2}, x \neq -3, -1, 1$

   b) $\frac{2}{(x - 2)(x - 5)}, x \neq -5, 2, 5$

4. a) $6x, x \neq 0$ b) $\frac{5}{6a}, a \neq 0$ c) $\frac{3x}{2y^2}, x \neq 0, y \neq 0$ d) $\frac{7a}{3}, a \neq 0, b \neq 0$

5. a) $\frac{(x - 1)}{9}, x \neq -1$ b) $3, a \neq -2, 2$ c) $-\frac{8x}{3}, x \neq 0, 2$ d) $\frac{21(m + 4)(m + 2)}{5(2m + 1)}, m \neq -4, -2, -\frac{1}{2}$

6. a) $\frac{2(x + 1)}{(x + 2)(x + 3)}, x \neq -3, -2, 3$

   b) $\frac{2(n - 2)}{5}, n \neq -2, 2, 3, 4$

c) $\dfrac{(x-1)(3x-1)(2x-1)}{(x-3)(x+2)(4x+5)}, x \neq -2, -\dfrac{5}{4}, -\dfrac{1}{2}, 3$

d) $\dfrac{-3(3y-2)}{2(3y+2)}, y \neq -\dfrac{2}{3}, 3$

7. a) $\dfrac{x+y}{x+7y}, x \neq -y, -7y, y, 4y$

b) $\dfrac{(a-3b)}{2a}, a \neq 0, 2b, 3b, 5b$

c) $\dfrac{4(5x-y)}{3x(3x-y)}, x \neq 0, -\dfrac{1}{2}y, -\dfrac{1}{3}y, \dfrac{1}{3}y$

d) $\dfrac{-(3m-n)(m-n)}{2(m+n)}, m \neq \dfrac{1}{7}n, -\dfrac{2}{5}n, -n$

8. $\dfrac{3x^2}{(x-1)(x+2)}$, restrictions: $x \neq -3, -2, 0, \dfrac{1}{2}, 1, 2$

9. $\dfrac{10x^2}{(x-9)(x+3)}, x \neq 9, 7, -3$

10. $\dfrac{3p+1}{p-1}, p \neq -1, -\dfrac{1}{3}, 1$

11. If $x = y$, then Liz is dividing by 0.

12. a) Then you can simplify and cancel common factors

b) Sometimes you cannot factor and you need to take into account the factors you cancel because they could make the denominator equal to 0.

c) Yes. Dividing is the same as multiplying by the reciprocal.

13. $\dfrac{m(m+2n)}{(m+n)(4m+n)}, m \neq -\dfrac{n}{3}, -\dfrac{3n}{2}, 2n, n, -n, -2n, -\dfrac{n}{4}$

14. 22 933.7

Lesson 2.7, pp. 128–130

1. a) $\dfrac{19}{12}$ c) $\dfrac{35x+4}{28x^3}, x \neq 0$

b) $\dfrac{17x}{5}$ d) $\dfrac{2x+6}{x^2}, x \neq 0$

2. a) $-\dfrac{1}{9}$ c) $\dfrac{25-21x^2}{15x^2}, x \neq 0$

b) $\dfrac{7y}{6}$ d) $\dfrac{6y-15x}{3xy^2}, x, y \neq 0$

3. a) $\dfrac{8x+18}{(x-3)(5x-1)}, x \neq \dfrac{1}{5}, 3$

b) $\dfrac{2x+1}{(x-3)(x+3)}, x \neq -3, 3$

c) $\dfrac{-4x+22}{(x-3)(x-1)^2}, x \neq 1, 3$

4. a) $\dfrac{13}{8}$ b) $\dfrac{3x+11}{(x-3(x+3)}$ c) $\dfrac{13}{8}$; same

5. a) $\dfrac{5x}{4}$ c) $\dfrac{40xy^3-15x^2y+36}{60y^4}, y \neq 0$

b) $\dfrac{30+5t^2-6t^3}{10t^4}, t \neq 0$ d) $\dfrac{n^2+m^2-m^2n}{mn}, m \neq 0, n \neq 0$

6. a) $\dfrac{9a-8}{a(a-4)}, a \neq 0, 4$ c) $\dfrac{12x+43}{(x+4)(x+3)}, x \neq -4, -3$

b) $\dfrac{18x-8}{3x-2}, x \neq \dfrac{2}{3}$ d) $\dfrac{-2n-18}{(2n-3)(n-5)}, n \neq \dfrac{3}{2}, 5$

e) $\dfrac{10x^2-30x}{(x+4)(x-6)}, x \neq -4, 6$

f) $\dfrac{78x-129}{10(x-3)(2x-3)}, x \neq \dfrac{3}{2}, 3$

7. a) $\dfrac{3x-8}{(x+1)(x-4)}, x \neq -1, 4$

b) $\dfrac{2t^2+3t}{(t-4)(t+4)}, t \neq -4, 4$

c) $\dfrac{8t-1}{(t+3)^2(t-2)}, t \neq -3, 2$

d) $\dfrac{x^2-32x}{(x+2)(x+4)(x-5)}, x \neq 5, -2, -4$

e) $\dfrac{2x^2+7x+23}{(x-3)(x+3)(x-2)}, x \neq -3, 2, 3$

f) $\dfrac{9t^2-14t+2}{4(t-3)(t-4)(t+1)}, t \neq -1, 3, 4$

8. a) $\dfrac{7x+4}{(x+1)(4x+3)^2}, x \neq -1, -\dfrac{3}{4}$

b) $\dfrac{a^2+4a-7}{(a-3)(a-5)(2a+1)}, a \neq -\dfrac{1}{2}, 3, 5$

c) $\dfrac{10x^2-5x+7}{(2x-1)(2x+1)^2}, a \neq -\dfrac{1}{2}, \dfrac{1}{2}$

9. a) $\dfrac{9x^3-10y^2}{15xy}, x \neq 0, y \neq 0$

b) $43x^2-84x-\dfrac{136}{4(x-3)(x+1)(x-2)}, x \neq -1, 2, 3$

c) $\dfrac{p^3+5p^2-25p-65}{(p+5)(p+7)(p-5)}, p \neq -7, -5, -4, 3, 5, 6$

d) $\dfrac{15m^2-2mn-3m-n^2-15n}{(2m+n)(3m+n)}, m \neq -\dfrac{1}{2}n, -\dfrac{1}{3}n, -5n, \dfrac{1}{2}n$

10. a) $\dfrac{23m+20}{10}$

b) $\dfrac{20x-3}{4x^3}, x \neq 0$

c) $\dfrac{-y-7}{(y+1)(y-2)}, y \neq -1, 2$

d) $\dfrac{2x^2+13x+15}{(x+3)(x-2)(x+4)}, x \neq -4, -3, 2$

11. $\dfrac{-s+t-1}{t+s}, t \neq -s$

12. a) $\dfrac{x-300}{6}$ b) $0 \leq x < 300$

13. $\dfrac{2kdx+kx^2}{d^2(d+x)^2}, d \neq 0, -x$

14. a) Answers will vary. For example,

i) $\dfrac{1}{2}+\dfrac{1}{4}$ iii) $\dfrac{1}{4}+\dfrac{1}{6}$

ii) $\dfrac{1}{3}+\dfrac{1}{4}$

b) Factor the quadratic denominators and find the common denominator from these factors.

15. **a)** $\frac{1}{n} - \frac{1}{n+1}$
$= \frac{1(n+1)}{n(n+1)} - \frac{1(n)}{n(n+1)}$
$= \frac{n+1-n}{n(n+1)}$
$= \frac{1}{n(n+1)}$

b) Answers may vary. For example, 4, 12, 3 or 5, 20, 4

16. **a)** Let x be the smaller of the two consecutive even or odd numbers. Then $x + 2$ is the larger of the two.
$\frac{1}{x} + \frac{1}{x+2} = \frac{x + (x+2)}{x(x+2)}$
$= \frac{2x+2}{x(x+2)}$
$(2x+2)^2 + (x^2+2x)^2$
$= 4x^2 + 8x + 4 + x^4 + 4x^3 + 4x^2$
$= x^4 + 4x^3 + 8x^2 + 8x + 4$
$= (x^2 + 2x + 2)^2$
So, $2x + 2$, $x^2 + 2x$, and $x^2 + 2x + 2$ are Pythagorean triples.

b) Answers may vary. For example, 16, 63, 65

Chapter Review, pp. 132–133

1. **a)** $12x^2 - 12x + 13$ **b)** $8a^2 - 6ab + 3b^2$

2. Answers will vary. For example, $f(x) = x^2 + x$, $g(x) = 2x$

3. **a)** no **b)** probably **c)** 32 or 33

4. **a)** $-84x^2 + 207x - 105$
b) $-3y^4 + 17y^3 - 38y^2 + 23y + 21$
c) $2a^3 + 6a^2b + 6ab^2 + 2b^3$
d) $12x^6 - 36x^5 - 21x^4 + 144x^3 - 60x^2 - 144x + 108$

5. $V = \left(\frac{1}{3}\right)\pi(r+x)^2(h+2x)$
$V = \left(\frac{1}{3}\right)\pi(2hrx + r^2h + 2r^2x + 4rx^2 + 2x^3 + x^2h)$

6. **a)** $8x^4 - x^3 + x^2 - 1$
b) $2x^4 + 5x^3 - 10x^2 - 20x + 8$
c) $-13x^3 + 8x^2 - 11x$
d) $-5x^6 - 6x^5 - 2x^4 + 22x^3 - 4x^2$
e) $-20x$
f) $2x^3 + 29x^2 - 4x - 252$
g) $x^4 + 10x^3 + 19x^2 - 30x + 9$

7. **a)** $6m^2n^2(2n + 3m)$ **d)** $2(5x + 6)(5x - 6)$
b) $(x - 5)(x - 4)$ **e)** $(3x - 1)^2$
c) $3(x + 3)(x + 5)$ **f)** $(2a - 1)(5a + 3)$

8. **a)** $2x^2y(y^3 - 3x^3y^2 + 4x)$ **d)** $(5x - 6)(3x - 7)$
b) $(x + 4)(2x + 3)$ **e)** $(a^2 + 4)(a + 2)(a - 2)$
c) $(x + 2)(x - 5)$ **f)** $-(m + 4n)(3m + 2n)$

9. **a)** $-2a^2 - 3c^2$; $b \neq 0$ **c)** $1 - z$; $x \neq 0, y \neq 0$
b) $3y^3 - 2z^2 + 5$; $x \neq 0$ **d)** $2r - 3p + 5k$; $m \neq 0, n \neq 0$

10. **a)** $\frac{8xy^3 + 12x^2y^2 - 3x^2}{y}$, $x \neq 0, y \neq 0$
b) $\frac{7}{2}$, $a \neq 2b$
c) $\frac{1}{m+7}$, $m \neq -7, -3$
d) $\frac{2x+1}{2x+3}$, $x \neq -\frac{3}{2}, \frac{3}{2}$
e) $\frac{3x(x-7)}{7(x-3)(x-1)}$, $x \neq 1, 3$
f) $\frac{x-y}{x+y}$, $x \neq -\frac{1}{3}y, -y$

11. Perhaps, but probably not. e.g., $\frac{x+1}{x-1}$ and $\frac{x+2}{x-1}$ are not equivalent.

12. **a)** $\frac{y}{2}$; $x \neq 0, y \neq 0$ **c)** $\frac{2b}{3c^2}$; $a \neq 0, b \neq 0, c \neq 0$
b) m; $m \neq 0, n \neq 0$ **d)** $\frac{5}{2p}$; $p \neq 0, q \neq 0$

13. **a)** $\frac{x}{36y}$; $x \neq 0, y \neq 0$
b) $\frac{(x-3)(x+1)}{(x-1)}$; $x \neq \pm 1, \pm 2, 5$
c) $\frac{x(1-x)}{-y(y+1)}$; $x \neq 0, -1, y \neq 0, \pm 1$
d) $x - y$; $y \neq \pm 2x, -x, -\frac{2}{3}x$

14. **a)** $\frac{2}{15x}$; $x \neq 0$
b) $\frac{3x-7}{(x+1)(x-1)}$; $x \neq 1, -1$
c) $\frac{2(x-2)}{(x+4)(x-1)(x-3)}$; $x \neq -4, 1, 3$
d) $\frac{5}{(x-2)(x+3)(x-3)}$; $x \neq \pm 3, 2$

15. **a)** $\frac{3x^2 - 14x + 24}{6x^3}$, $x \neq 0$
b) $\frac{7x^2 - 10x}{(x+2)(x-6)}$, $x \neq -2, 6$
c) $\frac{3x^2 + 30x}{(x-3)(x+4)(x-2)}$, $x \neq -4, 2, 3$
d) $\frac{6}{x+1}$, $x \neq -1, -5, 2$
e) $\frac{(x-2y)(x+y)}{(x-y)(x+3y)}$, $x \neq -3y, -y, y, 2y$
f) $\frac{3b^2 + 8b - 5}{(b-5)(b+3)}$, $x \neq 5, -3, -6$

16. 65

17. **a)** $\frac{6}{n^2 - 3n + 2}$ or $\frac{6}{(n-1)(n-2)}$, $n \neq 0, 1, 2$
b) **i)** $\frac{1}{2}$ **ii)** 1

Chapter Self-Test, p. 134

1. **a)** $x^2 - 5x$ **b)** $4m^2 - 3mn + 5n^2$

2. **a)** $72a^2 - 198a + 70$ **c)** $20x^3 - 57x^2 - 11x + 6$
b) $-7x^3y^3 + 10x^4y^2 - 12x^2y^4$ **d)** $9p^4 + 6p^3 - 11p^2 - 4p + 4$

3. no

4. **a)** $24n^2 + 48n + 26$ **b)** 866

5. **a)** $m(m - 1)$ **d)** $(x + 3)(3x - 2y - 1)$
b) $(x - 3)(x - 24)$ **e)** $(y - 2)(5x - 3)$
c) $(5x + y)(3x - 2y)$ **f)** $(p - m + 3)(p + m - 3)$

6. $x = -1, 1, 4$

Answers

7. **a)** $\frac{14}{3b^2}$, $a \neq 0$, $b \neq 0$

b) $\frac{2}{(x-2)(x+3)}$, $x \neq -3, 2, 4$

c) $\frac{6t-49}{(t+2)(t-9)}$, $t \neq -2, 9$

d) $\frac{-x^2-18x}{(3x+2)(2x+3)(2x-3)}$, $x \neq -\frac{3}{2}, -\frac{2}{3}, \frac{3}{2}$

8. yes (as long as there are no restrictions that were factored out)

9. yes

Chapter 3

Getting Started, p. 138

1. **a)** 0 **b)** -21 **c)** 0 **d)** -1 **e)** $-3k^2 + 4k - 1$ **f)** $-3k^2 - 4k - 1$

2. **a)** $f(x) = x^2 + 2x - 15$ **b)** $f(x) = 2x^2 + 12x$ **c)** $f(x) = -3x^2 - 12x - 9$ **d)** $f(x) = x^2 - 2x + 1$

3. **a)** vertex $(-3, -4)$, $x = -3$, domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \leq -4\}$

b) vertex $(5, 1)$, $x = 5$, domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq 1\}$

4. **a)** vertex $(0, 4)$, $x = 0$, opens up

b) vertex $(4, 1)$, $x = 4$, opens up

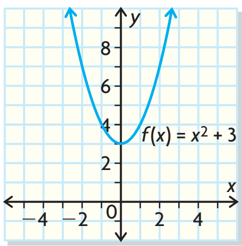
c) vertex $(-7, -3)$, $x = -7$, opens down

d) vertex $(1.5, 36.75)$, $x = 1.5$, opens down

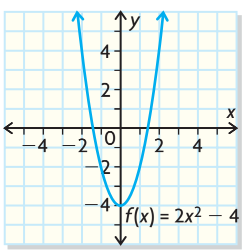
5. **a)** $x = 3$ or 8 **b)** $x = 0.55$ or 5.45 **c)** $x = -1$ or 1.67 **d)** $x = 0.5$ or 3

6. **a)** $(-3, 0)$, $(3, 0)$ **b)** $(-1.83, 0)$, $(9.83, 0)$ **c)** $(1.33, 0)$, $(2, 0)$ **d)** $(0, 0)$, $(3, 0)$

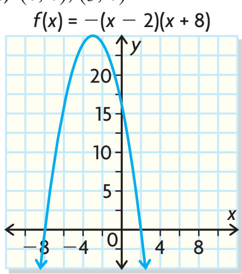
7. **a)**



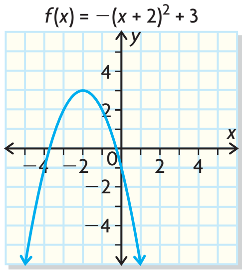
b)



c)



d)



8.

Definition:	**Characteristics:**
equation is of form $y = ax^2 + bx + c$ or equivalent	graph is a parabola function has two, one, or no zeros second differences are constant
Examples:	**Non-examples:**
$y = x^2$ $y = -4(x+3)^2 - 5$	$y = 5 - 4x$ $y = 2\sqrt{x} - 5$

Quadratic Function

Lesson 3.1, pp. 145–147

1. **a)** linear, first differences are constant

b) quadratic, second differences are constant

c) linear, first differences are constant

d) quadratic, second differences are constant

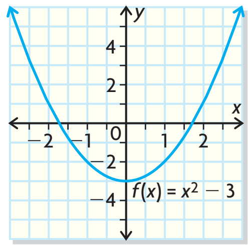
2. **a)** opens up **b)** opens down **c)** opens down **d)** opens up

3. **a)** zeros $x = 2$ or -6 **b)** opens down **c)** $x = -2$

4. **a)** vertex $(-2, 3)$ **b)** $x = -2$

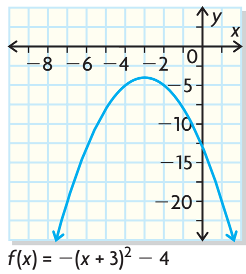
c) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \leq 3\}$

5. **a)**



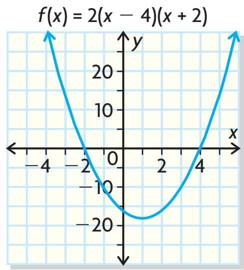
opens up, vertex $(0, -3)$, $x = 0$

b)



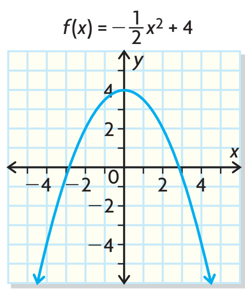
opens down, vertex $(-3, -4)$, $x = -3$

c)



opens up, vertex $(1, -18)$, $x = 1$

d)



opens down, vertex $(0, 4)$, $x = 0$

6. **a)** $f(x) = -3x^2 + 6x + 3$, $(0, 3)$

b) $f(x) = 4x^2 + 16x - 84$, $(0, -84)$

7. **a)** opens down

b) vertex $(-1, 8)$

c) $(-3, 0)$, $(1, 0)$

d) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \leq 8\}$

e) negative; parabola opens down

f) $f(x) = -2(x + 1)^2 + 8$ or $f(x) = -2(x + 3)(x - 1)$

8. **a)** opens up

b) vertex $(1, -3)$

c) $x = 1$

d) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \geq -3\}$

e) positive; parabola opens up

9. **a)** $x = 0$ **c)** $x = 12$ **e)** $x = -1.5$

b) $x = -7$ **d)** $x = -2$ **f)** $x = -\frac{5}{16}$

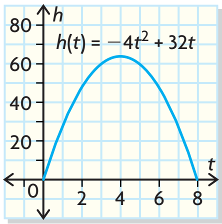
10. **a)**

x	-2	-1	0	1	2
$f(x)$	3	4	3	0	-5

b) First differences: $1, -1, -3, -5$; Second differences: -2; parabola opens down

c) $f(x) = -(x + 1)^2 + 4$

11. **a)**



b) 8 s; height starts at 0 m and is 0 m again after 8 s.

c) $h(3) = 60$ m

d) 64 m

12. $y = 30$

13. Similarities: both are quadratic; both have axis of symmetry $x = 1$. Differences: $f(x)$ opens up, $g(x)$ opens down; $f(x)$ has vertex $(1, -2)$, $g(x)$ has vertex $(1, 2)$

14.

x	-2	-1	0	1	2	3
$f(x)$	19	9	3	1	3	9
First Differences		-10	-6	-2	2	6
Second Differences		4	4	4	4	

15. $56 250

16. $y = -\frac{1}{110}(x + 7.5)^2 + \frac{1805}{88}$

Lesson 3.2, pp. 153–154

1. a) and c); (a) is negative.

2. **a)** vertex $(-5, -2)$, minimum value -2

b) vertex $(4, 8)$, maximum value 8

3. **a)** maximum: 6 **c)** maximum: 8

b) minimum: 0 **d)** minimum: -7

4. **a)** complete the square; minimum: -5

b) factor or complete the square; minimum: -4

c) factor or complete the square; minimum: -18

d) factor or complete the square; maximum: 27

e) use partial factoring; minimum: 2

f) use vertex form; maximum: -5

5. **a)** **i)** $R(x) = -x^2 + 5x$ **ii)** maximum revenue: $6250

b) **i)** $R(x) = -4x^2 + 12x$ **ii)** maximum revenue: $9000

c) **i)** $R(x) = -0.6x^2 + 15x$ **ii)** maximum revenue: $93 750

d) **i)** $R(x) = -1.2x^2 + 4.8x$ **ii)** maximum revenue: $4800

6. **a)** minimum: -2.08 **b)** maximum: 1.6

7. **a)** **i)** $P(x) = -x^2 + 12x - 28$ **ii)** $x = 6$

b) **i)** $P(x) = -2x^2 + 18x - 45$ **ii)** $x = 4.5$

c) **i)** $P(x) = -3x^2 + 18x - 18$ **ii)** $x = 3$

d) **i)** $P(x) = -2x^2 + 22x - 17$ **ii)** $x = 5.5$

8. **a)** 70 m **b)** 2 s **c)** 50 m

9. $562 500

10. Minimum value is 2, therefore $3x^2 - 6x + 5$ cannot be less than 1.

11. **a)** $5 450 000

b) Maximum profit occurs when $40 000 is spent on advertising.

c) $22 971

12. Is possible, because maximum rectangular area occurs when rectangle is 125 m by $\frac{250}{\pi}$ m.

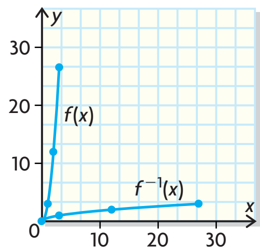
13. Possible response: Function is in standard form, so to find the minimum, we must find the vertex. Completing the square would result in fractions that are more difficult to calculate than whole numbers. Since this function will factor, putting the function in factored form and averaging the zeros to find the x-intercept of the vertex would be possible; however, there would still be fractions to work with. Using the graphing calculator to graph the function, then using CALC to find the minimum, would be the easiest method for this function.

14. $t = \frac{v_0}{9.8}$ seconds

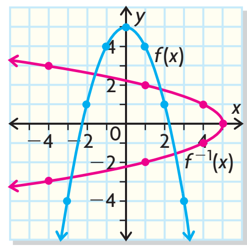
15. $9

Lesson 3.3, pp. 160–162

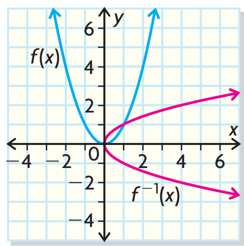
1. **a)**



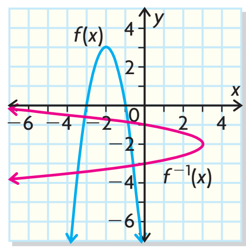
b)



2. **a)**



b)

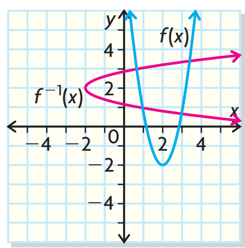


3. $f^{-1}(x) = \pm\sqrt{\frac{x + 1}{2}}$

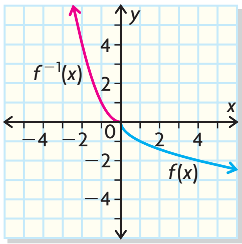
4. **a)** -1 **c)** 0 or 2

b) $1 \pm \sqrt{\frac{7 - x}{2}}$ **d)** $1 \pm \sqrt{-a}$

5. **a), b)**



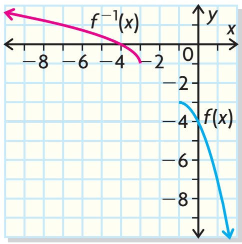
6. a), b)



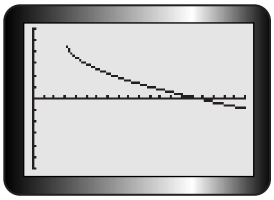
c) domain $= \{x \in \mathbf{R} \mid x \le 0\}$; range $= \{y \in \mathbf{R} \mid y \ge 0\}$

d) $g^{-1}(x) = (-x)^2$ or $g^{-1}(x) = x^2, x \le 0$

7. $f^{-1}(x) = -1 + \sqrt{-x - 3}$



8. $f^{-1}(x) = 5 - \sqrt{2x - 6}, x \le 3, x \le -3$



9. a) domain $= \{x \in \mathbf{R} \mid -2 < x < 3\}$;
range $= \{y \in \mathbf{R} \mid -3 \le y < 24\}$

b) $f^{-1}(x) = 1 + \sqrt{\dfrac{x + 3}{3}}, -3 \le x \le 24$

10. a) $h(t) = -5(t - 1)^2 + 40$

b) domain $= \{t \in \mathbf{R} \mid 0 \le t \le 3.83\}$;
range $= \{h \in \mathbf{R} \mid 0 \le h \le 40\}$

c) $$t = \begin{cases} 1 - \sqrt{\dfrac{40 - h}{5}}, & 35 < h \le 40 \\ 1 + \sqrt{\dfrac{40 - h}{5}}, & 0 \le h \le 35 \end{cases}$$

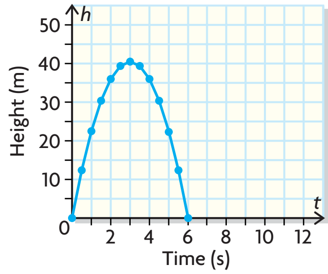
d) domain $= \{h \in \mathbf{R} \mid 0 \le h \le 40\}$;
range $= \{t \in \mathbf{R} \mid 0 \le t \le 3.83\}$

11. a)

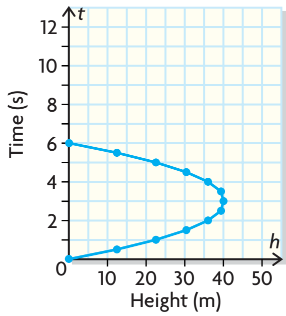
Time (s)	0	0.5	1	1.5	2	2.5
Height (m)	0	12.375	22.5	30.375	36.0	39.375

Time (s)	3	3.5	4	4.5	5	5.5	6
Height (m)	40.5	39.375	36.0	30.375	22.5	12.375	0

b)



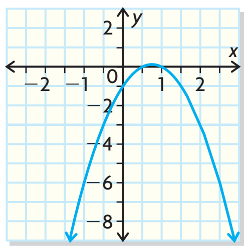
c)



d) The inverse is not a function. It does not pass the vertical-line test.

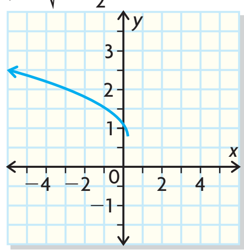
12. a) (0.75, 0.125)

b)



$f(x) = -2x^2 + 3x - 1$

c) $y = \sqrt{\dfrac{0.125 - x}{2}} + 0.75$

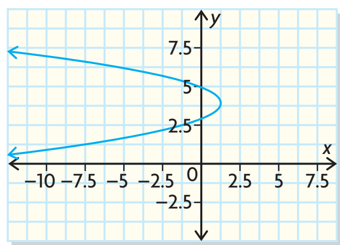


d) domain $= \{x \in \mathbf{R} \mid x \le 0.125\}$; range $= \{y \in \mathbf{R} \mid y \ge 0.75\}$

e) The y-values were restricted to ensure $f^{-1}(x)$ is a function.

13. a) i) $f(x) = -(x - 4)^2 + 1$

ii)

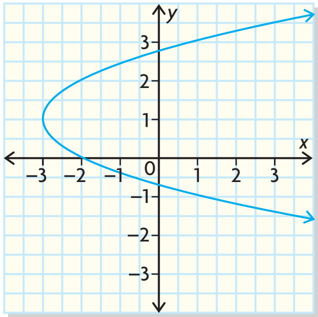


iii) Domain of f should be restricted to $\{x \in \mathbf{R} \mid x \ge 4\}$ or $\{x \in \mathbf{R} \mid x \le 4\}$

iv) f^{-1} is $y = \pm\sqrt{-x - 1} + 4$

b) i) $f(x) = (x - 1)^2 - 3$

ii)

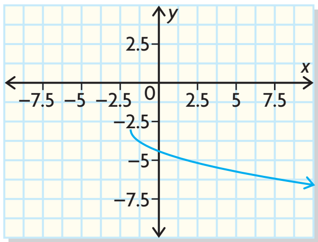


iii) Domain of f should be restricted to $\{x \in \mathbf{R} \mid x \le 1\}$ or $\{x \in \mathbf{R} \mid x \ge 1\}$

iv) f^{-1} is $y = \pm\sqrt{x + 3} + 1$

c) i) $f(x) = (x + 3)^2 - 2$ where $x \le -3$

ii)

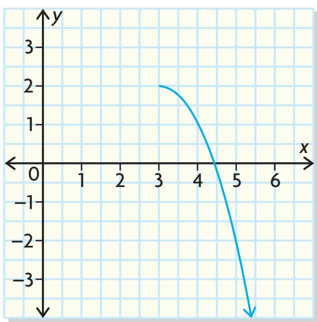


iii) No restrictions necessary.

iv) $f^{-1}(x) = \pm\sqrt{x + 2} - 3$

d) i) $f(x) = 3 + \sqrt{2 - x}$

ii)



iii) No restrictions necessary.

iv) f^{-1} is $y = -(x - 3)^2 + 2$ where $x \ge 3$

14. The original function must be restricted so that only one branch of the quadratic function is admissible. For example, if $f(x) = x^2$ had its domain restricted to $x \ge 0$, the inverse of $f(x)$ would be a function.

15. a) Possible response: Switch x and y and solve resulting quadratic equation for y, either by completing the square or by using the quadratic formula.

b) No, because the original function assigns some y-values to two x-values, so the inverse assigns two y-values to some x-values.

16. a) $P(x) = (x - 3.21)(14\,700 - 3040x)$

b) $P^{-1}(x) = 4.02 \pm \sqrt{\dfrac{-x + 2008}{3040}}$. This equation will take the total profit and determine the price per kilogram.

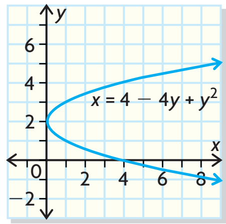
c) $P^{-1}(1900) = 4.02 \pm \sqrt{\dfrac{-1900 + 2008}{3040}}$

$= 4.21$ or 3.83

If the meat manager charges either \$4.21/kg or \$3.83/kg, she will make a profit of \$1900.

d) \$4.02/kg **e)** \$3.97/kg. Profit would be about \$2289.

17. a)



b) domain $= \{x \in \mathbf{R} \mid x \ge 0\}$; range $= \{y \in \mathbf{R}\}$

c) $y = (x - 2)^2$

d) Yes, the inverse is a function. Its graph will be a parabola, so it will pass the vertical-line test.

Lesson 3.4, pp. 167–168

1. a) $3\sqrt{3}$ **b)** $5\sqrt{2}$ **c)** $7\sqrt{2}$ **d)** $4\sqrt{2}$

2. a) $\sqrt{35}$ **b)** $\sqrt{66}$ **c)** $10\sqrt{6}$ **d)** $-32\sqrt{39}$

3. a) $7\sqrt{5}$ **b)** $5\sqrt{7}$ **c)** $-\sqrt{3} + 19\sqrt{2}$ **d)** $-\sqrt{2}$

4. a) $6\sqrt{3}$ **b)** $-25\sqrt{5}$ **c)** $20\sqrt{10}$ **d)** $-\sqrt{15}$ **e)** $2\sqrt{5}$ **f)** $-18\sqrt{3}$

5. a) $2\sqrt{3} - \sqrt{15}$ **b)** $2\sqrt{14} + 6\sqrt{6}$ **c)** 32 **d)** $-24\sqrt{3}$ **e)** $36\sqrt{2}$ **f)** -140

6. a) $-2\sqrt{2}$ **b)** $-\sqrt{3} + 8\sqrt{2}$ **c)** $-9\sqrt{2}$ **d)** $15\sqrt{2}$ **e)** $2\sqrt{13}$ **f)** $16\sqrt{3} - 4\sqrt{7}$

7. a) $18 + 12\sqrt{10} - 3\sqrt{5} - 10\sqrt{2}$

b) $31 + 12\sqrt{3}$

c) -3

d) $-7 - 2\sqrt{6}$

e) $83 - 12\sqrt{35}$

f) $4 + 3\sqrt{6} - 8\sqrt{3} - 13\sqrt{2}$

8. $4\sqrt{2}$ cm

9. $15\sqrt{2}$ cm

10. $3\sqrt{10}$ cm

11. $6\sqrt{2}$

12. Perimeter $= 8\sqrt{2} + 4\sqrt{5}$, Area $= 12$

13. $(\sqrt{a} + \sqrt{b})^2$

14. Possible response: $2\sqrt{50}$, $5\sqrt{8}$, $10\sqrt{2}$; The last one is in simplest radical form because the number under the radical sign cannot be simplified any further.

15. a) $a\sqrt{a}$ **b)** $x^2y^3\sqrt{x}$ **c)** $3n^3\sqrt{n}$ **d)** $-p + 2q - \sqrt{pq}$

16. $2\sqrt{2}$

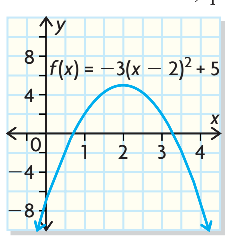
17. $x = 16$

Mid-Chapter Review, p. 170

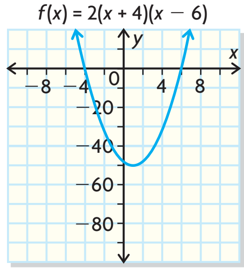
1. a) second differences $= -4$; quadratic

b) second differences $= 2$; quadratic

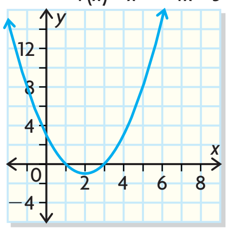
2. a)

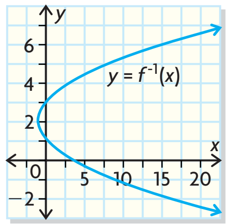


b) $f(x) = 2(x + 4)(x - 6)$



3. a) vertex (2, 5), $x = 2$, domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \le 5\}$
   b) vertex (1, −50), $x = 1$, domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \ge -50\}$
4. a) $f(x) = -3x^2 + 12x - 7$
   b) $f(x) = 2x^2 - 4x - 48$
5. a) Minimum value of −7   c) Maximum value of 12.5
   b) Minimum value of −50   d) Minimum value of −24.578 125
6. Maximum profit is $9000 when 2000 items are sold.
7. 2000 items/h
8. 64
9. a) $f^{-1}(x) = 2 \pm \sqrt{x + 1}$
   b) domain of $f(x) = \{x \in \mathbf{R}\}$, range of $f(x) = \{y \in \mathbf{R} \mid y \ge -1\}$; domain of $f^{-1}(x) = \{x \in \mathbf{R} \mid x \ge -1\}$, range of $f^{-1}(x) = \{y \in \mathbf{R}\}$
   c) $f(x) = x^2 - 4x + 3$





10. $x = 10 + \sqrt{\dfrac{R - 15}{-2.8}}$
11. Usually, the original function assigns some y-values to two x-values, so the inverse assigns two y-values to some x-values.
12. a) $\{x \in \mathbf{R} \mid x = -3\}$ $\{y \in \mathbf{R} \mid y \le 0\}$
    b) $f^{-1}(x) = x^2 - 3, x \ge 0\}$
13. a) $4\sqrt{3}$ c) $6\sqrt{5}$ e) $35\sqrt{2}$
    b) $2\sqrt{17}$ d) $-15\sqrt{3}$ f) $-16\sqrt{3}$
14. a) $7\sqrt{2}$ c) $-5\sqrt{3}$ e) $14 + 3\sqrt{3}$
    b) $30\sqrt{3}$ d) $9\sqrt{7} - 19\sqrt{2}$ f) $70 + 55\sqrt{2}$

Lesson 3.5, pp. 177–178

1. a) $x = -1$ or -4 b) $x = 2$ or 9 c) $x = \pm\dfrac{3}{2}$ d) $x = -\dfrac{1}{2}$ or 4
2. a) $x = 5.61$ or -1.61 c) no real roots
   b) $x = 1.33$ or -2 d) $x = -1.57$ or 5.97
3. a) $x = -1$ or -0.25 b) $x = 1$ or 4.5
4. a) i) Solve by factoring, function factors   ii) $x = 0$ or 10
   b) i) Quadratic formula, function does not factor
      ii) $x = \dfrac{-3 \pm \sqrt{5}}{4}$
   c) i) Quadratic formula, function does not factor
      ii) $x = -2 \pm \sqrt{7}$
   d) i) Quadratic formula, function does not factor
      ii) $x = -4 \pm \sqrt{7}$
   e) i) Solve by factoring, function factors
      ii) $x = -1$ or 10
   f) i) Quadratic formula, function does not factor
      ii) $x = 2 \pm \sqrt{19}$
5. a) (2.59, 0), (−0.26, 0)   b) (1, 0), $\left(\dfrac{21}{4}, 0\right)$
6. a) 14 000 b) 4000 or 5000 c) 836 or 10 164 d) 901 or 11 099
7. 1.32 s
8. a) 50 000   b) 290 000   c) 2017
9. 15 m by 22 m
10. −19, −18 or 18, 19
11. base = 8 cm, height = 24 cm
12. 2.1 m
13. a) after 1.68 s and again at 17.09 s
    b) The rocket will be above 150 m for 17.09 − 1.68 = 15.41 s.
14. $2.75
15. Factoring the function and finding the zeros; substituting the values of a, b, and c into the quadratic formula; graphing the function on a graphing calculator and using **CALC** to find the zeros
16. 10 cm, 24 cm, 26 cm
17. $x = 0$ or $-\dfrac{2}{3}$

Lesson 3.6, pp. 185–186

1. a) vertex (0, −5), up, 2 zeros   d) vertex (−2, 0), up, 1 zero
   b) vertex(0, 7), down, 2 zeros   e) vertex (−3, −5), down, no zeros
   c) vertex (0, 3), up, no zeros   f) vertex (4, −2), up, 2 zeros
2. a) 2 zeros   b) 2 zeros   c) 2 zeros   d) 1 zero
3. a) 2 zeros   b) no zeros   c) 1 zero   d) 1 zero
4. a) 2 zeros   b) 2 zeros   c) 2 zeros   d) no zeros
5. a) 2 break-even points   c) 1 break-even point
   b) Cannot break even   d) Cannot break even
6. $k = \dfrac{4}{3}$
7. $k < -2$ or $k > 2$
8. $k > \dfrac{4}{3}$, $k = \dfrac{4}{3}$, $k < \dfrac{4}{3}$
9. $k = -4$ or 8
10. No, resulting quadratic has no solutions.
11. Answers may vary. For example,
    a) $y = -2(x + 1)(x + 2)$
    b) $y = 2x^2 + 1$
    c) $y = -2(x - 2)^2$
12. A: break-even $x = 4.8$
    B: break-even $x = 0.93$
    C: break-even $x = 2.2$
    Buy Machine B. It has the earliest break-even point.
13. a) no effect   d) change from 1 to 2 zeros
    b) no effect   e) change from 1 to no zeros
    c) no effect   f) change from 1 to 2 zeros
14. 10.5
15. $f(x) = -(x - 3)(3x + 1) + 4$ is a vertical translation of 4 units up of the function $g(x) = -(x - 3)(3x + 1)$. Function $g(x)$ opens down and has 2 zeros. Translating this function 4 units up will have no effect on the number of zeros, so $f(x)$ has 2 zeros.
16. a) If the vertex is above the x-axis, the function will have 2 zeros if it opens down and no zeros if it opens up. If the vertex is below the x-axis, there will be 2 zeros if the function opens up and no zeros if it opens down. If the vertex is on the x-axis, there is only 1 zero.
    b) If the linear factors are equal or multiples of each other, there is 1 zero; otherwise, there are 2 zeros.
    c) If possible, factor and determine the number of zeros as in part (b). If not, use the value of $b^2 - 4ac$. If $b^2 - 4ac > 0$, there are 2 zeros, if $b^2 - 4ac = 0$, 1 zero, and if $b^2 - 4ac < 0$, no zeros.
17. $(x^2 - 1)k = (x - 1)^2$
    $kx^2 - k = x^2 - x - x + 1$
    $kx^2 - k = x^2 - 2x + 1$
    $0 = x^2 - kx^2 - 2x + 1 + k$
    $0 = x^2(1 - k) - 2x + (1 + k)$

The equation will have one solution when the discriminant is equal to zero.

$$\begin{aligned} b^2 - 4ac &= 0 \\ (-2)^2 - 4(1 - k)(1 + k) &= 0 \\ 4 - 4(1 - k^2) &= 0 \\ 4 - 4 + 4k^2 &= 0 \\ 4k^2 &= 0 \\ k^2 &= 0 \\ k &= 0 \end{aligned}$$

Therefore, the function will have one solution when $k = 0$.

18. Function has 2 zeros for all values of k.

Lesson 3.7, pp. 192–193

1. Same zeros, 3 and -4

2. Same vertex, $(2, -4)$; stretched vertically by different factors, opening in different directions

3. $(0, -7)$ (the y-intercept)

4. **a)** $f(x) = -\frac{7}{6}(x + 4)(x - 3)$ **c)** $f(x) = \frac{1}{6}(x^2 - 7)$

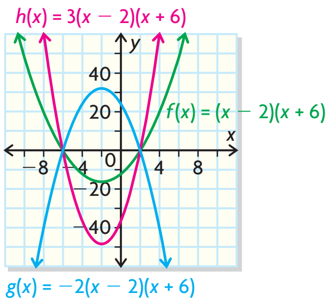
b) $f(x) = -\frac{6}{33}x(x - 8)$ **d)** $f(x) = -4(x^2 - 2x - 1)$

5. **a)** $f(x) = -\frac{13}{36}(x + 2)^2 + 5$ **c)** $f(x) = \frac{2}{25}(x - 4)^2 - 5$

b) $f(x) = -13(x - 1)^2 + 6$ **d)** $f(x) = \frac{8}{49}(x - 4)^2$

6. $f(x) = 5.5x^2 - 6x - 7$

7. **a)–c)**

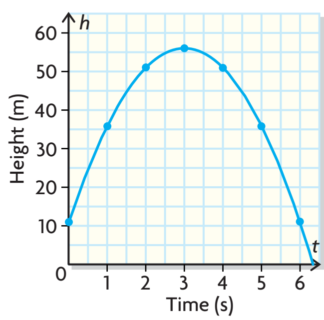


8. $f(x) = -\frac{6}{7}(x - 4)(x + 4)$

9. $f(x) = \frac{5}{33}(x^2 - 4x + 1)$

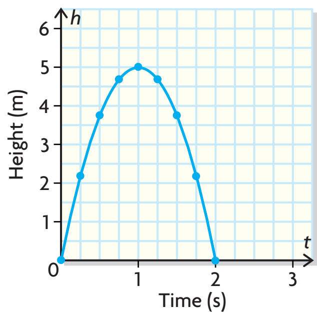
10. $f(x) = -\frac{3}{16}x(x - 12)$ Yes, because at a height of 5 m the bridge is 6.11 m wide.

11. **a), b)**



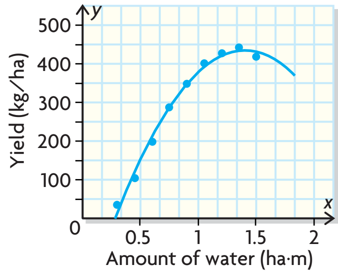
c) $h(t) = -5(t - 3)^2 + 56$

12. **a), b)**



c) $h(t) = -5t^2 + 10t$

13. **a)**

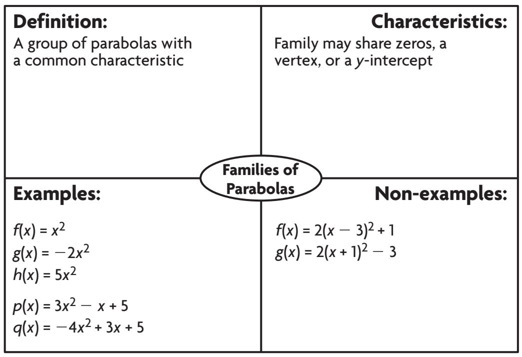


b) approximately $(1.35, 442)$

c) possible function (using $(0.60, 198)$ and vertex): $f(x) = 343(x - 1.35)^2 + 442$

14. $f(x) = -3(x + 3)(x + 1)$ or $f(x) = -3x^2 - 12x - 9$

15. Sample response:



16. 15.36 m

17. $f(x) = -\frac{1}{4}(x + 3)(x - 1)(x - 5)$

Lesson 3.8, pp. 198–199

1. **a)** $(3, 9)$ $(-2, 4)$ **b)** $(0, 3)$ $(-0.25, 2.875)$ **c)** no solutions

2. **a)** $(4, 3)$ $(6, -5)$ **b)** $(2, 7)$ $(-0.5, -0.5)$ **c)** no solutions

3. one solution

4. **a)** $(1.5, 8)$ $(-7, -43)$

b) $(1.91, 8.91)$ $(-1.57, 5.43)$

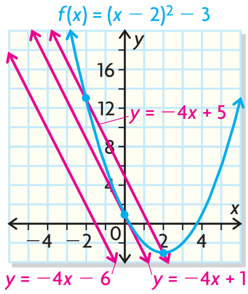
c) no solutions

d) $(-0.16, 3.2)$ $(-1.59, -3.95)$

5. 3 and 5 or -1 and 1

6. \$3.00

7. **a)**



b) $y = -4x - 6$, $y = -4x + 1$, $y = -4x + 5$

c) y-intercepts are all less than 1

8. $k = -5$

9. $k > \frac{73}{12}$

10. 7.20 s

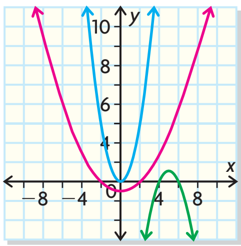
11. 10, −2

12. Yes, 0.18 s after kick at (0.18, 4.0)

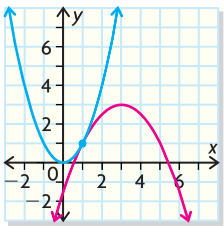
13. Plot graphs of functions and count points of intersection; calculate $b^2 - 4ac$, since there are two points of intersection when $b^2 - 4ac > 0$, one when $b^2 - 4ac = 0$, and none when $b^2 - 4ac < 0$

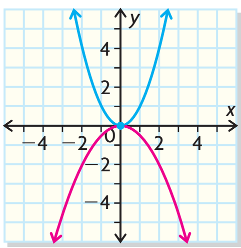
14. $(0, -2)$, $(4, -\frac{14}{3})$

15. Zero points of intersection:

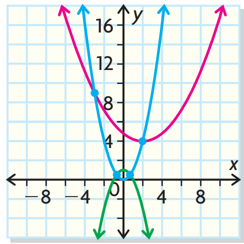


One point of intersection:





Two points of intersection:



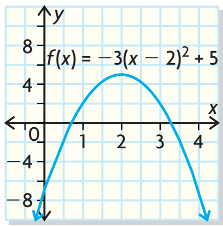
16. $y = 0.5x - 1$

Chapter Review, pp. 202–203

1. **a)** down, vertex (2, 5), $x = 2$

b) domain $= \{x \in \mathbf{R}\}$, range $= \{y \in \mathbf{R} \mid y \leq 5\}$

c)

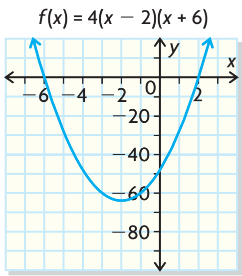


2. **a)** up, zeros 2 and −6

b) vertex (−2, −64)

c) domain $= \{x \in \mathbf{R}\}$; range $= \{y \in \mathbf{R} \mid y \geq -64\}$

d)



3. $x = -1$

4. **a)** Maximum value of 7 at $x = 4$

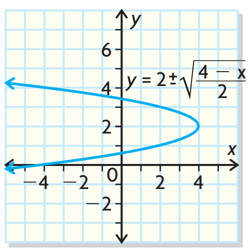
b) Minimum value of −36 at $x = -3$

5. 42 m after about 2.9 s

6. $g(x)$ and $h(x)$ are the two branches of the inverse of $f(x) = x^2$.

7. The inverse of a quadratic function is not a function, because it has two y-values for every x-value. It can be a function only if the domain of the original function has been restricted to a single branch of the parabola.

8. **a)**



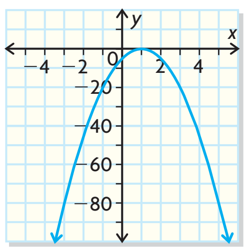
b) Domain $= \{x \in \mathbf{R} \mid x \leq 4\}$; Range $= \{y \in \mathbf{R}\}$

c) The inverse relation is not a function; it does not pass the vertical-line test.

9. a) $7\sqrt{2}$ c) $-4\sqrt{4}$
   b) $-20\sqrt{2}$ d) $37 - 12\sqrt{7}$
10. $2\sqrt{66}$
11. $(9 + 3\sqrt{5})$ cm
12. $\left(\frac{5}{2}, 0\right), (-3, 0)$
13. a) 52 428 b) 1990
14. 55.28 m by 144.72 m
15. Yes, because $14t - 5t^2 = 9$ has $b^2 - 4ac = 16 > 0$, so there are two roots. Because parabola opens down and is above t-axis for small positive t, at least one of these roots is positive.
16. $x < -0.5$ or $x > 3.5$
17. 4408 bikes
18. $f(x) = -\frac{5}{3}x^2 + \frac{20}{3}x - \frac{5}{3}$
19. The family of parabolas will all have vertex $(-3, -4)$; $f(x) = 10(x + 3)^2 - 4$
20. a) $f(x) = -\frac{7}{9}x^2 + 15$ b) 8.8 m
21. $(-5, 19), (1.5, -0.5)$
22. Yes, after 4 s. Height is 15 m.
23. a) No, they will not intersect. The discriminant of $f(x) - g(x)$ is -47. There are no real solutions for $f(x) - g(x)$, meaning that $f(x)$ and $g(x)$ do not intersect.
    b) Answers will vary. Use $g(x) = 3x - 5$.

Chapter Self-Test, p. 204

1. a) $f(x) = -5(x - 1)^2$, vertex $(1, 0)$
   b) zero at $x = 1$, axis of symmetry $x = 1$, opens down
   c) domain $= \{x \in \mathbf{R}\}$; range $= \{y \in \mathbf{R} \mid y \leq 0\}$
   d)



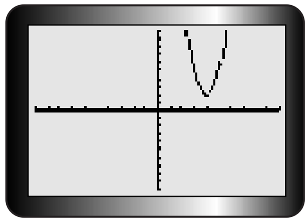
$f(x) = -5x^2 + 10x - 5$

2. a) Maximum value; complete the square.
   b) Minimum value; average the zeros.
3. a) Vertex form; vertex is visible in equation.
   b) Standard form; y-intercept is visible in equation.
   c) Factored form; zeros are visible in equation.
   d) Vertex form; use x-coordinate of vertex.
   e) Vertex form.; use vertex and direction of opening.
4. 360 000 m^2
5. $f^{-1} = 1 \pm \sqrt{\frac{x + 3}{2}}$
6. a) $2 - 4\sqrt{2}$
   b) $15 - 3\sqrt{10} + 5\sqrt{5} - 5\sqrt{2}$
   c) $\sqrt{8}$ can be simplified to $2\sqrt{2}$. This resulted in like radicals that could be combined.
7. $k = -2$ or 2
8. Intersects in 2 places, since $2x^2 - 3x + 2 = 6x - 5$ has $b^2 - 4ac > 0$; $(1, 1), (3.5, 16)$
9. $f(x) = -x^2 + 8x - 13$

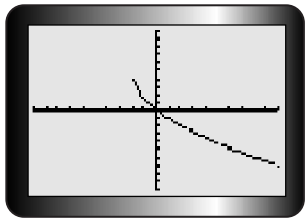
Cumulative Review Chapters 1–3, pp. 206–209

1. (c)	7. (d)	13. (d)	19. (d)	25. (b)	31. (b)
2. (b)	8. (b)	14. (b)	20. (d)	26. (c)	32. (c)
3. (b)	9. (a)	15. (a)	21. (c)	27. (d)	
4. (a)	10. (b)	16. (c)	22. (d)	28. (a)	
5. (b)	11. (c)	17. (d)	23. (b)	29. (b)	
6. (c)	12. (a)	18. (a)	24. (c)	30. (d)	

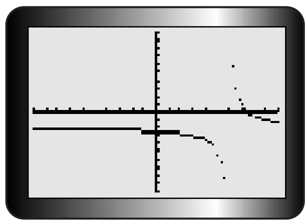
33. a) Domain: $\{x \in \mathbf{R}\}$, Range: $\{y \in \mathbf{R} \mid y \geq 2\}$; Parent function: $y = x^2$; Transformations: Vertical stretch by a factor of 3, horizontal translation 4 right, vertical translation 2 up; Graph:



    b) Domain: $\{x \in \mathbf{R} \mid x \geq -2\}$, Range: $\{y \in \mathbf{R} \mid y \leq 5\}$; Parent function: $y = \sqrt{x}$; Transformations: Vertical stretch by a factor of 2, reflection in the x-axis, horizontal compression by a factor of $\frac{1}{3}$, horizontal translation 2 left, vertical translation 5 up; Graph:



    c) Domain: $\{x \in \mathbf{R} \mid x \neq 6\}$, Range: $\{y \in \mathbf{R} \mid y \neq -2\}$; Parent function: $y = \frac{1}{x}$; Transformations: Horizontal stretch by a factor of 3, horizontal translation 6 to the right, vertical translation 2 down; Graph:



34. Jill: 3.6 km/h, 8 h 20 min; Sacha: 5 km/h; 6 h 20 min (including time to stop and talk with friend)
35. a) 8 students b) about \$700

Chapter 4

Getting Started, p. 212

1. a) 49 c) $\frac{1}{5}$ e) 10 000
   b) 32 d) 1 f) $\frac{1}{8}$

2. a) 9 c) -16 e) -125
b) -27 d) 16 f) -125

3. $(-5)^{120}$ will result in a positive answer since the exponent is an even number.

4. a) 81 c) 4096 e) 256
b) 5 764 801 d) $-1\ 000\ 000$ f) -1

5. a) 49 b) 24 c) 8 d) $\frac{1}{3}$

6. a) $\frac{55}{24}$ c) $\frac{21}{16}$ e) $\frac{62}{75}$
b) $-\frac{25}{24}$ d) $-\frac{3}{10}$ f) $-\frac{39}{16}$

7. a) a^7 b) b^4 c) c^{12} d) d^{10}

8. a) $x = 2$ b) $m = \frac{8}{3}$ c) $a = 3$ d) $r = 8$

9. a) $V \doteq 94.248\ \text{cm}^3$ b) $V \doteq 65.45\ \text{cm}^3$

10. a) first differences are all -5; linear function
b) first differences are 1, 2, 3, 4, 5; second differences are all 1; quadratic function

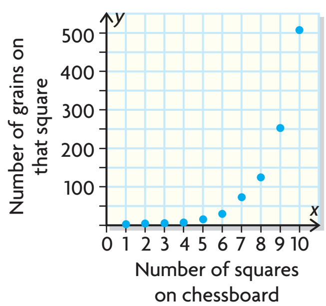
Lesson 4.1, p. 216

1. a) Both graphs decrease rapidly at the beginning, then continue to decrease less rapidly before levelling off.
b) 85 °C
c) 20 °C

2. a)

Number of Squares on the Chessboard	Number of Grains on that Square	First Differences
1	1	
2	2	1
3	4	2
4	8	4
5	16	8
6	32	16
7	64	32
8	128	64
9	256	128
10	512	256

b)



c) They are similar in their shape; that is both decrease rapidly at the beginning and then level off. They are different in that they have different y-intercepts and asymptotes.

Lesson 4.2, pp. 221–223

1. a) $\frac{1}{5^4}$ c) 2^4 e) $\frac{11}{3}$
b) $(-10)^3$ d) $-\left(\frac{5}{6}\right)^3$ f) $\frac{8}{7^2}$

2. a) $(-10)^0 = 1$ c) 2^{13} e) $-\frac{1}{9^4}$
b) $\frac{1}{6^2}$ d) $\frac{1}{11^8}$ f) $\frac{1}{7^{12}}$

3. $2^{-5} = \frac{1}{2^5}$ is less than $\left(\frac{1}{2}\right)^{-5} = 2^5$

4. a) $2^4 = 16$ c) $5^{-2} = \frac{1}{25}$ e) $4^3 = 64$
b) $(-8)^0 = 1$ d) $3^{-2} = \frac{1}{9}$ f) $7^{-2} = \frac{1}{49}$

5. a) $3^1 = 3$ c) $12^0 = 1$ e) $3^{-2} = \frac{1}{9}$
b) $9^0 = 1$ d) $5^0 = 1$ f) $9^1 = 9$

6. a) $10^3 = 1000$ c) $6^{-1} = \frac{1}{6}$ e) $2^{-3} = \frac{1}{8}$
b) $8^{-1} = \frac{1}{8}$ d) $4^2 = 16$ f) $13^1 = 13$

7. a) $-\frac{3}{16}$ c) 1 e) $\frac{1}{1000}$
b) $\frac{1}{2}$ d) 9 f) $-\frac{1}{12}$

8. a) $\frac{1}{400}$ c) $-\frac{1}{3}$ e) $\frac{1}{16}$
b) 2 d) $\frac{1}{9}$ f) 125

9. a) $-\frac{1}{64}$ c) $-\frac{1}{125}$ e) $-\frac{1}{216}$
b) $\frac{1}{16}$ d) $-\frac{1}{25}$ f) $-\frac{1}{36}$

10. 5^{-2}, 10^{-1}, 3^{-2}, 2^{-3}, 4^{-1}, $(0.1)^{-1}$; If the numerators of the numbers are all the same (1), then the larger the denominator, the smaller the number.

11. a) $\frac{1}{36}$ b) $-\frac{9}{2}$ c) $-\frac{2}{3}$ d) $\frac{1}{576}$

12. a) Erik: $3^{-1} \neq -\frac{1}{3}$ (negative exponents do not make numbers negative)
Vinn: $3 = 3^1$ and he did not add the exponents correctly.

b) Correct solution:
$$\boxed{\begin{aligned} & 3^{-2} \times 3 \\ &= \frac{1}{3^2} \times 3 \\ &= \frac{1}{9} \times 3 \\ &= \frac{3}{9} \\ &= \frac{1}{3} \end{aligned}}$$

13. a) $\frac{1}{8}$ c) $\frac{9}{4}$ e) 6 g) $\frac{1}{6}$ i) $-\frac{1}{9}$
b) $\frac{1}{6}$ d) $\frac{17}{4}$ f) $\frac{1}{26}$ h) $\frac{57}{34}$

14. a) 4 c) $\frac{1}{9}$ e) 9 g) 9
b) 9 d) $\frac{2}{3}$ f) $\frac{1}{81}$ h) 9

15. a) $(-10)^3$ is -10 multiplied by itself three times. 10^{-3} is the reciprocal of 10 cubed.
b) $(-10)^4$ is -10 multiplied by itself four times. -10^4 is the negative of 10^4.

16. a) $x = -1$ c) $x = 0$ e) $n = -2$
b) $x = -2$ d) $n = -2$ f) $w = -2$

17. $10^{-y} = \frac{1}{5}$

18. a) x^{10-2r} c) b^{m+2n} e) a^{10-2p}
b) b^{4m-n} d) x^{21-2r} f) $3^{6-m}x^{24-5m}$

Lesson 4.3, pp. 229–230

1. a) $\sqrt{49} = 7$ c) $\sqrt[3]{-125} = -5$ e) $\sqrt[4]{81} = 3$
b) $\sqrt{100} = 10$ d) $\sqrt[4]{16} = 2$ f) $-\sqrt{144} = -12$

2. a) $512^{\frac{1}{9}} = 2$ d) $(-216)^{\frac{5}{3}} = 7776$
b) $(-27)^{\frac{1}{3}} = -3$ e) $\left(\frac{-32}{243}\right)^{\frac{1}{5}} = \frac{-2}{3}$
c) $27^{\frac{2}{3}} = 9$ f) $\left(\frac{16}{81}\right)^{\frac{-1}{4}} = \frac{3}{2}$

3. a) 8^1 c) $(-11)^{\frac{11}{4}}$ e) $9^{\frac{-13}{15}}$
b) $8^{\frac{1}{3}}$ d) 7 f) $10^{-\frac{7}{5}}$

4. a) 5 b) -2 c) 4 d) 3

5. a) 11 c) $\frac{47}{3}$ e) $\frac{253}{4}$
b) -18 d) $-\frac{255}{32}$ f) 3

6. a) $4^{\frac{1}{2}} = 2$ c) 4 e) $16^{-\frac{1}{4}} = \frac{1}{2}$
b) $100^{\frac{-1}{2}} = \frac{1}{10}$ d) $\frac{1}{3}$ f) 64

7. a) 4.996 c) 1.262 e) 5.983
b) 6.899 d) 2.999 f) 98.997

8. 0.25 m

9. $27^{\frac{4}{3}} = 81$, $27^{1.3333} \doteq 80.991\ 101\ 73$ The values are not equal as $\frac{4}{3} \neq 1.3333$.

10. $0.2 = \frac{1}{5}$, an odd root , $0.5 = \frac{1}{2}$, an even root. Even root of a negative number is not real.

11. $125^{\frac{-2}{3}} = \frac{1}{125^{\frac{2}{3}}} = \frac{1}{(125^{\frac{1}{3}})^2} = \frac{1}{5^2} = \frac{1}{25}$

12. a) -8 c) 0.0081 e) 0.008
b) 39.0625 d) 2.25 f) 1 679 616

13. $4^{2.5} = 4^{\frac{5}{2}} = (\sqrt{4})^5$. Change 2.5 to a fraction as $\frac{5}{2}$. This is the same as taking the square root of the four and then taking the fifth power of that result.

14. a) false c) false e) false
b) false d) true f) true

15. a) n cannot be zero or an even number. b) n cannot be zero.

16. The value of x can equal the value of y. Also $x = 4$ and $y = 2$.

17. Yes this works. The value of i is approximately 0.017.

18. a) $x = \frac{1}{6}$ b) $x = \frac{27}{2}$

Lesson 4.4, pp. 235–237

1. a) x^7 c) m^8 e) y^6
b) p^2 d) $\frac{1}{a^2}$ f) $\frac{1}{k^{12}}$

2. a) $\frac{1}{y^2}$ c) $\frac{1}{n^{24}}$ e) 1
b) x^4 d) $\frac{1}{w}$ f) $\frac{1}{b^{19}}$

3. a) 36 b) $x^2y^2 - 36$
c) Usually it is faster to substitute numbers into the simplified form.

4. a) p^2q c) $\frac{1}{a^2b^7}$ e) wx^4
b) $\frac{y^2}{x^6}$ d) $\frac{n^6}{m^4}$ f) $\frac{a^6}{b^4}$

5. a) $72x^8y^{11}$ c) $\frac{y^6}{150x^4}$ e) $\frac{r^4}{p^7}$
b) $\frac{a^5}{b^2}$ d) $\frac{3m^{10}}{4n^2}$ f) $\frac{y^4}{x^5}$

6. a) 1 c) $\frac{5}{6m^{11}}$ e) $\frac{1}{4}$
b) $\frac{9c}{2}$ d) $10x^4$ f) $\frac{1}{x}$

7. a) $4x^3y^2 = 32$ c) $\frac{9y^2}{8x} = \frac{45}{16}$
b) $\frac{1}{2p^3} = \frac{1}{54}$ d) $\frac{1}{35b} = \frac{1}{350}$

8. a) $1000x^{\frac{3}{4}} = 8000$ c) $\frac{-5}{8a^4} = \frac{-5}{8}$
b) $\frac{x^3}{16} = \frac{125}{16}$ d) $\frac{12n^3}{m^2} = \frac{3}{25}$

9. a) $18m^5n^5$ b) $\frac{27y}{2}$ c) $4a^{10}$ d) $\frac{1}{\sqrt{3x^{\frac{5}{2}}}}$

10. $M = \frac{x^4}{16y^9}$
a) Answers may vary. For example, $x = 2, y = 1$
b) Answers may vary. For example, $x = 3, y = 1$
c) Answers may vary. For example, $y = 1, x = 1$
d) impossible, always positive

11. a) $\frac{2h + 2r}{hr}$ b) $\frac{SA}{V} = \frac{8}{3} \doteq 2.67\ \text{cm}^{-1}$

12. These simplify to $\frac{y}{x} = -\frac{3}{2}, \frac{x^2}{y^2} = \frac{4}{9}, \frac{x^2}{y^3} = \frac{4}{27}$, respectively. Switch second and third for proper order.

13. Algebraic and numerical expressions are similar in the following way: when simplifying algebraic or numerical expressions, you have to follow the order of operations. When simplifying algebraic expressions, you can only add or subtract like terms—while unlike terms may be multiplied. In this way algebraic expressions are different than numerical expressions.

14. a) $r = \sqrt[3]{\frac{3V}{4\pi}} = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$ b) 4 m

15. x

Answers

Mid-Chapter Review, p. 239

1. **a)** 5^5 **c)** 9^{18} **e)** $\left(\frac{1}{10}\right)^2$
b) $-\frac{1}{8}$ **d)** 3^2 **f)** 7^2

2. **a)** $-\frac{1}{16}$ **b)** $\frac{24}{25}$ **c)** $\frac{4}{25}$ **d)** $\frac{-7}{64}$

3. **a)** $\frac{16}{49}$ **b)** $-\frac{8}{125}$ **c)** $-\frac{27}{8}$ **d)** -27

4. x cannot be zero or a negative number for $x^{\frac{-1}{2}}$. x can be zero for $x^{\frac{1}{2}}$, but not negative.

5. **a)** $\frac{7}{9}$ **c)** $\frac{3}{4}$ **e)** $\frac{1}{3}$
b) $\frac{10}{11}$ **d)** $-\frac{1}{125}$ **f)** 2

6.

	Exponential Form	Radical Form	Evaluation of Expression
a)	$100^{\frac{1}{2}}$	$\sqrt{100}$	10
b)	$16^{0.25}$	$\sqrt[4]{16}$	2
c)	$121^{\frac{1}{2}}$	$\sqrt{121}$	11
d)	$(-27)^{\frac{5}{3}}$	$\sqrt[3]{(-27)^5}$	-243
e)	$49^{2.5}$	$\sqrt{49^5}$	16 807
f)	$1024^{\frac{1}{10}}$	$\sqrt[10]{1024}$	2

7. **a)** -33.068 **b)** 31.147 **c)** 0.745 **d)** -63.096

8. $-8^{\frac{4}{3}} = -(\sqrt[3]{8})^4 = -(2)^4 = -16$ and $(-8)^{\frac{4}{3}} = (\sqrt[3]{-8})^4 = (-2)^4 = 16$
The second expression has an even root so the negative sign is eliminated.

9. **a)** $\frac{1}{x^5}$ **c)** $\frac{1}{y^6}$ **e)** $\frac{3}{a^8}$
b) n^4 **d)** $-x^5$ **f)** -8

10. **a)** $\frac{x^{0.2}}{y^{0.7}}$ **c)** $\frac{y^5}{x}$ **e)** 1
b) $\frac{n}{m}$ **d)** $\frac{4a^6b^{10}}{c^2}$ **f)** $\frac{2}{5}$

11. **a)** $\frac{2b}{a} = 3$ **b)** $\frac{3}{a^2b^2} = \frac{1}{12}$

12. **a)** a^{p+2} **b)** $2^{3-2m}x^{6-6m}$ **c)** 1 **d)** x^{n-4m}

Lesson 4.5, p. 243

1. **a)** quadratic **b)** exponential **c)** exponential **d)** linear

2. **a)** exponential; the values increase at a fast rate
b) exponential; the values increase at a fast rate
c) linear; straight line
d) quadratic; graph is a parabolic shape

Lesson 4.6, pp. 251–253

1. **a)** The function moves up 3 units. It is a vertical translation.
b) The function moves to the left 3 units. It is a horizontal translation.
c) The function values are decreased by a factor of $\frac{1}{3}$. It is a vertical compression.
d) The x-values are increased by a factor of 3. It is a horizontal stretch.

2. **a)** The base function is 4^x. Horizontal translation left 1 unit, vertical stretch of factor 3, and reflection in x-axis.
b) The base function is $\left(\frac{1}{2}\right)^x$. Vertical stretch factor 2, horizontal compression of factor $\frac{1}{2}$, and vertical translation of 3 units up.
c) The base function is $\left(\frac{1}{2}\right)^x$. Vertical stretch factor 7, and vertical translation of 1 unit down and 4 units to the right.
d) The base is 5^x. Horizontal compression by a factor of $\frac{1}{3}$ and a translation 2 units to the right.

3.

Function	y-intercept	Asymptote	Domain	Range
$y = 3^x + 3$	4	$y = 3$	$x \in \mathbf{R}$	$y > 3, y \in \mathbf{R}$
$y = 3^{x+3}$	27	$y = 0$	$x \in \mathbf{R}$	$y > 0, y \in \mathbf{R}$
$y = \frac{1}{3}(3^x)$	$\frac{1}{3}$	$y = 0$	$x \in \mathbf{R}$	$y > 0, y \in \mathbf{R}$
$y = 3^{\frac{x}{3}}$	1	$y = 0$	$x \in \mathbf{R}$	$y > 0, y \in \mathbf{R}$
$y = -3(4^{x+1})$	-12	$y = 0$	$x \in \mathbf{R}$	$y < 0, y \in \mathbf{R}$
$y = 2\left(\frac{1}{2}\right)^{2x} + 3$	5	$y = 3$	$x \in \mathbf{R}$	$y > 3, y \in \mathbf{R}$
$y = 7(0.5^{x-4}) - 1$	111	$y = -1$	$x \in \mathbf{R}$	$y > -1, y \in \mathbf{R}$
$y = 5^{3x-6}$	6.4×10^{-5}	$y = 0$	$x \in \mathbf{R}$	$y > 0, y \in \mathbf{R}$

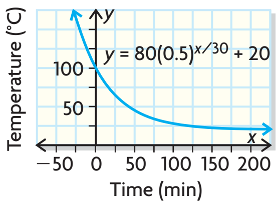
4. **a)** horizontal compression of factor $\frac{1}{2}$, reflect in x-axis
b) vertical stretch of factor 5, reflect in y-axis, translate 3 units right
c) vertical stretch of factor 4, horizontal compression of factor $\frac{1}{3}$, reflect in the x-axis, translate 3 units left and 6 units down

5.

	Function	Transformations	y-intercept	Asymptote	Domain	Range
a)	$0.5f(-x) + 2$	• vertical compression by a factor of $\frac{1}{2}$ • reflection in the y-axis • translation of 2 units up	2.5	$y = 2$	$x \in \mathbf{R}$	$y > 2, y \in \mathbf{R}$
b)	$-f(0.25x + 1) - 1$	• reflection in the x-axis • horizontal stretch of 4 • translation 1 down and 4 left	-5	$y = -1$	$x \in \mathbf{R}$	$y < -1, y \in \mathbf{R}$
c)	$-2f(2x - 6)$	• reflection in the x-axis • vertical stretch of 2 • horizontal compression by factor of $\frac{1}{2}$ • translation 3 units right	$\frac{-2}{4^6}$	$y = 0$	$x \in \mathbf{R}$	$y < 0, y \in \mathbf{R}$
d)	$f(-0.5x + 1)$	• reflection in the y-axis • horizontal stretch of 2 • horizontal translation of 2 units right	4	$y = 0$	$x \in \mathbf{R}$	$y > 0, y \in \mathbf{R}$

6. Both functions have the y-intercept of 1 and the asymptote is $y = 0$. Their domains are all real numbers and their ranges are $y > 0$. The second function will increase at a faster rate.

7. **Water Temperature vs. Time**



The y-intercept represents the initial temperature of 100 °C. The asymptote is the room temperature and the lower limit on the temperature of the water.

8. **a)** If the doubling time were changed to 9, the exponent would change from $\frac{t}{3}$ to $\frac{t}{9}$. The graph would not rise as fast.

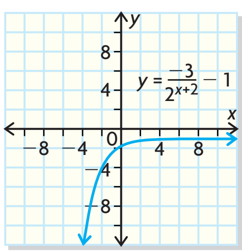
b) domain is $t \geq 0$, $t \in \mathbf{R}$; range is $N > N_0$, $N \in \mathbf{R}$

9. **a)** (iii) **b)** (ii) **c)** (iv) **d)** (i)

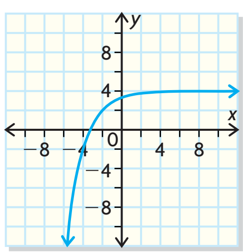
10. **a)** $y = -2^{2x} + 6$ **b)** $y = 2^{-x-3} - 2$

11. Translate down 5 units, translate right 1 unit, and vertically compress by factor $\frac{1}{4}$.

12.



13.



$y = 4 - 2\left(\frac{1}{3}\right)^{0.5x+1}$

14. Reflect in the y-axis and translate 2 units up.

Lesson 4.7, pp. 261–264

1. **a)** 407.22 **b)** 35.16 **c)** 378.30 **d)** 13 631.85

2.

	Function	Exponential Growth or Decay?	Initial Value	Growth or Decay Rate
a)	$V = 20(1.02)^t$	growth	20	2%
b)	$P = (0.8)^n$	decay	1	−20%
c)	$A = 0.5(3)^x$	growth	0.5	200%
d)	$Q = 600\left(\frac{5}{8}\right)^w$	decay	600	−37.5%

3. **a)** 1250 persons; it is the value for a in the general exponential function.
b) 3%; it is the base of the exponent minus 1. **c)** 1730 **d)** 2012

4. **a)** \$1500; it is the value for a in the general exponential function.
b) −5%; it is the base of the exponent minus 1.
c) \$437.98
d) 10^{th} month after purchase

5. **a)** 6% **c)** 15
b) \$1000 **d)** $V = 1000(1.06)^n$, \$2396.56

6. **a)** The doubling period is 10 hours.
b) 2 represents the fact that the population is doubling in number (100% growth rate).
c) 500 is the initial population.
d) 1149 bacteria
e) 2639 bacteria
f) The population is 2000 at 8 a.m. the next day.

7. **(c)** and **(d)** have bases between 0 and 1.

8. **a)** 15361
b) During the 29th year the population will double.
c) 17 years ago
d) $\{n \in \mathbf{R}\}$; $\{P \in \mathbf{R} \mid P \geq 0\}$

9. **a)** 82 °C **b)** 35 °C **c)** after 25 min

10. **a)** $C = 100(0.99)^w$.
100 refers to the percent of the colour at the beginning.
99 refers to the fact that 1% of the colour is lost during every wash.
w refers to the number of washes.
b) $P = 2500(1.005)^t$
2500 refers to the initial population.
1.005 refers to the fact that the population grows 0.5% every year.
t refers to the number of years after 1990.
c) $P = P_0(2)^t$
2 refers to the fact that the population doubles in one day.
t refers to the number of days.

11. **a)** 100% **d)** 226
b) $P = 80(2^t)$ **e)** 13.6 h
c) 5120 **f)** $\{t \in \mathbf{R} \mid t \geq 0\}$; $\{P \in \mathbf{R} \mid p \geq 80\}$

12. **a)** $V = 5(1.06^t)$ **b)** \$0.36 **c)** \$0.91

13. **a)** $I = 100(0.91^d)$ **b)** 49.3%

14. **a)** $P = 100(0.01^a)$ **b)** 6 applications

15. approximately 2.7%

16. **a)** $P = 200(1.75^{\frac{t}{3}})$
b) 200 refers to the initial count of yeast cells.
1.75 refers to the fact that the cells grow by 75% every 3 h.
$\frac{t}{3}$ refers to the fact that the cells grow every 3 h.

17. **a)** It could be a model of exponential growth.
b) $y = 4.25^x$ may model the situation.
c) There are too few pieces of data to make a model and the number of girls may not be the same every year.

18. **a)** exponential decay **b)** 32.3%

Chapter Review, pp. 267–269

1. **a)** $x^2 > x^{-2}$, if $x > 1$; If $x > 1$, then $x^{-2} = \frac{1}{x^2}$ will be less than one and x^2 will be greater than one.
b) $x^{-2} > x^2$, if $-1 < x < 1$ and $x \neq 0$, then $\frac{1}{x^2}$ will be greater than one and x^2 will be less than one.

2. **a)** $(7)^{-1} = -\frac{1}{7}$ **c)** $5^0 = 1$ **e)** $11^2 = 121$
b) $(-2)^5 = -32$ **d)** $4^4 = 256$ **f)** $(-3)^3 = -27$

3. **a)** $x^{\frac{7}{3}}$ **c)** $p^{\frac{11}{2}}$
b) $\sqrt[5]{y^8}$ **d)** $\sqrt[4]{m^5}$

4. **a)** $\frac{125}{8}$ **c)** $-\frac{1}{15}$ **e)** -64
b) $\frac{15}{4}$ **d)** 81 **f)** 2

5. **a)** 1 **c)** $\frac{1}{c^{\frac{1}{6}}}$ **e)** e^{14}
b) b **d)** $d^{\frac{13}{2}}$ **f)** $\frac{1}{f^{\frac{1}{5}}}$

6. Let $a = 9$ and $b = 16$; then $\sqrt{9 + 16} = \sqrt{25} = 5$ but $\sqrt{9} + \sqrt{16} = 7$, and $5 \neq 7$ so $\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$

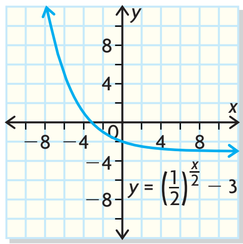
7. **a)** $200x^5 = -6400$ **c)** $\frac{3}{2w^3} = \frac{-1}{18}$ **e)** $\frac{x^{12}}{6} = \frac{2048}{3}$
b) $\frac{64}{m^2}$ **d)** $3y^5 = -96$ **f)** $\frac{16}{3x} = \frac{32}{3}$

8. **a)** $3xy^3$ **c)** $\frac{1}{m^2 n^{\frac{1}{2}}}$ **e)** $-\frac{1}{x^{1.8}}$
b) $\frac{b}{a}$ **d)** x^9 **f)** $\frac{x^9}{y}$

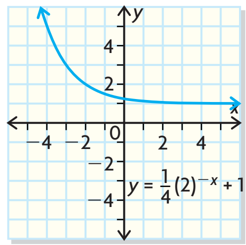
9. **a)** quadratic **c)** exponential **e)** exponential
b) linear **d)** exponential **f)** exponential

10. **a)** exponential **b)** quadratic **c)** exponential

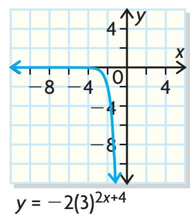
11. **a)** $y = \frac{1}{2}^x$; horizontal stretch by a factor of 2 and vertical translation of 3 down



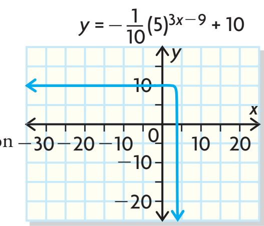
b) $y = 2^x$; vertical stretch of $\frac{1}{4}$, reflection in the y-axis, and vertical translation of 1 unit up



c) $y = 3^x$; reflection in the x-axis, vertical stretch by a factor of 2, horizontal compression by a factor of 2, and horizontal translation of 2 left



d) $y = 5^x$; reflection in the x-axis, vertical compression of $\frac{1}{10}$, horizontal compression by a factor of 3, horizontal translation of 3 units right, and vertical translation of 10 units up

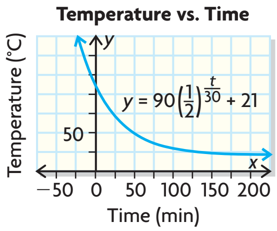


12. y-intercept $= 2$
asymptote: $y = 1$
equation: $y = 2^x + 1$

13.

	Function	Growth/ Decay	y-int	Growth/ Decay Rate
a)	$V(t) = 100(1.08)^t$	growth	100	8%
b)	$P(n) = 32(0.95)^n$	decay	32	−5%
c)	$A(x) = 5(3)^x$	growth	5	200%
d)	$Q(n) = 600\left(\frac{5}{8}\right)^n$	decay	600	−37.5%

14. a) The base of the exponent is less than 1.
b) 90 °C
c)



d) 44 °C
e) The 30 in the exponent would be a lesser number.
f) There would be a horizontal compression of the graph; that is, the graph would increase more quickly.

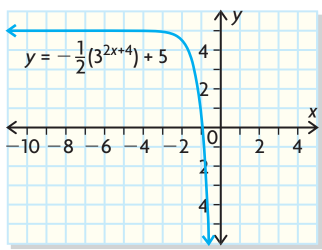
15. a) \$28 000 c) \$18 758 e) \$3500
b) 12.5% d) \$20 053 f) \$2052

16. a) $P = \frac{1}{3}(1.1)^n$
$\frac{1}{3}$ refers to the fact that the pond is $\frac{1}{3}$ covered by lilies.
1.1 refers to the 10% increase in coverage each week.
n refers to the number of weeks.
b) $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{4.5\times 10^9}}$
A_0 refers to the initial amount of U_{238}.
$\frac{1}{2}$ refers to the half-life of the isotope.
t refers to the number of years.
c) $I = 100(0.96)^n$
0.96 refers to the 4% decrease in intensity per gel.
n refers to the number of gels.

17. a) $P = 45\,000(1.03)^n$ c) during 2014
b) 74 378 d) 7.2%

Chapter Self-Test, p. 270

1. a) There is a variable in the exponent part of the equation, so it's an exponential equation.
b) You can tell by the second differences.
c) reflection in the x-axis, vertical compression of $\frac{1}{2}$, horizontal compression by a factor of 2, and translations of 4 left and 5 up



2. a) $-\frac{1}{125}$
b) 9

3. a) $-243y^5$
b) $\frac{1}{3125a^3b}$
c) $2x$
d) $\frac{1}{4xy^4}$

4. a) $I = 100(0.964)^n$
b) 89.6%
c) As the number of gels increases the intensity decreases exponentially.

5. a) $P = 2(1.04)^n$, where P is population in millions and n is the number of years since 1990
b) 18 years after 1990 or in 2008

6. (d)

7. $n \neq 0$; n must be odd because you cannot take even roots of negative numbers.

Chapter 5

Getting Started, p. 274

1. a) $c = 13$ m
b) $f = \sqrt{57}$ m

2. a) $\sin A = \frac{5}{13}$, $\cos A = \frac{12}{13}$, $\tan A = \frac{5}{12}$
b) $\sin D = \frac{8}{11}$, $\cos D = \frac{\sqrt{57}}{11}$, $\tan D = \frac{8\sqrt{57}}{57}$

3. a) 67°
b) 43°

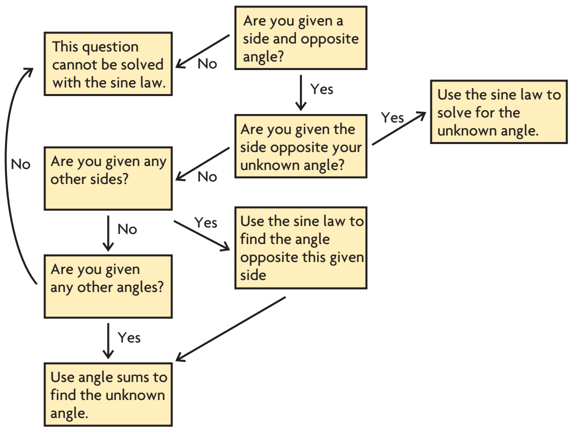
4. a) 0.515
b) 0.342

5. a) 71°
b) 45°
c) 48°

6. 61 m

7. 25.4 m

8. Answers may vary. For example,



Lesson 5.1, pp. 280–282

1. $\sin A = \frac{5}{13}$, $\cos A = \frac{12}{13}$, $\tan A = \frac{5}{12}$,
$\csc A = \frac{13}{5}$, $\sec A = \frac{13}{12}$, $\cot A = \frac{12}{5}$

2. $\csc \theta = \frac{17}{8}$, $\sec \theta = \frac{17}{15}$, $\cot \theta = \frac{15}{8}$

3. a) $\csc \theta = 2$ b) $\sec \theta = \frac{4}{3}$ c) $\cot \theta = \frac{2}{3}$ d) $\cot \theta = 4$

4. a) 0.83 b) 1.02 c) 0.27 d) 1.41

Answers

5. a) i) $\csc\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = \frac{4}{3}$

ii) $\csc\theta = \frac{12}{8.5}$, $\sec\theta = \frac{12}{8.5}$, $\cot\theta = 1$

iii) $\csc\theta = \frac{3.6}{3}$, $\sec\theta = \frac{3.6}{2}$, $\cot\theta = \frac{2}{3}$

iv) $\csc\theta = \frac{17}{8}$, $\sec\theta = \frac{17}{15}$, $\cot\theta = \frac{15}{8}$

b) i) 37° ii) 56° iii) 45° iv) 28°

6. a) 17° b) 52° c) 46° d) 60°

7. a) 5.2 m b) 6.4 m

8. a) 1.2 cm b) 8.0 km

9. a) For any right triangle with acute angle θ, $\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$.

Case 1: If neither the adjacent side nor the opposite side is zero, the hypotenuse is always greater than either side and $\csc\theta > 1$.
Case 2: If the adjacent side is reduced to zero, each time you calculate $\csc\theta$, you get a smaller and smaller value until $\csc\theta = 1$.
Case 3: If the opposite side is reduced to zero, each time you calculate $\csc\theta$, you get a greater and greater value until you reach infinity.
So for all possible cases in a right triangle, cosecant is always greater than or equal to 1.

b) For any right triangle with acute angle θ, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$.

Case 1: If neither the adjacent side nor the opposite side is zero, the hypotenuse is always greater than either side and $\cos\theta < 1$.
Case 2: If the opposite side is reduced to zero, each time you calculate $\cos\theta$, you get a greater and greater value until $\cos\theta = 1$.
Case 3: If the adjacent side is reduced to zero, each time you calculate $\cos\theta$, you get a smaller and smaller value until $\cos\theta = 0$.
So for all possible cases in a right triangle, cosine is always less than or equal to 1.

10. $\theta = 45°$ and adjacent side = opposite side

11. a) and b) 13.1 m

12. 7.36 m

13. (b) a right triangle with two 45° angles would have the greatest area, at an angle of 41°, (b) is closest to 45° and will therefore have the greatest area of those triangles.

14. 4.5 m

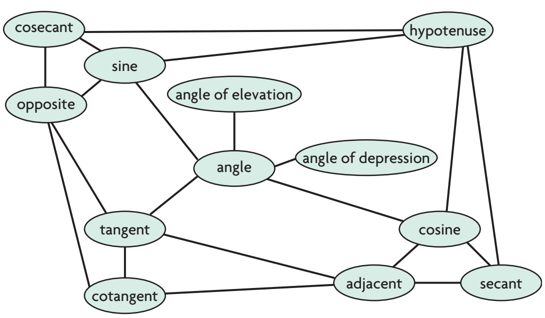
15. 8.15 m

16. a) Answers will vary. For example, 10° b) 7°

c) $\sin\theta = \frac{3}{\sqrt{634}}$, $\cos\theta = \frac{25}{\sqrt{634}}$, $\tan\theta = \frac{3}{25}$,

$\csc\theta = \frac{\sqrt{634}}{3}$, $\sec\theta = \frac{\sqrt{634}}{25}$, $\cot\theta = \frac{25}{3}$

17. Answers will vary. For example,



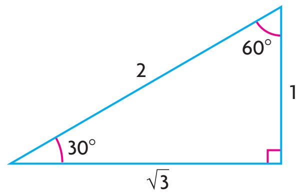
18. $p = 53$ cm, $q = 104$ cm, $\angle P = 27°$, $\angle Q = 63°$

19. Since $\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$, the adjacent side must be the smallest side.

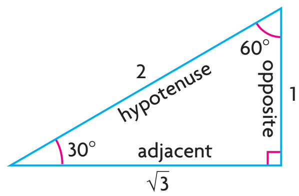
20. (csc and cot) 0°, (sec) 90°

Lesson 5.2, pp. 286–288

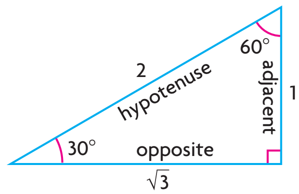
1. a)



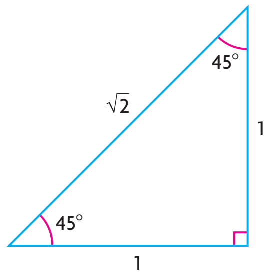
b)



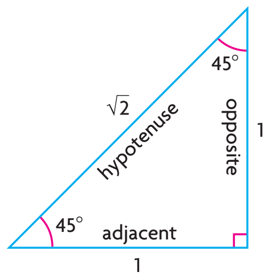
c)



2. a)



b)



3. a) $\frac{\sqrt{3}}{2}$ b) $\frac{\sqrt{3}}{2}$ c) 1 d) $\frac{\sqrt{2}}{2}$

4. a) 0 b) 1 c) $-\frac{1}{6}$ d) 0

5. a) $\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$ c) $\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$

b) $\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1$

6. a) $\dfrac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} = \dfrac{\sqrt{3}}{3}$ b) $\dfrac{\left(\frac{\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)} = 1$ c) $\dfrac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = \sqrt{3}$

7. a) 60° b) 30° c) 45° d) 30°

8. $\dfrac{5\sqrt{3}}{2}$ m, assuming that the wall is perpendicular to the floor

9. $\dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{3}} = \dfrac{1}{\frac{1}{2} \times \frac{\sqrt{3}}{2}} = \dfrac{4\sqrt{3}}{3}$

10. a) Use the proportions of the special triangle $45° - 45° - 90°$, given that the two smaller sides are 27.4 m.
  b) 38.7 m

11. a) $3(6 + 6\sqrt{3})$ square units
  b) $\dfrac{169}{8}(3 + \sqrt{3})$ square units

12. a) 2.595 b) $\dfrac{2\sqrt{2} - \sqrt{6} + 10}{4}$
  c) Megan didn't use a calculator. Her answer is exact, not rounded off.

13. $\dfrac{2\sqrt{(3 - 3}}{4}$

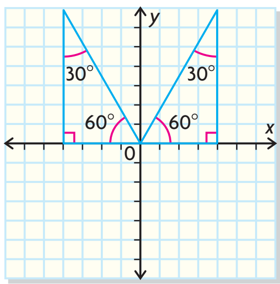
14. $\dfrac{1}{4}$

15. a) $1 + \left(\dfrac{3}{\sqrt{3}}\right)^2 = (2)^2$ c) $1 + \left(\dfrac{3}{\sqrt{3}}\right)^2 = \left(\dfrac{2}{\sqrt{3}}\right)^2$
  b) $1 + (1)^2 = \left(\dfrac{2}{\sqrt{2}}\right)^2$

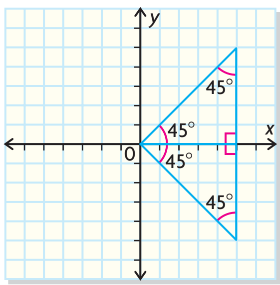
Lesson 5.3, p. 292

1. a) 135° c) 210°
  b) 120°, 240° d) 45°, 225°

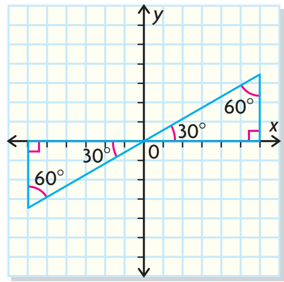
2. a)



b)



c)



3. a) 45°
  b) $\tan\theta = 1$, $\cos\theta = -\dfrac{\sqrt{2}}{2}$, $\sin\theta = -\dfrac{\sqrt{2}}{2}$

4.

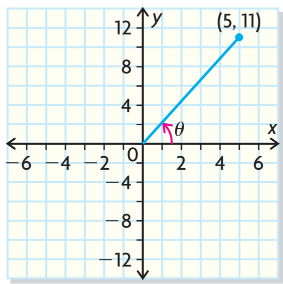
Trigonometric Ratio	Quadrant 1	2	3	4
sine	+	+	−	−
cosine	+	−	−	+
tangent	+	−	+	−

Lesson 5.4, pp. 299–301

1.

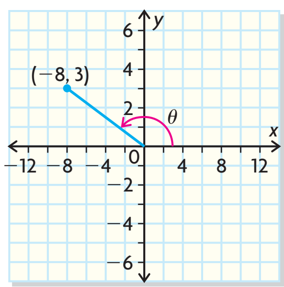
	quadrant	β	sign
a)	4	45°	−
b)	2	70°	−
c)	4	75°	+
d)	3	45°	+

2. a) i)



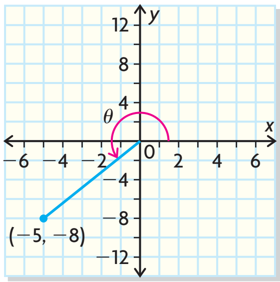
  ii) $r = 12.1$
  iii) $\sin\theta = \dfrac{11}{12.1}$, $\cos\theta = \dfrac{5}{12.1}$, $\tan\theta = \dfrac{11}{5}$
  iv) $\theta = 66°$

  b) i)



  ii) $r = 8.5$
  iii) $\sin\theta = \dfrac{3}{8.5}$, $\cos\theta = \dfrac{-8}{8.5}$, $\tan\theta = \dfrac{3}{-8}$
  iv) $\theta = 159°$

c) i)

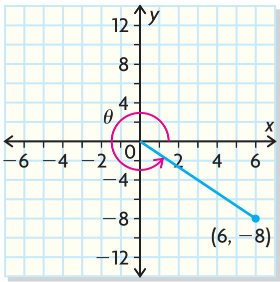


ii) $r = 9.4$

iii) $\sin\theta = \frac{-8}{9.4}$, $\cos\theta = \frac{-5}{9.4}$, $\tan\theta = \frac{8}{5}$

iv) $\theta = 238°$

d) i)



ii) $r = 10$

iii) $\sin\theta = \frac{-8}{10}$ or $\frac{-4}{5}$, $\cos\theta = \frac{6}{10}$ or $\frac{3}{5}$, $\tan\theta = \frac{-8}{6}$ or $\frac{-4}{3}$

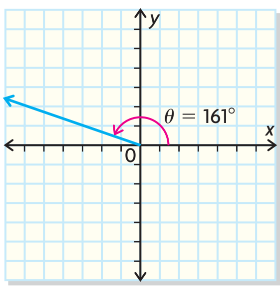
iv) $\theta = 307°$

3. **a)** $\sin 180° = 0$, $\cos 180° = -1$, $\tan 180° = 0$
b) $\sin 270° = -1$, $\cos 270° = 0$, $\tan 270°$ is undefined
c) $\sin 360° = 0$, $\cos 360° = 1$, $\tan 360° = 0$

4. **a)** $\sin 20°$ **b)** $\cos 60°$ **c)** $\tan 290°$ **d)** $\sin 190°$

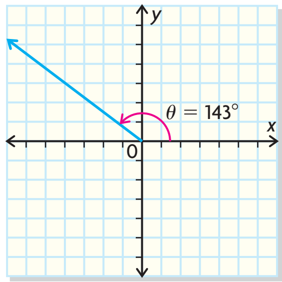
5. **a) i)** $\sin\theta$
ii) $\sin 165° = 0.26$, $\cos 165° = -0.97$, $\tan 165° = -0.27$
b) i) $\tan\theta$
ii) $\sin(-125°) = -0.82$, $\cos(-125°) = -0.57$, $\tan(-125°) = 1.43$
c) i) $\sin\theta$
ii) $\sin(-251°) = 0.95$, $\cos(-251°) = -0.33$, $\tan(-251°) = -2.90$
d) i) $\cos\theta$
ii) $\sin 332° = -0.47$, $\cos 332° = 0.88$, $\tan 332° = -0.53$

6. **a) i)** $x = -2\sqrt{2}$, $y = 1$, $r = 3$
ii)



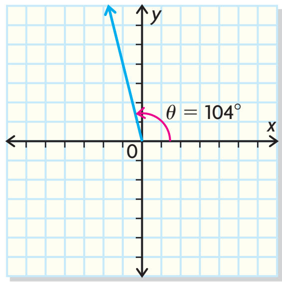
iii) $\theta = 161°$, $\beta = 19°$

b) i) $x = -4$, $y = 3$, $r = 5$
ii)



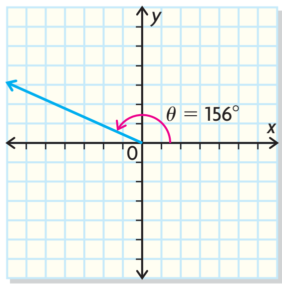
iii) $\theta = 143°$, $\beta = 37°$

c) i) $x = -1$, $y = \sqrt{15}$, $r = 4$
ii)



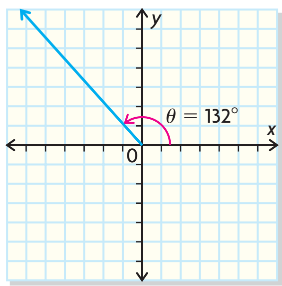
iii) $\theta = 104°$, $\beta = 76°$

d) i) $x = -\sqrt{21}$, $y = 2$, $r = 5$
ii)



iii) $\theta = 156°$, $\beta = 24°$

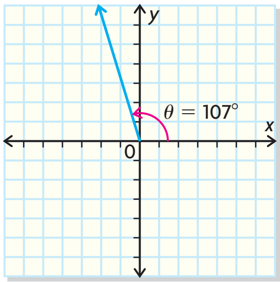
e) i) $x = -10$, $y = 11$, $r = \sqrt{221}$
ii)



iii) $\theta = 132°$, $\beta = 48°$

f) i) $x = -2, y = 3\sqrt{5}, r = 7$

ii)

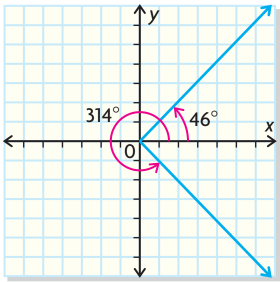


iii) $\theta = 107°, \beta = 73°$

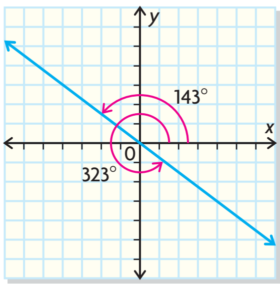
7. **a)** $-199°$ **b)** $-217°$ **c)** $-256°$ **d)** $-204°$ **e)** $-228°$ **f)** $-253°$

8. **a)** 29°, 151°, **b)** 171°, 351° **c)** 151°, 209° **d)** 7°, 187° **e)** 205°, 335° **f)** not possible

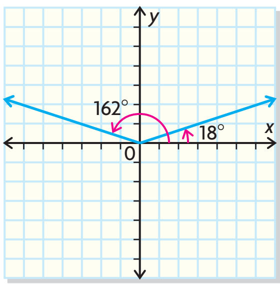
9. **a)** 46°, 314°



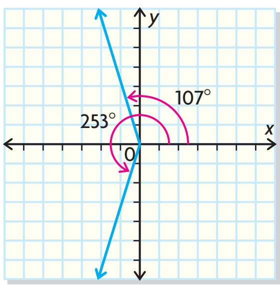
b) 143°, 323°



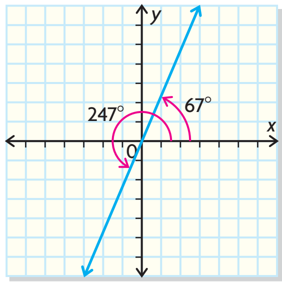
c) 18°, 162°



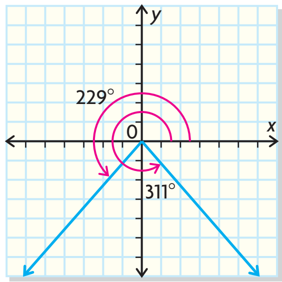
d) 107°, 253°



e) 67°, 247°



f) 229°, 311°



10. **a)** **i)** 225°, $-135°$

ii) $\sin\theta = \frac{-\sqrt{2}}{2}$, $\cos\theta = \frac{-\sqrt{2}}{2}$, $\tan\theta = 1$

b) **i)** 270°, $-90°$

ii) $\sin\theta = -1$, $\cos\theta = 0$, $\tan\theta$ is undefined

c) **i)** 180°, $-180°$

ii) $\sin\theta = 0$, $\cos\theta = -1$, $\tan\theta = 0$

d) **i)** 0°, $-360°$

ii) $\sin\theta = 0$, $\cos\theta = 1$, $\tan\theta = 0$

11. You can't draw a right triangle if $\theta \geq 90°$.

12. **a)** quadrant 2 or 3

b) Quadrant 2: $\sin\theta = \frac{\sqrt{119}}{12}$, $\cos\theta = \frac{-5}{12}$, $\tan\theta = \frac{\sqrt{119}}{-5}$

Quadrant 3: $\sin\theta = \frac{-\sqrt{119}}{12}$, $\cos\theta = \frac{-5}{12}$, $\tan\theta = \frac{\sqrt{119}}{5}$

c) 115°, 245°

13. $\alpha = 180°$

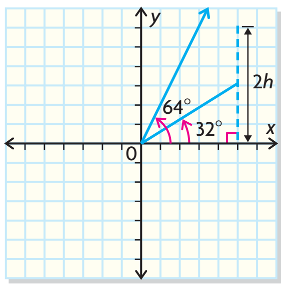
14. Answers may vary. For example, given $P(x, y)$ on the terminal arm of angle θ, $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$.

15. **a)** 25°, 155°, 205°, 335° **b)** 148°, 352° **c)** 16°, 106°, 196°, 286°

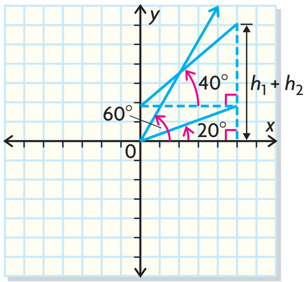
16. **a)** θ could lie in quadrant 3 or 4. $\theta = 233°$ or $307°$

b) θ could lie in quadrant 2 or 3. $\theta = 139°$ or $221°$

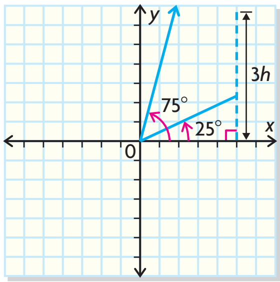
17. **a)**



b)



c)



Mid-Chapter Review, p. 304

1. **a)** 2.9238 **b)** 3.8637 **c)** 5.6713 **d)** 1.0125

2. **a)** 49° **b)** 76° **c)** 38° **d)** 66°

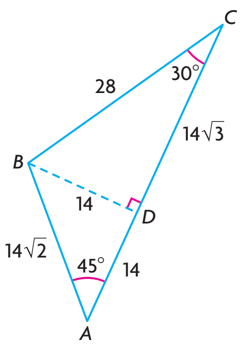
3. tan 54°, csc 46°, sec 44°, or cot 36°

4. 16.6 m

5. $45° < \theta < 90°$

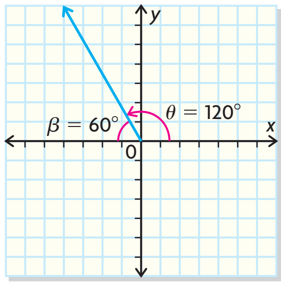
6. **a)** $\frac{\sqrt{3}}{2}$ **b)** 1 **c)** 2 **d)** $\sqrt{2}$

7. **a)**

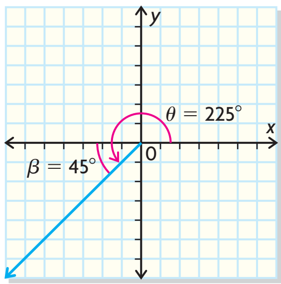


b) $\sin A = \frac{\sqrt{2}}{2}$, $\cos A = \frac{\sqrt{2}}{2}$, $\tan A = 1$, $\sin DBC = \frac{\sqrt{3}}{2}$, $\cos DBC = \frac{1}{2}$, $\tan DBC = \sqrt{3}$

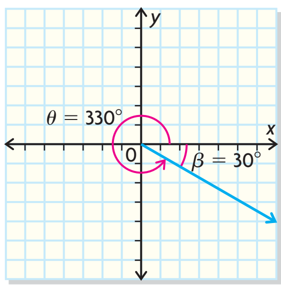
8. **a)** **i)** $\frac{\sqrt{3}}{2}$ **ii)** 60°



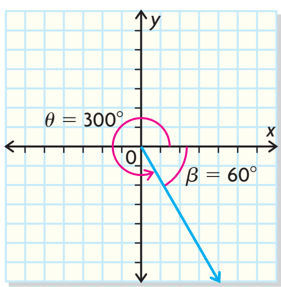
b) **i)** $\frac{-\sqrt{2}}{2}$ **ii)** 135°

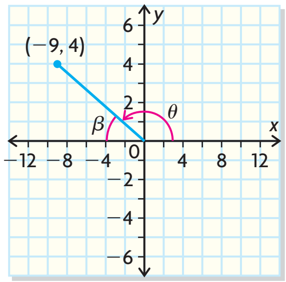


c) **i)** $\frac{-\sqrt{3}}{3}$ **ii)** 150°



d) **i)** $\frac{1}{2}$ **ii)** 60°



9. a) 

b) 24°

c) 156°

10. No, the only two possible angles within the given range are 37° and 323°.

11. a) $\sin\theta = \dfrac{15}{17}$, $\cos\theta = \dfrac{-8}{17}$, $\csc\theta = \dfrac{17}{15}$, $\sec\theta = \dfrac{17}{-8}$, $\cot\theta = \dfrac{-8}{15}$

b) 118°

12. 235°, 305°

13. a), b), c), e), and f) must be false.

a) $-1 \le \cos\theta \le 1$ **b)** $\tan\theta < 0$ **c)** $\sec\theta < 0$

e) $\cot\theta < 0$ **f)** $-1 \le \sin\theta \le 1$

Lesson 5.5, pp. 310–311

1. a) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

$$\text{L.S.} = \cot\theta = \frac{x}{y}$$

$$\text{R.S.} = \frac{\cos\theta}{\sin\theta} = \frac{x}{r} \div \frac{y}{r} = \frac{x}{r} \times \frac{r}{y} = \frac{x}{y}$$

$\therefore \cot\theta = \dfrac{\cos\theta}{\sin\theta}$ for all angles θ where $0° \le \theta \le 360°$ except 0°, 180°, and 360°.

b) $\tan\theta\cos\theta = \sin\theta$

$$\text{L.S.} = \tan\theta\cos\theta = \frac{y}{x} \times \frac{x}{r} = \frac{y}{r}$$

$$\text{R.S.} = \sin\theta = \frac{y}{r}$$

$\therefore \tan\theta\cos\theta = \sin\theta$ for all angles θ where $0° \le \theta \le 360°$.

c) $\csc\theta = \dfrac{1}{\sin\theta}$

$$\text{L.S.} = \csc\theta = \frac{r}{y}$$

$$\text{R.S.} = \frac{1}{\sin\theta} = 1 \div \frac{y}{r} = 1 \times \frac{r}{y} = \frac{r}{y}$$

$\therefore \csc\theta = \dfrac{1}{\sin\theta}$ for all angles θ where $0° \le \theta \le 360°$ except 0°, 180°, and 360°.

d) $\cos\theta\sec\theta = 1$

$$\text{L.S.} = \cos\theta\sec\theta = \frac{x}{r} \times \frac{r}{x} = 1$$

$$\text{R.S.} = 1$$

$\therefore \cos\theta\sec\theta = 1$ for all angles θ where $0° \le \theta \le 360°$.

2. a) $\cos^2\alpha$ **b)** $\sec\alpha$ or $\dfrac{1}{\cos\alpha}$ **c)** 1 **d)** $\cos\alpha$

3. a) $(1 - \cos\theta)(1 + \cos\theta)$ **c)** $(\sin\theta - 1)^2$

b) $(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)$ **d)** $\cos\theta(1 - \cos\theta)$

4. $\dfrac{\cos^2\theta}{1 - \sin\theta} = 1 + \sin\theta$

$$\cos^2\theta = (1 + \sin\theta) \times (1 - \sin\theta)$$
$$\cos^2\theta = 1 + \sin\theta - \sin\theta - \sin^2\theta$$
$$\cos^2\theta = 1 - \sin^2\theta$$
$$\cos^2\theta = \cos^2\theta$$

5. a) $\dfrac{\sin x}{\tan x} = \cos x$

$$\text{L.S.} = \frac{\sin x}{\tan x} = \sin x \div \frac{\sin x}{\cos x} = \sin x \times \frac{\cos x}{\sin x} = \cos x$$

= R.S., for all angles x where $0° \le x \le 360°$ except 0°, 90°, 180°, 270°, and 360°.

b) $\dfrac{\tan\theta}{\cos\theta} = \dfrac{\sin\theta}{1 - \sin^2\theta}$

$$\text{L.S.} = \frac{\tan\theta}{\cos\theta} = \frac{\sin\theta}{\cos\theta} \div \cos\theta = \frac{\sin\theta}{\cos\theta} \times \frac{1}{\cos\theta} = \frac{\sin\theta}{\cos^2\theta} = \frac{\sin\theta}{1 - \sin^2\theta}$$

= R.S., for all angles θ where $0° \le \theta \le 360°$ except 90° and 270°.

c) $\dfrac{1}{\cos\alpha} + \tan\alpha = \dfrac{1 + \sin\alpha}{\cos\alpha}$

$$\text{L.S.} = \frac{1}{\cos\alpha} + \tan\alpha = \frac{1}{\cos\alpha} + \frac{\sin\alpha}{\cos\alpha} = \frac{1 + \sin\alpha}{\cos\alpha}$$

= R.S., for all angles α where $0° \le \alpha \le 360°$ except 90° and 270°.

d) $\sin\theta\cos\theta\tan\theta = 1 - \cos^2\theta$

L.S. $= \sin\theta\cos\theta\tan\theta$

$= \frac{y}{r} \times \frac{x}{r} \times \frac{y}{x}$

$= \frac{y^2}{r^2}$

$= \sin^2\theta$

$= 1 - \cos^2\theta$

= R.S., for all angles θ where $0° \le \theta \le 360°$ except $90°$ and $270°$.

6. You need to prove that the equation is true for *all* angles specified, not just one.

7. **a)** $\cos\theta(1 - \sin\theta)$ **b)** $-\sin^2\theta$ **c)** 1 **d)** $\frac{\csc\theta - 2}{\csc\theta + 1}$, where $\csc\theta \ne 1$

8. **a)** $\frac{\sin^2\theta}{1 - \cos\theta} = 1 + \cos\theta$

L.S. $= \frac{\sin^2\theta}{1 - \cos\theta}$

$= \frac{1 - \cos^2\theta}{1 - \cos\theta}$

$= \frac{(1 - \cos\theta)(1 + \cos\theta)}{1 - \cos\theta}$

$= 1 + \cos\theta$

= R.S., where $\cos\theta \ne 1$

b) $\frac{\tan^2\alpha}{1 + \tan^2\alpha} = \sin^2\alpha$

L.S. $= \frac{\tan^2\alpha}{1 + \tan^2\alpha}$

$= \frac{\tan^2\alpha}{\sec^2\alpha}$

$= \frac{\sin^2\alpha}{\cos^2\alpha} \div \frac{1}{\cos^2\alpha}$

$= \frac{\sin^2\alpha}{\cos^2\alpha} \times \cos^2\alpha$

$= \sin^2\alpha$

= R.S., where $\tan\alpha \ne -1$

c) $\cos^2 x = (1 - \sin x)(1 + \sin x)$

R.S. $= (1 - \sin x)(1 + \sin x)$

$= 1 - \sin x + \sin x - \sin^2 x$

$= 1 - \sin^2 x$

$= \cos^2 x$

= L.S.

d) $\sin^2\theta + 2\cos^2\theta - 1 = \cos^2\theta$

L.S. $= \sin^2\theta + 2\cos^2\theta - 1$

$= \sin^2\theta + \cos^2\theta + \cos^2\theta - 1$

$= 1 + \cos^2\theta - 1$

$= \cos^2\theta$

= R.S.

e) $\sin^4\alpha - \cos^4\alpha = \sin^2\alpha - \cos^2\alpha$

L.S. $= \sin^4\alpha - \cos^4\alpha$

$= (\sin^2\alpha - \cos^2\alpha)(\sin^2\alpha + \cos^2\alpha)$

$= (\sin^2\alpha - \cos^2\alpha) \times 1$

$= \sin^2\alpha - \cos^2\alpha$

= R.S.

f) $\tan\theta + \frac{1}{\tan\theta} = \frac{1}{\sin\theta\cos\theta}$

L.S. $= \tan\theta + \frac{1}{\tan\theta}$

$= \frac{\sin\theta}{\cos\theta} + \left(1 \div \frac{\sin\theta}{\cos\theta}\right)$

$= \frac{\sin\theta}{\cos\theta} + \left(1 \times \frac{\cos\theta}{\sin\theta}\right)$

$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$

$= \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\cos\theta\sin\theta}$

$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$

$= \frac{1}{\sin\theta\cos\theta}$

= R.S., where $\tan\theta \ne 0$, $\sin\theta \ne 0$, and $\cos\theta \ne 0$.

9. **a)** Farah's method only works for equations that don't have a trigonometric ratio in the denominator.

b) If an equation has a trigonometric ratio in the denominator that can't equal zero, Farah's method doesn't work.

10. not an identity; $\csc^2 45 + \sec^2 45 = 4$ is not an identity

11. $\sin^2 x\left(1 + \frac{1}{\tan^2 x}\right) = 1$

L.S. $= \sin^2 x\left(1 + \frac{1}{\tan^2 x}\right)$

$= \sin^2 x\left(1 + 1 \div \frac{\sin^2 x}{\cos^2 x}\right)$

$= \left(\sin^2 x + \sin^2 x \div \frac{\sin^2 x}{\cos^2 x}\right)$

$= \left(\sin^2 x + \sin^2 x \times \frac{\cos^2 x}{\sin^2 x}\right)$

$= \sin^2 x + \cos^2 x$

$= 1$

= R.S.

12. **a)** $\frac{\sin^2\theta + 2\cos\theta - 1}{\sin^2\theta + 3\cos\theta - 3} = \frac{\cos^2\theta + \cos\theta}{-\sin^2\theta}$

L.S. $= \frac{\sin^2\theta + 2\cos\theta - 1}{\sin^2\theta + 3\cos\theta - 3}$

$= \frac{(1 - \cos^2\theta) + 2\cos\theta - 1}{(1 - \cos^2\theta) + 3\cos\theta - 3}$

$= \frac{-\cos^2\theta + 2\cos\theta}{-\cos^2\theta + 3\cos\theta - 2}$

$= \frac{\cos\theta \times (2 - \cos\theta)}{(2 - \cos\theta)(\cos\theta - 1)}$

$= \frac{\cos\theta}{\cos\theta - 1}$

R.S. $= \frac{\cos^2\theta + \cos\theta}{-\sin^2\theta}$

$= \frac{\cos^2\theta + \cos\theta}{\cos^2\theta - 1}$

$= \frac{\cos\theta \times (\cos\theta + 1)}{(\cos\theta + 1)(\cos\theta - 1)}$

$= \frac{\cos\theta}{\cos\theta - 1}$

= L.S., where $\sin\theta \ne 0$, $\cos\theta \ne 1$

b) $\sin^2\alpha - \cos^2\alpha - \tan^2\alpha = \dfrac{2\sin^2\alpha - 2\sin^4\alpha - 1}{1 - \sin^2\alpha}$

$$\begin{aligned}
\text{L.S.} &= \sin^2\alpha - \cos^2\alpha - \tan^2\alpha \\
&= \sin^2\alpha - \cos^2\alpha - \frac{\sin^2\alpha}{\cos^2\alpha} \\
&= \frac{\cos^2\alpha \times \sin^2\alpha}{\cos^2\alpha} - \frac{\cos^4\alpha}{\cos^2\alpha} - \frac{\sin^2\alpha}{\cos^2\alpha} \\
&= \frac{\cos^2\alpha \times \sin^2\alpha - \cos^4\alpha - \sin^2\alpha}{1 - \sin^2\alpha} \\
&= \frac{\sin^2\alpha \times (1 - \sin^2\alpha) - (1 - \sin^2\alpha)^2 - \sin^2\alpha}{1 - \sin^2\alpha} \\
&= \frac{-\sin^4\alpha - (1 - 2\sin^2\alpha + \sin^4\alpha)}{1 - \sin^2\alpha} \\
&= \frac{2\sin^2\alpha - 2\sin^4\alpha - 1}{1 - \sin^2\alpha} \\
&= \text{R.S., where } \sin\theta \neq 1
\end{aligned}$$

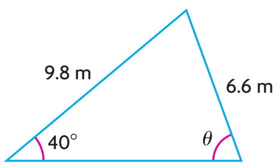
13. Answers may vary. For example, $\dfrac{\sin^3\theta}{\cos\theta} + \sin\theta\cos\theta = \tan\theta$ by multiplying by $\dfrac{\sin\theta}{\cos\theta}$

14. **a)** (iii)
b) **i)** $\sin^2 x \neq 1$
iv) $\sin\beta \neq \cos\beta$
v) $\sin\beta \neq 0$, $\cos\beta \neq -1$
vi) $\cos x \neq 1$

Lesson 5.6, pp. 318–320

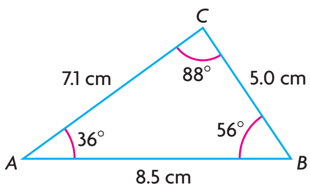
1. **a)** 47° **b)** 128°

2. **a)**



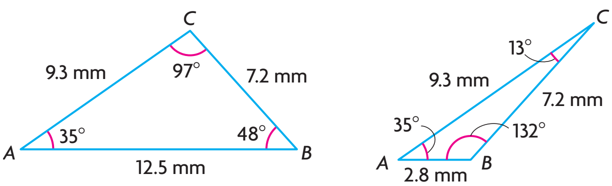
b) $h = 6.3$ m, h is less than either given side
c) two lengths (9.5 m or 5.6 m)

3. **a)** no triangle exists
b) no triangle exists
c)

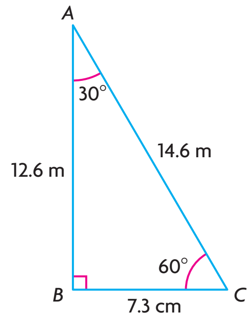


4. **a)** 40° **b)** 68° or 23°

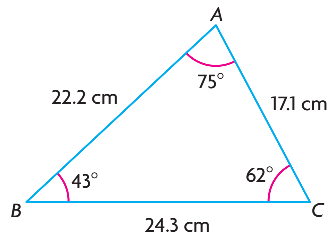
5. **a)**



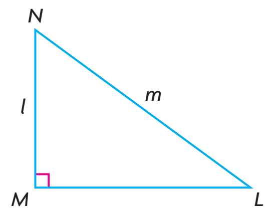
b)



c) no triangle exists
d)

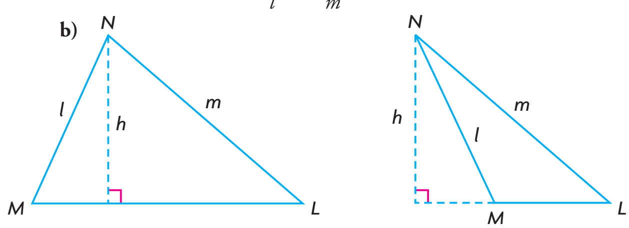


6. 34° **7.** 257.0 m **8.** 7499 m
9. 299.8 m **10.** 25 m
11. Carol only on same side is 66°. This is 11 m.
a) 28 m **b)** 31 m **c)** 52 m;
Carol only on same side as 66°. Other distance is 11 m.
a) 6 m **b)** 6 m **c)** 2 m;
All on same side. Distance to 66° is 11 m.
a) 28 m **b)** 37 m **c)** 16 m;
All on same side. Distance to 35° is 11 m.
a) 19 m **b)** 24 m **c)** 7 m
12. 481 m
13. (35° opposite 430 m side) 515 m, 8°, and 137°
14. **a)**



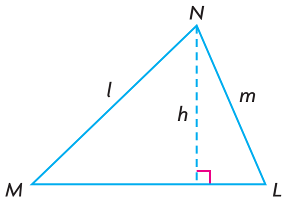
(right triangle) $l < m$, $\dfrac{\sin L}{l} = \dfrac{1}{m}$, (height) $h = l = m\sin L$

b)



$h < l < m$, (height) $h = m\sin L$

c)



(acute triangle) $l > m$, (height) $h = m\sin L$

15. **a)** (lighthouse A) 3 km, (lighthouse B) 13 km **b)** 5.4 km or 5 km

16. **a)** 366 m **b)** no

17. (lower guy wire) 264 m, (upper guy wire) 408 m

Lesson 5.7, pp. 325–327

1. **a)** 6.2 **b)** 18.7

2. **a)** 35° **b)** 40°

3. **a)** 8.0 **b)** 104° **c)** 100° **d)** 69.4

4. **a)** $m \doteq 15.0$ cm, $\angle L \doteq 46°$, $\angle N \doteq 29°$
b) $\angle R = 32°$, $t \doteq 13.9$ cm, $r \doteq 15.7$ cm
c) $\angle A \doteq 98°$, $\angle B \doteq 30°$, $\angle C \doteq 52°$
d) $\angle X = 124°$, $y \doteq 8.1$ cm, $z \doteq 12.9$ cm

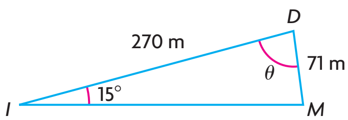
5. 11°

6. 138 m

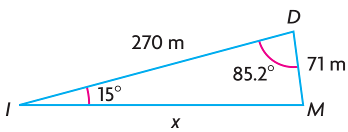
7. 1.4

8. (tower A) 31.5 km, (tower B) 22.3 km

9. **a)** Answers may vary. For example, Mike is standing on the other road and is 71 m from Darryl. From Darryl's position, what angle, to the nearest degree, separates the intersection from Mike?



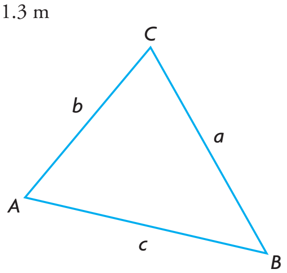
b) Answers may vary. For example, How far, to the nearest metre, is Mike from the intersection?



(Answer: 424 m)

10. 101.3 m

11.



a) a, b, and c or b, c, and $\angle A$

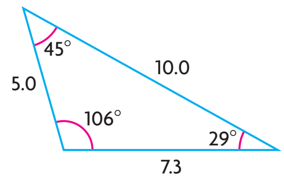
$$a^2 = b^2 + c^2 - 2bc\cos A$$

b) a, b, and $\angle A$

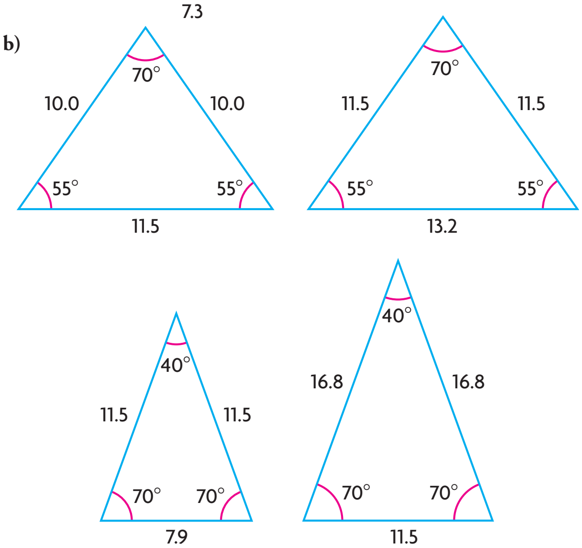
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

12. 35 cm

13. **a)**



b)



14. **a)** 2.4 km **b)** A is higher by 0.3 km.

Lesson 5.8, pp. 332–335

1. Answers may vary. For example, use primary trigonometric ratios to calculate the hypotenuse of each right triangle. Add the results together to get the length of line needed.

2. **a)** Answers may vary. For example, if you use the sine law, you don't have to solve a quadratic equation.
b) Answers may vary. For example, use a right triangle with acute angles 40° and 50°. Then, solve $\cos 50° = \frac{2.5}{x}$.

3. **a)** 15 cm **b)** 37.9 cm **c)** 17 cm **d)** 93°

4. **a)** 520.5 m
b) Use the sine law, then trigonometric ratios.

5. Yes, the distance is about 7127 m.

6. 258 m **7.** 24 m **8.** 736 m **9.** 47°

10. 4.5 m, 2.0 m, 6.0 m piece fits in 2.6 m × 2.1 m × 6.0 m vehicle. Other two pieces fit in 2.5 m × 2.1 m × 4.0 m vehicle.

11. Yes, the height is about 23 m.

12. **a)** You can't solve the problem.
b) You need the altitude of the balloon and angle of depression from the balloon to Bill and Chris.

13. **a)** 39 km **b)** 34 km

14. 524 m

15. 148.4 km

16. **a)** 84.4 m **b)** 64.2 m

Chapter Review, pp. 338–339

1. **a)** **i)** $\csc\theta = \frac{\sqrt{233}}{13}$, $\sec\theta = \frac{\sqrt{233}}{8}$, $\cot\theta = \frac{8}{13}$

ii) 58°

b) **i)** $\csc\theta = \frac{37}{12}$, $\sec\theta = \frac{37}{35}$, $\cot\theta = \frac{35}{12}$

ii) 19°

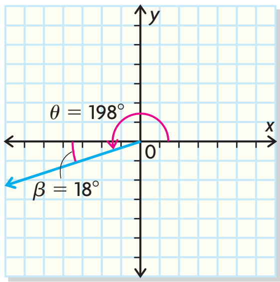
c) **i)** $\csc\theta = \frac{39}{23}$, $\sec\theta = \frac{39}{4\sqrt{62}}$, $\cot\theta = \frac{4\sqrt{62}}{23}$

ii) 36°

2. **a)** $\frac{3}{4}$ **b)** 0 **c)** $\frac{2\sqrt{3} + 3\sqrt{2}}{6}$

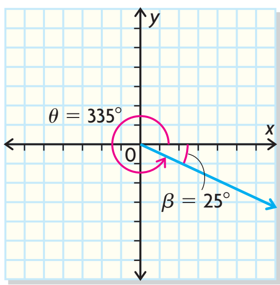
3. **a)** **i)** +; $\tan 18° = 0.3249$

ii) $\theta = 18°$ is the principal and related angle.



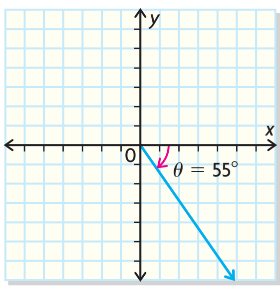
b) **i)** −; $\sin 205° = -0.4226$

ii) principal angle is $\theta = 205°$, $\beta = 25°$ is the related angle



c) **i)** +; $\cos(-55°) = 0.5736$

ii) principal angle is $\theta = 305°$, $\beta = 55°$ is the related angle



4. **a)** $\sin\theta = \frac{5\sqrt{29}}{29}$, $\cos\theta = \frac{-2\sqrt{29}}{29}$, $\tan\theta = \frac{5}{-2}$

b) $\sin\theta = \frac{-\sqrt{2}}{2}$, $\cos\theta = \frac{\sqrt{2}}{2}$, $\tan\theta = -1$

c) $\sin\theta = \frac{-5\sqrt{41}}{41}$, $\cos\theta = \frac{-4\sqrt{41}}{41}$, $\tan\theta = \frac{5}{4}$

5. **a)** quadrant 2 or 3

b) quadrant 2: $\sin\phi = \frac{2}{\sqrt{53}}$, $\tan\phi = \frac{2}{-7}$, $\csc\phi = \frac{\sqrt{53}}{2}$, $\sec\phi = \frac{\sqrt{53}}{-7}$, $\cot\phi = \frac{-7}{2}$; quadrant 3: $\sin\phi = \frac{-2}{\sqrt{53}}$, $\tan\phi = \frac{2}{7}$, $\csc\phi = \frac{\sqrt{53}}{-2}$, $\sec\phi = \frac{\sqrt{53}}{-7}$, $\cot\phi = \frac{7}{2}$

c) quadrant 2: 164°, quadrant 3: 196°

6. The equation is an identity.

7. **a)**
$$\begin{aligned}\tan\alpha\cos\alpha &= \left(\frac{\sin\alpha}{\cos\alpha}\right)(\cos\alpha)\\ &= \sin\alpha\end{aligned}$$

b)
$$\begin{aligned}\frac{1}{\cot\phi} &= \tan\phi\\ &= \frac{\sin\phi}{\cos\phi}\\ &= \sin\phi\left(\frac{1}{\cos\phi}\right)\\ &= \sin\phi\sec\phi\end{aligned}$$

ϕ cannot be equal to 90° or 270°.

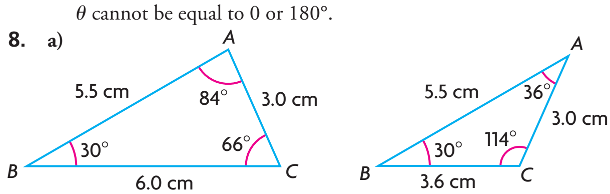
c)
$$\begin{aligned}1-\cos^2 x &= \sin^2 x\\ &= \sin^2 x\left(\frac{\cos x}{\cos x}\right)\\ &= \sin x\ \sin x\left(\frac{\cos x}{\cos x}\right)\\ &= \sin x\ \cos x\left(\frac{\sin x}{\cos x}\right)\\ &= \sin x\ \cos x\ \tan x\\ &= \sin x\ \cos x\left(\frac{1}{\cot x}\right)\\ &= \frac{\sin x\cos x}{\cot x}\end{aligned}$$

x cannot be equal to 0 or 180°.

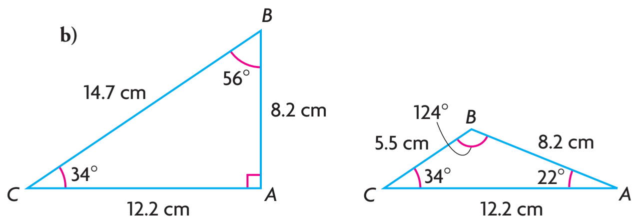
d) $\sec\theta\cos\theta + \sec\theta\sin\theta$
$$\begin{aligned}&= \left(\frac{1}{\cos\theta}\right)\cos\theta + \left(\frac{1}{\cos\theta}\right)\sin\theta\\ &= 1 + \frac{\sin\theta}{\cos\theta}\\ &= 1 + \tan\theta\end{aligned}$$

θ cannot be equal to 0 or 180°.

8. **a)**



b)



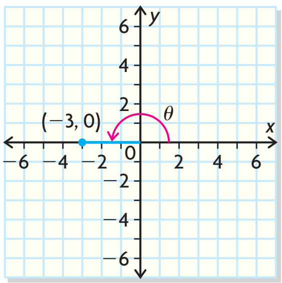
c) no triangle exists

9. 30.5 km

10. **a)** 15.5
b) 8.4
c) 5.2
11. 4.4 m
12. 13 m
13. 18°

Chapter Self-Test, p. 340

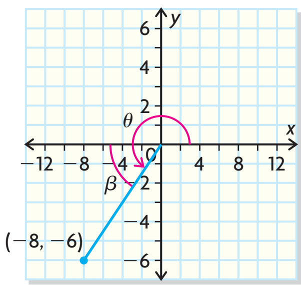
1. **a)** **i)**



$\sin\theta = 0$ $\quad\csc\theta$ is undefined
$\cos\theta = -1$ $\quad\sec\theta = -1$
$\tan\theta = 0$ $\quad\cot\theta$ is undefined

ii) $\theta = 180°$ is the principal angle; related angle is 0°

b) **i)**



$\sin\theta = \dfrac{-3}{5}$ $\quad\csc\theta = \dfrac{5}{-3}$

$\cos\theta = \dfrac{-4}{5}$ $\quad\sec\theta = \dfrac{5}{-4}$

$\tan\theta = \dfrac{3}{4}$ $\quad\cot\theta = \dfrac{4}{3}$

ii) $\theta = 217°$ is the principal angle, $\beta = 37°$ is the related angle

2. **a)** 210°, 330° **c)** 135°, 315°
b) 30°, 330° **d)** 120°, 240°

3. **a)** $\dfrac{-60}{169}$ **b)** 1

4. **i)** **a)** $\sin^2\theta + \cos^2\theta = 1$

$$\frac{1}{\cos^2\theta}(\sin^2\theta + \cos^2\theta) = \left(\frac{1}{\cos^2\theta}\right)(1)$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

b) $\sin^2\theta + \cos^2\theta = 1$

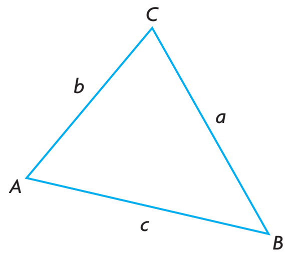
$$\frac{1}{\sin^2\theta}(\sin^2\theta + \cos^2\theta) = \left(\frac{1}{\sin^2\theta}\right)(1)$$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2 = \csc^2\theta$$

ii) These identities are derived from $\sin^2\Phi + \cos^2\Phi = 1$

5. **a)**

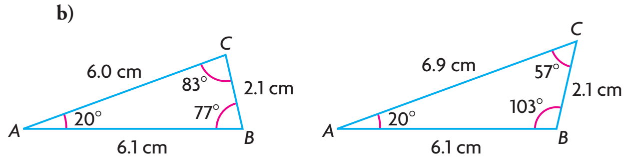


b) $a^2 = b^2 + c^2 - 2bc\cos A$
$b^2 = a^2 + c^2 - 2ac\cos B$
$c^2 = a^2 + b^2 - 2ab\cos C$

c) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ or $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

6. **a)** 97.4 m **b)** 1.6 m

7. **a)** no triangle exists
b)



8. 22 m

Chapter 6

Getting Started, p. 344

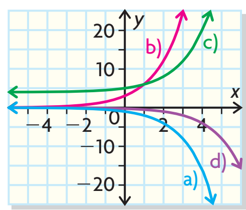
1. **a)** x represents the number of times the price is reduced by \$2. The factor $(30 - 2x)$ represents the price of one T-shirt in terms of the number of times the price is reduced; the factor $(100 + 20x)$ represents the total number of T-shirts sold in terms of the number of times the price is reduced.
b) 15 times **d)** \$4000 **f)** 200 T-shirts
c) 5 times **e)** \$20

2. **a)** 360 cm **b)** 0.25 s **c)** 720 cm/s
d) domain: $\{t \in \mathbf{R} \mid 0 \le t \le 0.5\}$;
range: $\{d \in \mathbf{R} \mid 0 \le d \le 180\}$

3. **a)** 32° **b)** 154°

4. 3.2 m

5.



6. 25 m

7. Answers will vary and may include the following:
- Vertical translation $y = x^2 + c$
  $y = x^2 - c$
- Horizontal translation $(x + d)^2$
  $(x - d)^2$
- Vertical stretch ≠ compression $y = ax^2$

Lesson 6.1, pp. 352–356

1. **a)** is periodic because the cycle repeats;
b) is not periodic because the cycle does not repeat;
c) is periodic because the cycle repeats;
d) is not periodic because the cycle does not repeat

2. range: $\{y \in \mathbf{R} \mid 2 \le y \le 10\}$; period: 4; axis: $y = 6$; amplitude: 4

3. **a)** 1 s
b) 1.5 cm
c) range: $\{d \in \mathbf{R} \mid 0 \le d \le 1.5\}$
d) domain: $\{t \in \mathbf{R} \mid 0 \le t \le 5\}$
e) $y = 0.75$
f) 0.75 cm
g) horizontal component of graph: device is not in motion, it remains fixed at 1.5 cm. Component of the graph with negative slope: the device is approaching the appliance and simultaneously attaching the bolt. Component of the graph with the positive slope; the device has finished attaching the bolt and is moving away from the house appliance.

4. **a)** period: 6
b) period: 2
c) not periodic
d) not periodic
e) period: 5.5
f) period: 20

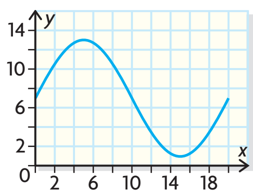
5. (a), (b), and (f)

6. (b) repeating cycle

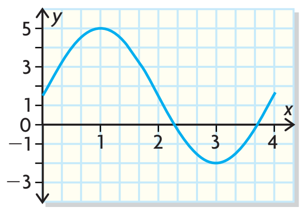
7. **a)** yes
b) 10 cm
c) 2 min
d) 8 min
e) range: $\{d \in \mathbf{R} \mid 4 \le d \le 10\}$
f) 7 cm
g) At $t = 6$ min and every 8 min from that time
h) 10 cm

8. **a)** 8 s; one rotation of Ferris wheel
b) $h = 4$
c) 3 m
d) $\{h \in \mathbf{R} \mid 1 \le h \le 7\}$
e) 32 s
f) 4 s, 12 s, 20 s, 28 s
g) at $t = 26$ s and $t = 30$ s

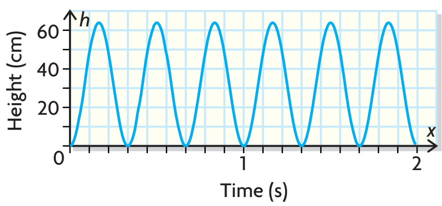
9.



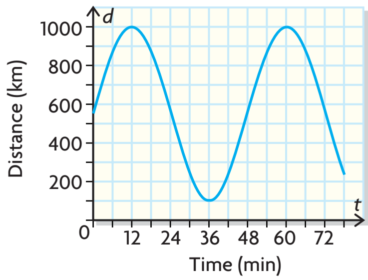
10.



11.

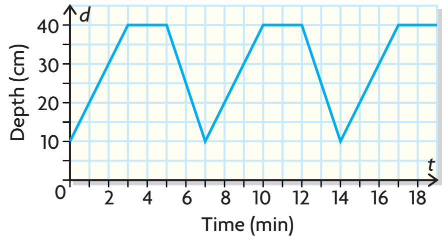


12. **a)**



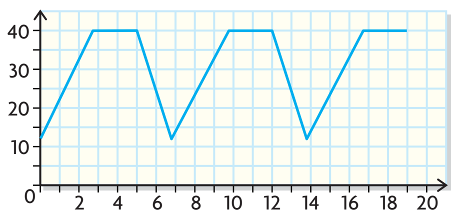
b) yes
c) 48 min: time to complete one orbit
d) approximately 900 km
e) At $t = 12$ min and every 48 min after that time
f) $\{t \in \mathbf{R} \mid 0 \le t \le 288\}$

13. **a)**

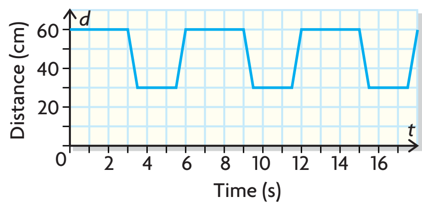


b) yes
c) period: 7; axis: $d = 25$; amplitude: 15
d) 10 cm/min
e) 15 cm/min
f) no, never intersects the t-axis

14. A periodic function is a function that produces a graph that has a regular repeating pattern over a constant interval. It describes something that happens in a cycle, repeating in the same way over and over.
Example:



15. **a)**



b) 6 s
c) range: $\{d \in \mathbf{R} \mid 30 \le d \le 60\}$; domain: $\{t \in 0 \le t \le 18\}$

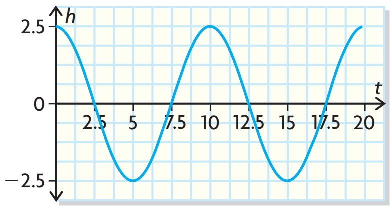
16. At time $t = 0$, the paddle is 40 cm in front of the CBR and doesn't move for 1 s. At 1 s, the paddle moves 30 cm away from the CBR and then returns to its original position of 40 cm in front of the CBR at 1.5 s. For 1 s, the paddle doesn't move. At $t = 2.5$ s, the paddle moves 30 cm away from the CBR and then returns to its original position of 40 cm in front of the CBR at $t = 3$ s where it remains for 1 s until 4 s.

Lesson 6.2, pp. 363–364

1. **a)** amplitude: 3; period: 180°; axis: $y = 1$
b) amplitude: 4; period: 720°; $y = -2$

2. **a)** -0.18 **b)** 90°, 270°

3. **a)**



b) 10 s **c)** -1 m **d)** 4 s

4. (1.29, 1.53)

5. **a)** periodic and sinusoidal
b) neither
c) periodic
d) neither
e) periodic and sinusoidal
f) neither

6. not necessarily periodic or sinusoidal, answers may vary

7. **a)** $g(90) = 1$; when $x = 90$, $y = 1$, or the sine of (y-coordinate of a point on the unit circle) at 90 is 1.
b) $h(90) = 0$; when $x = 90$, $y = 0$, or the cosine of (x-coordinate of a point on the unit circle) at 90 is 0.

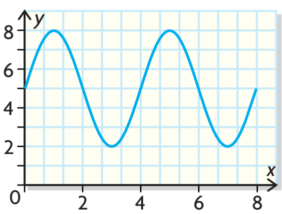
8. **a)** amplitude: 2; period: 360; increasing interval: 0 to 90, 270 to 360; decreasing interval: 90 to 270; axis: $y = 3$
b) amplitude: 3; period: 360; increasing interval: 0 to 90, 270 to 360; decreasing interval: 90 to 270; axis: $y = 1$
c) amplitude: 1; period: 720; increasing interval: 0 to 180, 540 to 720; decreasing interval: 180 to 540; axis: $y = 2$
d) amplitude: 1; period: 180; increasing interval: 0 to 45, 135 to 180; decreasing interval: 45 to 135; axis: $y = -1$
e) amplitude: 2; period: 1440; increasing interval: 0 to 360, 1080 to 1440; decreasing interval: 360 to 1080; axis: $y = 0$
f) amplitude: 3; period: 720; increasing interval: 0 to 180, 540 to 720; decreasing interval: 180 to 540; axis: $y = 2$

9. **a)** 0.82 **b)** 0.34 **c)** 1.5 **d)** 180° **e)** 270°

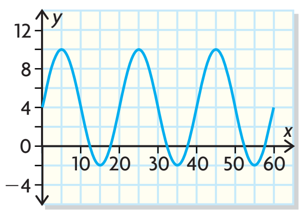
10. $x = -315°, -135°, 45°, 225°$

11. **a)** (0.91, 0.42) **b)** (0.87, 4.92) **c)** $(-2, 3.46)$ **d)** $(-1.93, -2.30)$

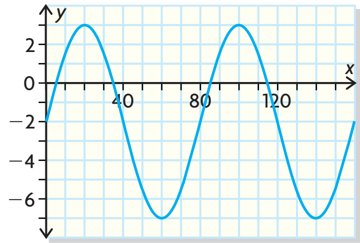
12. **a)**



b)



c)



13. **a)** Where Jim is on the Ferris wheel at 10 s, $h(10) = -5$. Ferris wheel is at lowest point.
b) Now $h(10) = 0$, Jim is at the midpoint.

14. Same period, amplitude, and axis. Different starting point for each circle.

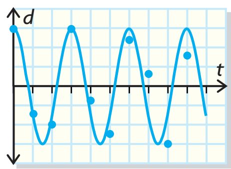
15. $y = \sin x + 0.5$

16. **a)**

t(s)	$d(t)$ (cm)
0	0.5
0.5	0.25
2	−0.25
1.5	−0.5
2	−0.25
2.5	0.25
3	0.5
3.5	0.25
4	−0.25
4.5	−0.5

t(s)	$d(t)$ (cm)
5	−0.25
5.5	0.25
6	0.5
6.5	0.25
7	−0.25
7.5	−0.5
8	−0.25
8.5	0.25
9	0.5

b)



c) The function repeats itself every 3 s.
d) The amplitude and the displacement from rest are the same.

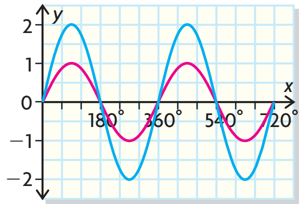
Lesson 6.3, pp. 370–373

1. **a)** $y = 8$; resting position of the swing
b) 6 m
c) 4 s; time to complete one full swing
d) 2 m
e) No, 2 s is not long enough to run safely.
f) Amplitude would increase with each swing. It would not be sinusoidal because the amplitude is changing.

2.

	Period	Amplitude	Axis	Maximum	Minimum	Speed (m/s)
A	12 s	3 m	$y = 2$	5	−1	1.57
B	16 s	4 m	$y = 3$	7	−1	1.57

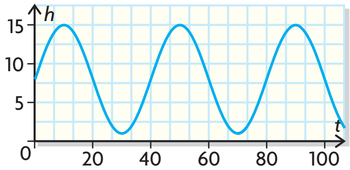
3. Answers may vary. For example,



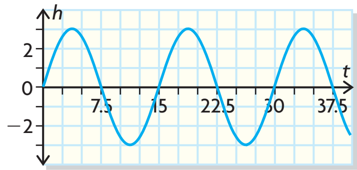
4. **a)** 1 in.
b) 0.04 s, how long it takes the blade to make a full rotation
c) 4 in.; radius of saw blade
d) 628 in./s

5. **a)** about 0.035 s **b)** $y = 0$ amperes **c)** 4.5 amperes

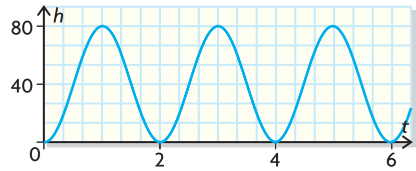
6. **a)**



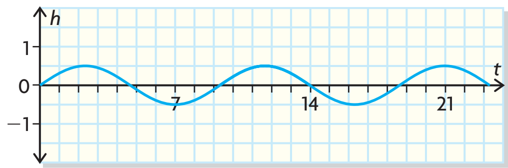
b)



c)

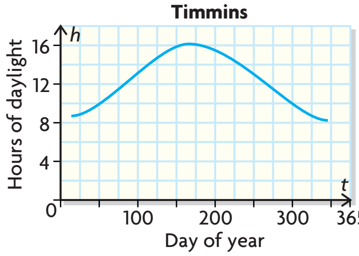


d)

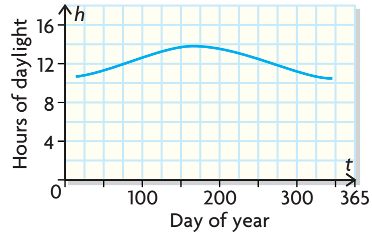


7. **a)**

Timmins

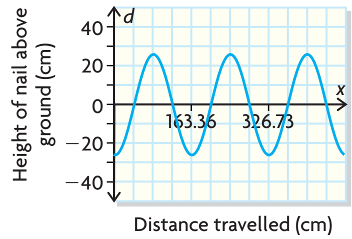


Miami



b) Timmins: period: 365; amplitude: 3.9 h; axis: $h = 12.2$ h
Miami: period: 365; amplitude: 1.7 h; axis: $h = 12.2$ h
c) The farther north one goes, more extreme differences occur in the hours of daylight throughout the year.

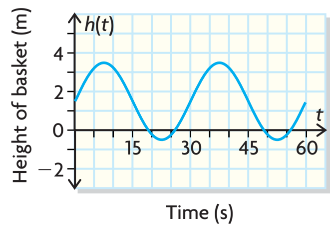
8. **a)**



b) approximately 25 cm
c) approximately 687 cm
d) The driver doesn't spin the wheels.

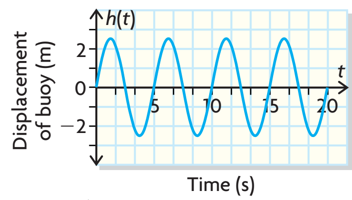
9. Have the same period $\left(\frac{1}{3}\text{ s}\right)$ and equation of the axis ($d = 0$); have different amplitudes (3 and 2). Conjecture: lower the wind speed and you decrease the distance the post shakes back and forth

10. **a)**



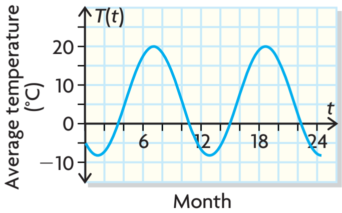
b) 30 s; period, one revolution
c) 2 m; amplitude, radius
d) $y = 1.5$ m; equation of axis
e) 3.232; at $t = 10$ s, the basket is at a height of 3.232 m.

11. **a)**



b) 5 s; period
c) 12 waves; graph completes 12 cycles in 1 min
d) 5 m; vertical distance between the maximum and minimum values of h

12. a)

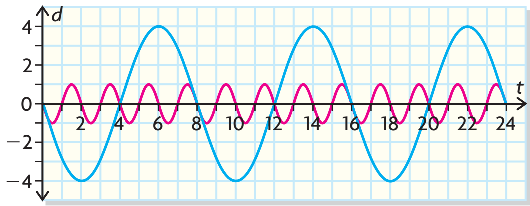


b) The period represents 12 months, or 1 year.
c) between an average high of 20.1 °C and an average low of −8.3 °C
d) 5.9 °C
e) 17.4; on the 30th month, the average monthly temperature is 17.4 °C.

13. a) A: period: 8 s; B: period: 6 s; time for the wrecking ball to complete a swing back and forth
b) A and B: equation of axis $d = 0$; resting position of each wrecking ball
c) A: amplitude: 4 m; B: amplitude: 3 m; maximum distance balls swing back and forth from the resting position
d) A: $\{y \in \mathbf{R} \mid -4 \leq y \leq 4\}$; B: $\{y \in \mathbf{R} \mid -3 \leq y \leq 3\}$
e) The wrecking ball modelled using the red curve sways at a faster rate but doesn't swing as far.

14. You need the amplitude, where it starts on the graph, axis, and period.

15. a) clockwise
b) 8 s
c)



d) 0.5 m
e) 0.81 m
f) 0 m

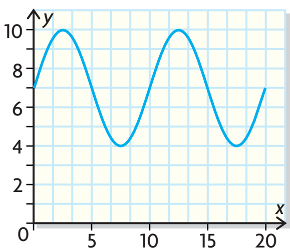
Time (s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
d small gear	0	−1	0	1	0	−1	0	1	0	−1	0	1

Time (s)	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0	10.5	11.0	11.5	12.0
d small gear	0	−1	0	1	0	−1	0	1	0	−1	0	1	0

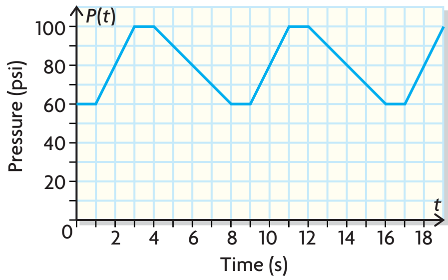
Time (s)	0	2.0	4.0	6.0	8.0	10.0	12.0	14.0	16.0	18.0	20.0	22.0	24.0
d large gear	0	−4	0	4	0	−4	0	4	0	−4	0	4	0

Mid-Chapter Review, p. 376

1. Answers may vary. For example,



2. a)



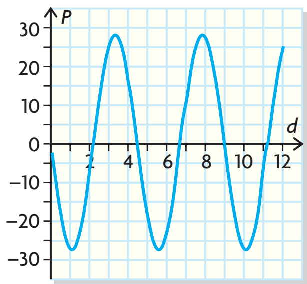
b) cycle repeats
c) 8 s
d) $P = 80$
e) 20
f) 20 psi/s
g) 10 psi/s
h) No, because its lowest pressure value is 60 psi.

3. a) period: 180; axis: $g = 7$; amplitude: 5; range: $\{g \in \mathbf{R} \mid 2 \leq g \leq 12\}$
b) smooth, repeating waves
c) 5.3
d) 0°, 180°, 360°

4. (3.1, 6.3)

5. a) Both have a period of 0.25; the time for the tire to complete one revolution.
b) Both have same equation of the axis, $h = 30$; the height of the axle.
c) 1: amplitude: 30; 2: amplitude: 20; distance from white mark to the centre of the wheel
d) 1: $\{h \in \mathbf{R} \mid 0 \leq h \leq 60\}$; 2: $\{h \in \mathbf{R} \mid 10 \leq h \leq 50\}$
e) 1: 754 cm/s; 2: 502 cm/s
f) This graph would be periodic in nature and have a smaller amplitude than the graph of Mark 2 (the red graph).

6. a)



b) period: 365 days or 1 year
c) axis: $P = 0$; the position is 0° with respect to due west
d) amplitude: 28; maximum number of degrees north or south of due west the Sun can be at sunset for this particular latitude
e) $\{P \in \mathbf{R} \mid -28 \leq P \leq 28\}$
f) −16.3°

Lesson 6.4, p. 379

1. a) vertical stretch of 3
b) horizontal translation of 50°
c) reflection in the *x*-axis
d) horizontal compression of $\frac{1}{5}$
e) vertical translation of −6
f) horizontal translation of −20°

2. a) axis: $y = 2$
b) amplitude: 4
c) period: 45°
d) horizontal translation of −30°; period: 180°
e) amplitude: 0.25
f) period: 720°

3. (a), (e)

Lesson 6.5, pp. 383–385

1. **a)** horizontal compression: $\frac{1}{4}$, vertical translation: 2

b) horizontal translation: 20, vertical compression: $\frac{1}{4}$

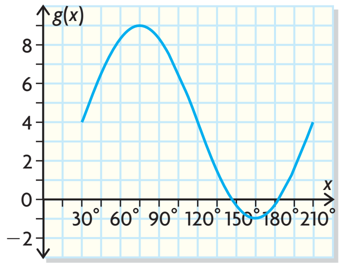
c) horizontal stretch: 2; reflection in x-axis

d) horizontal compression: $\frac{1}{18}$; vertical stretch: 12; vertical translation: 3

e) horizontal stretch: 3; horizontal translation: 40; vertical stretch: 20; reflection in x-axis

2. period: 120°; amplitude: 4; axis: $y = 6$; domain: $\{x \in \mathbf{R} \mid 0° \leq x \leq 240°\}$; range: $\{y \in \mathbf{R} \mid 2 \leq y \leq 10\}$

3.



4. **a)** horizontal translation: −10; vertical stretch: 2; reflection in x-axis

b) horizontal compression: $\frac{1}{5}$; vertical translation: 7

c) horizontal compression: $\frac{1}{2}$; horizontal translation: −6; vertical stretch: 9; vertical translation: −5

d) horizontal translation: 15; vertical compression: $\frac{1}{5}$; vertical translation: 1

e) horizontal stretch: 4; horizontal translation: −37; reflection in x-axis; vertical translation: −2

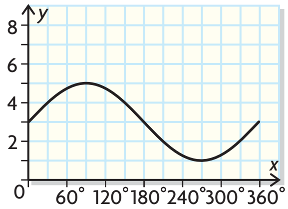
f) horizontal compression: $\frac{1}{3}$; vertical stretch: 6; reflection in x-axis; vertical translation: 22

5. **a)** (ii) **b)** (iii) **c)** (i)

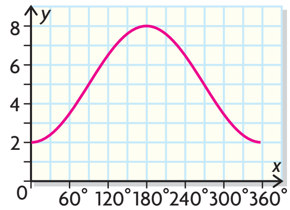
6.

	Period	Amplitude	Equation of the Axis	Domain	Range
a)	360°	3	$y = 2$	$\{x \in \mathrm{R} \mid 0° \leq x \leq 1080°\}$	$\{y \in \mathrm{R} \mid -1 \leq y \leq 5\}$
b)	180°	4	$g = 7$	$\{x \in \mathrm{R} \mid 0° \leq x \leq 540°\}$	$\{g \in \mathrm{R} \mid 3 \leq g \leq 11\}$
c)	360°	$\frac{1}{2}$	$h = -5$	$\{t \in \mathrm{R} \mid 0° \leq t \leq 1080°\}$	$\{h \in \mathrm{R} \mid -5.5 \leq h \leq -4.5\}$
d)	90°	1	$h = -9$	$\{x \in \mathrm{R} \mid 0° \leq x \leq 270°\}$	$\{h \in \mathrm{R} \mid -10 \leq h \leq -8\}$
e)	2°	10	$d = -30$	$\{t \in \mathrm{R} \mid 0° \leq t \leq 6°\}$	$\{d \in \mathrm{R} \mid -40 \leq d \leq -20\}$
f)	180°	$\frac{1}{2}$	$j = 0$	$\{x \in \mathrm{R} \mid 0° \leq x \leq 540°\}$	$\{j \in \mathrm{R} \mid -0.5 \leq j \leq 0.5\}$

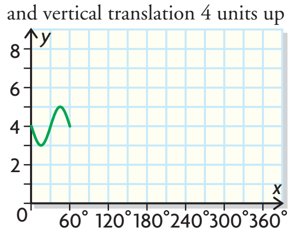
7. **a)** vertical stretch by a factor of 2 and vertical translation 3 units up



b) vertical stretch by a factor of 3, reflection in the x-axis, and vertical translation 5 units up

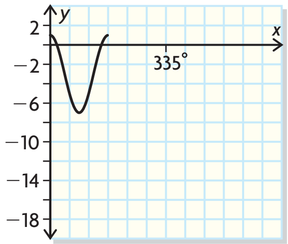


c) horizontal compression by a factor of $\frac{1}{6}$, reflection in the x-axis, and vertical translation 4 units up

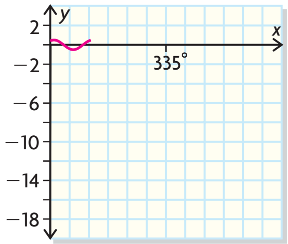


Answers

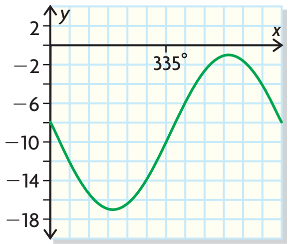
d) vertical stretch by a factor of 4, horizontal compression by a factor of $\frac{1}{2}$, and vertical translation 3 units down



e) vertical compression by a factor of $\frac{1}{2}$, horizontal compression by a factor of $\frac{1}{3}$, and horizontal translation 40° to the right



f) vertical stretch by a factor of 8, reflection in the x-axis, horizontal stretch by a factor of 2, horizontal translation 50° to the left, and vertical translation 9 units down



8.

	X min	X max	Y min	Y max
a)	0°	180°	5	5
b)	0°	720°	15	25
c)	0°	40°	75	89
d)	0°	1°	−26.5	−27.5

9. a) period: 1.2 s; one heart beat
b) $\{P \in \mathbf{R} \mid 80 \leq P \leq 120\}$, maximum blood pressure of 120, minimum blood pressure of 80

10. a) $y = 4\sin\left(\frac{1}{2}x\right) + 3$
b) $y = 4\cos\left(\frac{1}{2}x - 90\right) + 3$

11. Reflection, amplitude of $\frac{1}{2}$, vertical translation 30 upward, horizontal compression of 120 resulting in a period of 3.

12. horizontal translation: −45°

13. a) The number of hours of daylight increases to a maximum and decreases to a minimum in a regular cycle as Earth revolves around the Sun.
b) Mar. 21: 12 h; Sept. 21: 12 h, spring and fall equinoxes
c) June 21: 16 h; Dec. 21: 8 h, longest and shortest days of year; summer and winter solstices
d) 12 is the axis of the curve representing half the distance between the maximum and minimum hours of daylight.

Lesson 6.6, pp. 391–393

1. a) $y = 2\cos(4x) + 6$
b) $y = \cos(x - 90°) + 2$
c) $y = 2\cos(3x) - 2$

2. $y = 2\cos(2x) + 7$

3. $y = 4\cos(3x) + 5$

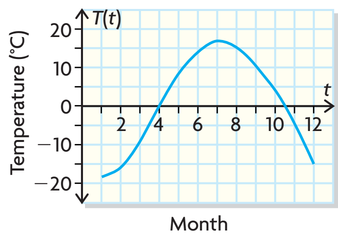
4. a) i) $y = 3\cos(60(x - 4°)) + 5$;
ii) $y = -0.5\cos(120x) + 1$;
iii) $y = \cos(90(x - 3°)) - 2$
b) i) $y = 5\cos(180(x - 1.5°)) + 25$;
ii) $y = 5\cos(120(x - 2°)) + 10$;
iii) $y = 10\cos(360x) - 5$

5. a) $y = \cos(3x) + 2$
b) $y = 4\cos\left(\left(\frac{1}{2}\right)(x - 180°)\right) + 17$
c) $y = 3\cos\left(\left(\frac{3}{2}\right)(x - 60°)\right) - 4$
d) $y = 3\cos(3(x - 10°)) + 2$

6. a) $y = 3\cos x + 11$
b) $y = 4\cos(2(x - 30°)) + 15$
c) $y = 2\cos(9(x - 7°))$
d) $y = 0.5\cos\left(\left(\frac{1}{2}\right)(x + 56°)\right) - 3$

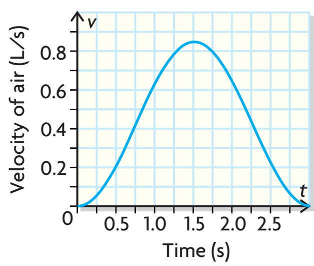
7. $y = -6\cos(8x) + 7$

8. a)



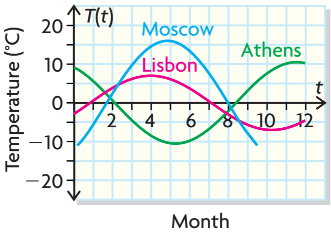
b) sinusoidal model because it changes with a cyclical pattern over time
c) $T(t) = -17.8\cos 30t - 0.8$
d) 8.1 °C

9. a) The respiratory cycle is an example of a periodic function because we inhale, rest, exhale, rest, inhale, and so on in a cyclical pattern.
b) $v = -0.425\cos(120t)° + 0.425$



c) The equation is almost an exact fit on the scatter plot.
d) 0 L/s; period is 3; troughs occur at 0, 3, and 6 s
e) $t = 0.8$ s and 2.3 s

10. a)



b) Athens: $T(t) = -10.5 \cos 30t + 22.5$;
Lisbon: $T(t) = -7 \cos 30\text{t} + 20$;
Moscow: $T(t) = -16 \cos 30t + 7$

c) latitude affects amplitude and vertical translation

d) Athens and Lisbon are close to the same latitude; Moscow is farther north.

11. a) $y = 3 \cos(9000(x - 0.01))° + 8$
b) maximum equivalent stress
c) 6.64 MPa

12. Find the amplitude. Whatever the amplitude is, a in the equation $y = a \cos(k(x - d)) + c$ will be equal to it. Find the period. Whatever the period is, k in the equation $y = a \cos(k(x - d)) + c$ will be equal to 360 divided by it. Find the equation of the axis. Whatever the equation of the axis is, c in the equation $y = a \cos(k(x - d)) + c$ will be equal to it. Find the phase shift. Whatever the phase shift is, d in the equation $y = a \cos(k(x - d)) + c$ will be equal to it. Determine if the function is reflected in its axis. If it is, the sign of a will be negative; otherwise, it will be positive. Determine if the function is reflected in the y-axis. If it is, the sign of k will be negative; otherwise, it will be positive.

13. $y = -30 \cos(1.909\,859x)° + 30$; 23.4 cm

14. $h = 7 \cos(15.8t)° + 8$, t in seconds, h in metres

Lesson 6.7, pp. 398–401

1. a) $d = 1.5$ m, distance between tail lights and the curb if the trailer isn't swinging back and forth
b) amplitude: 0.5 m, distance the trailer swings to the left and right
c) period: 2 s, the time it takes for the trailer to swing back and forth
d) $d = -0.5 \cos(180t)° + 1.5$; $\{d \in \mathbf{R} \mid 1 \le d \le 2\}$
e) range is the distance the trailer swings back and forth; domain is time
f) 1.2 m

2. a) $h = 10$ m, axle height
b) amplitude: 7 m, length of blade
c) period: 20 s, time in seconds to complete revolution
d) domain: $\{t \in \mathbf{R} \mid 0 \le t \le 140\}$;
range: $\{h \in \mathbf{R} \mid 3 \le h \le 17\}$
e) $h = -7 \cos(18x)° + 10$
f) period would be larger

3. $d = 4 \cos(90(t - 1))° + 8$

4. a) same period (24), same horizontal translation (12), different amplitude (2.5 and 10), different equations of the axis ($T = 17.5$ and $T = -20$). The top one is probably the interior temperature (higher, with less fluctuation).
b) domain (for both): $\{t \in \mathbf{R} \mid 0 \le t \le 48\}$;
range (top): $\{T \in \mathbf{R} \mid 15 \le T \le 20\}$;
range: (bottom): $\{T \in \mathbf{R} \mid -30 \le T \le -10\}$
c) blue: $T = 2.5 \cos(15(h - 12))° + 17.5$;
red: $T = 10 \cos(15(h - 12))° - 20$

5. a) $d = -30 \cos(18t)°$
b) $d = 9 \cos(36(t - 12))°$

6. a) $d = -0.3 \cos(144t)° + 1.8$
b) amplitude: 0.3, height of crest relative to normal water level
c) 2 m
d) 16
e) $d = -0.3 \cos(120t)° + 1.8$

7. $C = 4.5 \cos(21\,600t)°$

8. a) $h = 8 \cos(450(t - 0.2))° + 12$
b) domain: $\{\text{t} \in \mathbf{R}\}$; range: $\{h \in \mathbf{R} \mid 4 \le h \le 20\}$
c) $h = 12$cm, resting position of the spring
d) 6.3 cm

9. a) $h = -30 \cos[(0.025)x]° + 40$
b) domain: $\{d \in \mathbf{R} \mid 0 \le d \le 400\pi\}$;
range: $\{h \in \mathbf{R} \mid 10 \le h \le 70\}$
c) 69.7 cm

10. The periods and the horizontal translations are the same. As the rabbit population goes down, so does the fox population. As the rabbit population goes up, so does the fox population. The amplitudes differ (rabbits are higher). The axes are different (rabbit times 10 that of fox).

11. The period and amplitude, as well as where it starts on the x-axis and the position on the y-axis when it started

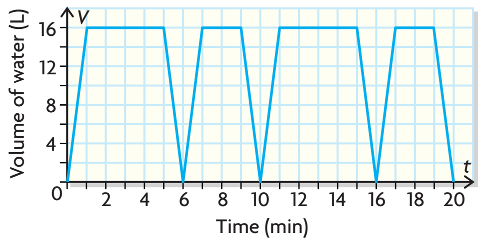
12. $h = -6\cos(0.5x)° + 13$

13. a) $f(x) = -3 \cos x - 1$ **b)** -3.8 **c)** (i) **d)** (iv)

14. 0°, 180°, 360°

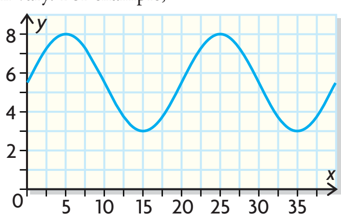
Chapter Review, pp. 404–405

1. a)

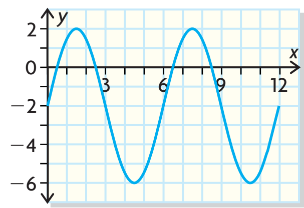


b) yes
c) period: 10 min, how long it takes for the dishwasher to complete one cycle
d) $y = 8$ L
e) 8 L
f) $\{V \in \mathbf{R} \mid 0 \le V \le 16\}$

2. Answers will vary. For example,



3. Answers will vary. For example,



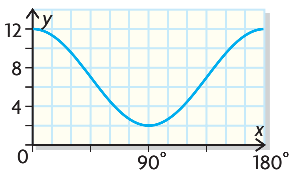
4. **a)** 80 s, time to complete one revolution
b) $y = 16$ m, height of axle above the ground
c) 9 m, radius of wheel
d) yes, graph started at maximum height opposed to boarding height
e) 0.71 m/s
f) $\{h \in \mathbf{R} \mid 7 \leq h \leq 25\}$
g) boarding height: 1 m

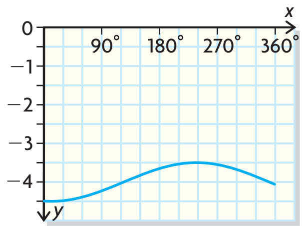
5. **a)** period: 120°; axis: $h = 9$; amplitude: 4; $\{h \in \mathbf{R} \mid 5 \leq h \leq 13\}$
b) yes
c) 6.2
d) 60°, 180°, 300°

6. **a)** 12 s, time between each wave
b) 5 m
c) 5.5 m
d) $\{d \in \mathbf{R} \mid 3 \leq d \leq 7\}$
e) 9 s

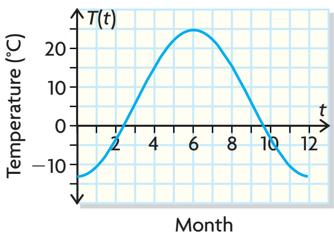
7. (3.63, 1.69)

8. **a)** axis $y = -3$
b) period: 90°
c) amplitude: 7
d) none

9. **a)**


b)


10. **a)** $\{y \in \mathbf{R} \mid -1 \leq y \leq 5\}$
b) $\{y \in \mathbf{R} \mid -0.5 \leq y \leq 0.5\}$

11. **a)**


b) On a yearly basis, the average temperature of each month will be roughly the same. It fluctuates in a cyclical pattern over a specific period of time.
c) max: 24.7 °C; min: −13.1 °C
d) 12; the curve repeats after 12 months, representing one year
e) $T = 5.8°$
f) 6 units right
g) $T(t) = -18.9 \cos 30t + 5.8$
h) −3.7 °C; month 38 is February, and the table shows a temperature close to that value.

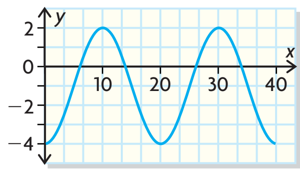
12. **a)** $y = \sin(0.5(\theta - 180)) + 2.5$
b) $y = 2 \sin[2(\theta + 90)] + 4$

13. **a)** period: 1.5 s, time to rock back and forth
b) 26 cm
c) $\{t \in \mathbf{R} \mid 0 \leq t \leq 60\}$
d) $\{d \in \mathbf{R} \mid 18 \leq d \leq 34\}$
e) 8 cm, maximum distance the chair rocks to the front or back from its resting position
f) $y = 8 \cos(240(t - 1.75))° + 26$
g) 30 cm

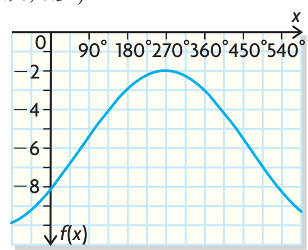
14. To determine the equation of a sinusoidal function, first calculate the period, amplitude, and equation of the axis. This information will help you determine the values of k, a, and c, respectively, in the equations $g(x) = a \sin(k(x - d)) + c$ and $h(x) = a \cos(k(x - d)) + c$.

Chapter Self-Test, p. 406

1. **a)** 40 s, time for stair to return to its initial position
b) $h = 2$m
c) the height at the top of the escalator
d) $\{h \in \mathbf{R} \mid -1 \leq h \leq 5\}$
e) $\{t \in \mathbf{R} \mid 0 \leq t \leq 400\}$
f) No, 300 is not a multiple of 40. Since it started at the ground, it would need to be for this to be true.

2.


3. (2.96, 6.34)

4. **a)**


b) amplitude: 4; period: 720°; axis: $y = -6$
c) −4.47
d) $\{y \in \mathbf{R} \mid -10 \leq y \leq -2\}$

5. **a)** minimum distance between the tip of the metre stick and the edge of the plywood
b) periods are the same; even though you are tracking different ends of the metre stick, the ends do belong to the same metre stick
c) 180 cm
d) the amplitudes are 30cm and 70cm, distance from nail to the ends of the metre stick
e) range 1: $\{d \in \mathbf{R} \mid 150 \leq d \leq 210\}$;
range 2: $\{d \in \mathbf{R} \mid 110 \leq d \leq 250\}$
f) $\{t \in \mathbf{R} \mid 0 \leq t \leq 25\}$
g) $d = 30 \cos(72t)° + 180$; $d = -70 \cos(72t)° + 180$
h) 189.3 cm

Chapters 4–6 Cumulative Review, pp. 408–411

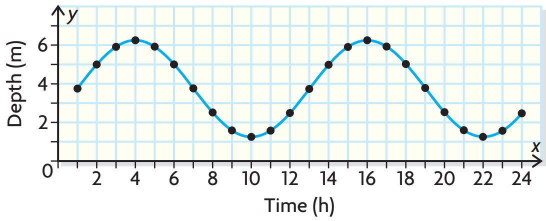
1. (b)	**7.** (b)	**13.** (a)	**19.** (b)	**25.** (c)
2. (a), (d)	**8.** (c)	**14.** (d)	**20.** (d)	**26.** (c)
3. (c)	**9.** (a)	**15.** (d)	**21.** (c)	**27.** (a)
4. (a)	**10.** (c)	**16.** (a)	**22.** (a)	**28.** (c)
5. (a)	**11.** (b)	**17.** (c)	**23.** (a)	
6. (c)	**12.** (a)	**18.** (a)	**24.** (b)	

29. 23 folds

30.

Angle (degrees)	Length of *BC* (cm)
105	80.7
125	62.6
145	43.5
165	26.1
175	20.8

31. **a)**



$f(h) = 2.5\cos(30(h - 4))° + 3.75$ or $f(h) = 2.5\sin(30(h - 1))° + 3.75$

b) 6.25 m

c) The minimum depth of the water at this location is 1.25 m. Therefore, since the hull of the boat must have a clearance of at least 1 m at all times, if the bottom of the hull is more than 0.25 m below the surface of the water, then this location is not suitable for the dock. However, if the bottom of the hull is less than or equal to 0.25 m below the surface of the water, then this location is suitable for the dock.

Chapter 7

Getting Started, p. 414

1. **a)** $y = -\frac{2}{5}x + 8$
b) $y = -9x + 49$
c) $y = \frac{7}{5}x - 7$
d) $y = -2x - 7$

2. **a)** 6 **b)** $\frac{13}{10}$ or 1.3 **c)** 0 **d)** 4

3. **a)** linear **b)** neither **c)** quadratic

4. **a)** $x = 5$ **b)** $x = -5$ **c)** $x = \frac{33}{16}$ **d)** $x = 1.53$

5. about 2.2 g

6. 51.2%

7.

Exponential Functions

Definition:
A function of the form $f(x) = a \times b^x$, where a and b are constants.
Constant changes in the independent variable result in the dependent variable being multiplied by a constant.

Rules/Method:
The graph has a horizontal asymptote, and the graph looks like one of these shapes:

In a table of values, look at the 1st ratios. If they are constant, the function is an exponential.

Examples:
$f(x) = 9 \times 5^x$
$f(x) = \frac{2}{3} \times \left(\frac{5}{11}\right)^x$

Non-examples:
$y = \frac{2}{3}x - 7$ (linear function)
$y = x^3$ (cubic function)
7×2^x (exponential expression)

Lesson 7.1, pp. 424–425

1. **a)** arithmetic, $d = 4$
b) not arithmetic
c) not arithmetic
d) arithmetic, $d = -11$

2. **a)** General term: $t_n = 14n + 14$
Recursive formula: $t_1 = 28$, $t_n = t_{n-1} + 14$, where $n > 1$
b) General term: $t_n = 57 - 4n$
Recursive formula: $t_1 = 53$, $t_n = t_{n-1} - 4$, where $n > 1$
c) General term: $t_n = 109 - 110n$
Recursive formula: $t_1 = -1$, $t_n = t_{n-1} - 110$, where $n > 1$

3. $t_{12} = 53$

4. $t_{15} = 323$

5.

	i)	ii)
a)	arithmetic	General term: $t_n = 3n + 5$ Recursive formula: $t_1 = 8$, $t_n = t_{n-1} + 3$, where $n > 1$
b)	not arithmetic	—
c)	not arithmetic	—
d)	not arithmetic	—
e)	arithmetic	General term: $t_n = 11n + 12$ Recursive formula: $t_1 = 23$, $t_n = t_{n-1} + 11$, where $n > 1$
f)	arithmetic	General term: $t_n = \left(\frac{1}{6}\right)n$ Recursive formula: $t_1 = \frac{1}{6}$, $t_n = t_{n-1} + \frac{1}{6}$, where $n > 1$

6. **a)** Recursive formula: $t_1 = 19$, $t_n = t_{n-1} + 8$, where $n > 1$
General term: $t_n = 8n + 11$
b) Recursive formula: $t_1 = 4$, $t_n = t_{n-1} - 5$, where $n > 1$
General term: $t_n = 9 - 5n$
c) Recursive formula: $t_1 = 21$, $t_n = t_{n-1} + 5$, where $n > 1$
General term: $t_n = 5n + 16$
d) Recursive formula: $t_1 = 71$, $t_n = t_{n-1} - 12$, where $n > 1$
General term: $t_n = 83 - 12n$

7.

	i)	ii)
a)	arithmetic	13, 27, 41, 55, 69; $d = 14$
b)	not arithmetic	—
c)	not arithmetic	—
d)	arithmetic	1, 2, 3, 4, 5; $d = 1$

8.

	i)	ii)	iii)
a)	$t_n = 5n + 30$	$t_1 = 35$, $t_n = t_{n-1} + 5$, where $n > 1$	$t_{11} = 85$
b)	$t_n = 42 - 11n$	$t_1 = 31$, $t_n = t_{n-1} - 11$, where $n > 1$	$t_{11} = -79$
c)	$t_n = -17 - 12n$	$t_1 = -29$, $t_n = t_{n-1} - 12$, where $n > 1$	$t_{11} = -149$
d)	$t_n = 11$	$t_1 = 11$, $t_n = t_{n-1}$, where $n > 1$	$t_{11} = 11$
e)	$t_n = \left(\frac{1}{5}\right)n + \frac{4}{5}$	$t_1 = 1$, $t_n = t_{n-1} + \frac{1}{5}$, where $n > 1$	$t_{11} = 3$
f)	$t_n = 0.17n + 0.23$	$t_1 = 0.4$, $t_n = t_{n-1} + 0.17$, where $n > 1$	$t_{11} = 2.1$

9.

	i)	ii)
a)	arithmetic	6, 4, 2, 0, −2; $d = -2$
b)	not arithmetic	—
c)	arithmetic	$\frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}$; $d = \frac{1}{4}$
d)	not arithmetic	—

10. **a)** 90 seats **b)** 23 rows

11. 63rd month

12. 16 years

Answers

13. **a)** 29 **b)** 38 **c)** 15 **d)** 14 **e)** 18 **f)** 9

14. $t_{100} = (t_8 - t_4) \times 23 + t_8$
The 4th and 8th terms differ by $4d$. The 8th and 100th term differ by $92d = 23 \times 4d$.

15. $t_n = 7n - 112$, where $n \in \mathbf{N}$

16. **a)** Answers will vary. For example,
20, 50, 80, … with $a = 20$ and $d = 30$
50, 20, −10, … with $a = 50$ and $d = -30$
5, 20, 35, 50, … with $a = 5$ and $d = 15$

b) The common difference must divide 30 (50 − 20) or −30 (20 − 50) evenly. The first term must be an integer multiple of the common difference away from 20 and 50.

17. $t_{100} = 112, 211, 310, 409, 607$, or 1201

18. yes

Lesson 7.2, pp. 430–432

1. **a)** not geometric **b)** geometric, $r = 3$ **c)** not geometric **d)** geometric, $r = \frac{1}{2}$

2. **a)** General term: $t_n = 9 \times 4^{n-1}$
Recursive formula: $t_1 = 9$, $t_n = 4t_{n-1}$, where $n > 1$

b) General term: $t_n = 625 \times 2^{n-1}$
Recursive formula: $t_1 = 625$, $t_n = 2t_{n-1}$, where $n > 1$

c) General term: $t_n = 10\,125 \times \left(\frac{2}{3}\right)^{n-1}$
Recursive formula: $t_1 = 10\,125$, $t_n = \left(\frac{2}{3}\right)t_{n-1}$, where $n > 1$

3. $t_{33} = 9963$

4. $t_{10} = 180$

5.

	i)	ii)
a)	geometric	General term: $t_n = 12 \times 2^{n-1}$ or $t_n = 3 \times 2^{n+1}$ Recursive formula: $t_1 = 12$, $t_n = 2t_{n-1}$, where $n > 1$
b)	not geometric	—
c)	not geometric	—
d)	geometric	General term: $t_n = 5 \times (-3)^{n-1}$ Recursive formula: $t_1 = 5$, $t_n = -3t_{n-1}$, where $n > 1$
e)	not geometric	—
f)	geometric	General term: $t_n = 125 \times \left(\frac{2}{5}\right)^{n-1}$ Recursive formula: $t_1 = 125$, $t_n = \left(\frac{2}{5}\right)t_{n-1}$, where $n > 1$

6.

	i)	ii)	iii)
a)	$t_n = 4 \times 5^{n-1}$	$t_1 = 4$, $t_n = 5t_{n-1}$, where $n > 1$	$t_6 = 12\,500$
b)	$t_n = -11 \times 2^{n-1}$	$t_1 = -11$, $t_n = 2t_{n-1}$, where $n > 1$	$t_6 = -352$
c)	$t_n = 15 \times (-4)^{n-1}$	$t_1 = 15$, $t_n = -4t_{n-1}$, where $n > 1$	$t_6 = -15\,360$
d)	$t_n = 896 \times \left(\frac{1}{2}\right)^{n-1}$ or $t_n = 7 \times 2^{8-n}$	$t_1 = 896$, $t_n = \left(\frac{1}{2}\right)t_{n-1}$, where $n > 1$	$t_6 = 28$
e)	$t_n = 6 \times \left(\frac{1}{3}\right)^{n-1}$ or $t_n = 2 \times 3^{2-n}$	$t_1 = 6$, $t_n = \left(\frac{1}{3}\right)t_{n-1}$, where $n > 1$	$t_6 = \frac{2}{81}$
f)	$t_n = 0.2^{n-1}$	$t_1 = 1$, $t_n = 0.2t_{n-1}$, where $n > 1$	$t_6 = 0.000\,32$

7.

	i)	ii)
a)	arithmetic	$t_n = 4n + 5$
b)	geometric	$t_n = 7 \times (-3)^{n-1}$
c)	geometric	$t_n = 18 \times (-1)^{n-1}$
d)	neither	—
e)	arithmetic	$t_n = 39 - 10n$
f)	geometric	$t_n = 128 \times \left(\frac{3}{4}\right)^{n-1}$

8. **a)** Recursive formula: $t_1 = 19$, $t_n = 5t_{n-1}$, where $n > 1$
General term: $t_n = 19 \times 5^{n-1}$

b) Recursive formula: $t_1 = -9$, $t_n = -4t_{n-1}$, where $n > 1$
General term: $t_n = -9 \times (-4)^{n-1}$

c) Recursive formula: $t_1 = 144$, $t_n = \left(\frac{1}{4}\right)t_{n-1}$, where $n > 1$
General term: $t_n = 144 \times \left(\frac{1}{4}\right)^{n-1}$ or $t_n = 9 \times 4^{3-n}$

d) Recursive formula: $t_1 = 900$, $t_n = \left(\frac{1}{6}\right)t_{n-1}$, where $n > 1$
General term: $t_n = 900 \times \left(\frac{1}{6}\right)^{n-1}$ or $t_n = 150 \times 6^{2-n}$

9.

	i)	ii)
a)	not geometric	—
b)	geometric	$-8, 24, -72, 216, -648$; $r = -3$
c)	geometric	$123, 41, \frac{41}{3}, \frac{41}{9}, \frac{41}{27}$; $r = \frac{1}{3}$
d)	geometric	$10, 20, 40, 80, 160$; $r = 2$

10.

	i)	ii)
a)	geometric	$4, 16, 64, 256, 1024$; $r = 4$
b)	not geometric	—
c)	not geometric	—
d)	geometric	$-\frac{7}{125}, \frac{7}{25}, -\frac{7}{5}, 7, -35$; $r = -5$
e)	not geometric	—
f)	geometric	$\frac{11}{13}, \frac{11}{169}, \frac{11}{2197}, \frac{11}{28\,561}, \frac{11}{371\,293}$; $r = \frac{1}{13}$

11. 1 474 560

12. 131 220 bacteria

13. $10 794.62

14. **a)** 65.61% **b)** 29 dosages

15. Yes, $t_{29} = t_7 \times \left(\frac{t_7}{t_5}\right)^{11}$. Use the formula for the general term to write t_{29} in terms of a and r. Then write the equation for t_{29} using the laws of exponents. Evaluate $\frac{t_7}{t_5}$ and then rewrite t_{29} in terms of t_5 and t_7.

16. **a)** 243 shaded triangles **b)** about 0.338 cm^2

17. Both sequences are recursive, so the recursive formulas look similar, except that you add with an arithmetic sequence but multiply with a geometric sequence. The general terms also look similar, except that with an arithmetic sequence, you add a multiple of the common difference to the first term but with a geometric sequence, you multiply a power of the common ratio with the first term.

18. $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}$; The sum gets closer to 2.

19. $t_{10} = 10\,752$, general term: $t_n = 2^{n-1}(2n + 1)$

20. Yes, only for the sequence $a, a, a, \ldots$, where $d = 0$ and $r = 1$.

21. Answers will vary. For example, **1**, **2**, 3, **4**, 5, 6, 7, **8**, … (arithmetic sequence). To form the geometric sequence (shown in red), the previous term determines which term you select.

22. 35.44 cm^2

Lesson 7.3, pp. 439–440

1. Yes, each term depends only on the two previous terms (difference), so the sequence will repeat. Check Sam's formula with other terms to see that it works.
2. $t_n = \dfrac{n}{n+1}$
3. a) $t_n = 2n + 1$ c) $t_n = 2n^2 + 2n$
   b) $t_n = 3n + 1$ d) (triangles) $t_4 = 2(4) + 1 = 9$ toothpicks
   (squares) $t_4 = 2(4)^2 + 2(4) = 40$ toothpicks
4. a) $t_n = -\left(\dfrac{n-1}{2}\right)$ if n is odd
   $= \dfrac{n}{2}$ if n is even
   b) Another rule is $t_1 = 0$, $t_n = \dfrac{n}{2}$ for even n, and $t_n = -t_{n-1}$ for odd n. The other rule is better because each term can be calculated directly, instead of having to calculate terms for even n before calculating terms for odd n.
   c) $t_{12345} = -6172$
5. $nx + \dfrac{1}{y^n}$
6. $t_n = \dfrac{N_n}{D_n}$, where $N_n = 3 \times 7^{n-1}$ and D_n has n fives or
   $t_n = \dfrac{3 \times 7^{n-1}}{\frac{5}{9}(10^n - 1)}$
7. a) 159, 319, 639 c) 34, 55, 89 e) −216, 343, −512
   b) 85, 79, 72 d) 54, 108, 110 f) 111, 223, 447
8. 9900 comparisons
9. 169 271, 846 354, 4 231 771
10. $t_{1000} = 20$
11. Answers will vary. For example, 2, 3, 6, 11, 18, 27, ... The 1st differences form a sequence of odd numbers. So to generate new terms, work backward from the 1st differences.

Lesson 7.4, p. 443

1. The ratio of consecutive terms tends toward the same value as the Fibonacci and Lucas sequences, and the sequences have similar identities.
2. The ratio of $\dfrac{F_n}{F_{n-1}}$ and $\dfrac{L_n}{L_{n-1}}$ get close to $r = \dfrac{1+\sqrt{5}}{2}$.
3. a) 1, 5, 7, 17, 31, 65, 127, 257, 511, and 1025
   b) 5, 1.4, 2.43, 1.82, 2.10, 1.95, 2.02, 1.99, and 2.00. Ratios get close to 2.
   c) $t_n = 2^n + (-1)^n$

Mid-Chapter Review, p. 447

1.

	i)	ii)	iii)
a)	$t_1 = 29$, $t_n = t_{n-1} - 8$, where $n > 1$	$t_n = 37 - 8n$	$t_{10} = -43$
b)	$t_1 = -8$, $t_n = t_{n-1} - 8$, where $n > 1$	$t_n = -8n$	$t_{10} = -80$
c)	$t_1 = -17$, $t_n = t_{n-1} + 8$, where $n > 1$	$t_n = 8n - 25$	$t_{10} = 55$
d)	$t_1 = 3.25$, $t_n = t_{n-1} + 6.25$, where $n > 1$	$t_n = 6.25n - 3$	$t_{10} = 59.5$
e)	$t_1 = \frac{1}{2}$, $t_n = t_{n-1} + \frac{1}{6}$, where $n > 1$	$t_n = \frac{1}{6}n + \frac{1}{3}$	$t_{10} = 2$
f)	$t_1 = x$, $t_n = t_{n-1} + 2x + 3y$, where $n > 1$	$t_n = (2n-1) \times x + 3(n-1)y$	$t_{10} = 19x + 27y$

2. a) Recursive formula: $t_1 = 17$, $t_n = t_{n-1} + 11$, where $n > 1$
   General term: $t_n = 11n + 6$
   b) Recursive formula: $t_1 = 38$, $t_n = t_{n-1} - 7$, where $n > 1$
   General term: $t_n = 45 - 7n$
   c) Recursive formula: $t_1 = 55$, $t_n = t_{n-1} + 18$, where $n > 1$
   General term: $t_n = 18n + 37$
   d) Recursive formula: $t_1 = 42$, $t_n = t_{n-1} - 38$, where $n > 1$
   General term: $t_n = 80 - 38n$
   e) Recursive formula: $t_1 = 159$, $t_n = t_{n-1} - 17$, where $n > 1$
   General term: $t_n = 176 - 17n$
3. 315 seats
4.

	i)	ii)
a)	arithmetic	General term: $t_n = 15n$ Recursive formula: $t_1 = 15$, $t_n = t_{n-1} + 15$, where $n > 1$ $t_6 = 90$
b)	geometric	General term: $t_n = 640 \times \left(\frac{1}{2}\right)^{n-1}$ Recursive formula: $t_1 = 640$, $t_n = \left(\frac{1}{2}\right)t_{n-1}$, where $n > 1$ $t_6 = 20$
c)	geometric	General term: $t_n = 23 \times (-2)^{n-1}$ Recursive formula: $t_1 = 23$, $t_n = -2t_{n-1}$, where $n > 1$ $t_6 = -736$
d)	geometric	General term: $t_n = 3000 \times (0.3)^{n-1}$ Recursive formula: $t_1 = 3000$, $t_n = 0.3t_{n-1}$, where $n > 1$ $t_6 = 7.29$
e)	arithmetic	General term: $t_n = 1.2n + 2.6$ Recursive formula: $t_1 = 3.8$, $t_n = t_{n-1} + 1.2$, where $n > 1$ $t_6 = 9.8$
f)	geometric	General term: $t_n = \left(\frac{1}{2}\right) \times \left(\frac{2}{3}\right)^{n-1}$ Recursive formula: $t_1 = \left(\frac{1}{2}\right) \times \left(\frac{2}{3}\right)t_{n-1}$, where $n > 1$ $t_6 = \frac{16}{243}$

5.

	i)	ii)
a)	geometric	5, 25, 125, 625, 3125
b)	geometric	$\frac{3}{7}, \frac{3}{19}, \frac{3}{67}, \frac{3}{259}, \frac{3}{1027}$
c)	arithmetic	5, −7, −19, −31, −43
d)	geometric	−2, 4, −8, 16, −32
e)	arithmetic	8, 11, 14, 17, 20

6. 45 weeks
7. 349, 519, 737
   The 3rd differences are constant, so use them to determine terms.
8. $t_n = x^n + ny$
9. a) 1, 8, 27 c) $t_n = n^3$
   b) 64, 125, 216 d) 3375
10. a) $t_{15} = 1453$
    b) $t_n = t_{n-2} + t_{n-1}$

Lesson 7.5, pp. 452–453

1. a) $S_{10} = 815$ c) $S_{10} = -1345$
   b) $S_{10} = -50$ d) $S_{10} = 210$
2. $S_{20} = 2450$
3. 670 bricks

4.

	i)	ii)
a)	arithmetic	$S_{25} = 1675$
b)	not arithmetic	—
c)	not arithmetic	—
d)	arithmetic	$S_{25} = 1650$
e)	arithmetic	$S_{25} = -1925$
f)	not arithmetic	—

5. a) $t_{12} = 81, S_{12} = 708$
b) $t_{12} = -134, S_{12} = -882$
c) $t_{12} = 48, S_{12} = 180$
d) $t_{12} = \frac{57}{10}$ or 5.7, $S_{12} = \frac{177}{5}$ or 35.4
e) $t_{12} = 15.51, S_{12} = 112.2$
f) $t_{12} = 12p + 22q, S_{12} = 78p + 132q$

6. a) $S_{20} = 1110$ b) $S_{20} = 1400$ c) $S_{20} = -1980$ d) $S_{20} = 2570$ e) $S_{20} = 2410$ f) $S_{20} = 1630$

7. a) $S_{20} = 970$ b) $S_{26} = 4849$ c) $S_{24} = -168$ d) $S_{711} = 760\,770$ e) $S_{16} = -1336$ f) $S_{22} = 0$

8. a) $D_n = \frac{n(n-3)}{2}$ b) 14 diagonals

9. $5630

10. 1102.5 m

11. 2170 toys

12. 3050 s or 50 min 50 s

13. 700 km

14. Two copies of the first representation fit together to form a rectangle $t_1 + t_n$ by n, yielding the formula $S_n = \frac{n(t_1 + t_n)}{2}$.
Two copies of the second representation fit together to form a rectangle. You can see a and d and get the formula $S_n = \frac{n[2a + (n-1)d]}{2}$.

15. $t_{25} = 79$

16. 26 terms

Lesson 7.6, p. 459–461

1. a) $S_7 = 6558$
b) $S_7 = \frac{3175}{16}$ or 198.4375
c) $S_7 = 4376$
d) $S_7 = \frac{127}{192}$

2. $S_6 = 15\,015$

3. a) $t_6 = 18\,750, S_6 = 23\,436$
b) $t_6 = -2673, S_6 = -4004$
c) $t_6 = 6720, S_6 = 26\,248\,320$
d) $t_6 = \frac{128}{1215}, S_6 = \frac{532}{243}$
e) $t_6 = -138.859, S_6 = -92.969$
f) $t_6 = 243x^{10}, S_6 = \frac{729x^{12} - 1}{3x^2 - 1}$

4.

	i)	ii)
a)	arithmetic	—
b)	geometric	$S_8 = 22\,960$
c)	geometric	$S_8 = \frac{13\,107}{8}$ or 1638.375
d)	neither	—
e)	geometric	$S_8 \doteq 12.579$
f)	arithmetic	—

5. a) $S_7 = 253\,903$
b) $S_7 = 1397$
c) $S_7 = \frac{163\,830}{1024}$ or about 159.990
d) $S_7 = 2186$
e) $S_7 = 49\,416$
f) $S_7 = \frac{645}{48}$ or 13.4375

6. a) $S_8 = 335\,923$
b) $S_7 = 1905$
c) $S_{10} = -250\,954$
d) $S_6 = 28\,234.9725$
e) $S_8 = 78\,642$
f) $S_{13} = \frac{8191}{1024}$

7. about 10.8 m

8. If $r = 1$, all the terms are the same, $a + a + a + \ldots$. So the sum of n terms would be $S_n = na$.

9. 1 048 575 line segments

10. 12.25 m^2

11. 5465 employees

12. 14 559 864

13. Answers will vary. For example, the first prize is $1829, each prize is 3 times the previous one, and there are 7 prizes altogether. The total value of the prizes is $1 999 097.

14.

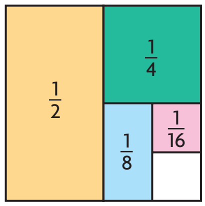
Arithmetic	Geometric	Similarities	Differences
$S_n = \frac{n(t_1 + t_n)}{2}$ Write the terms of S_n twice, once forward, then once backward above each other. Then add pairs of terms. The sum of the pairs is constant.	$S_n = \frac{t_{n+1} - t_1}{r - 1}$, where $r \neq 1$ Write the terms of S_n and rS_n above each other. Then subtract pairs of terms. The difference of all middle pairs is zero.	Both general formulas involve two "end" terms of the series.	You add with formula for arithmetic series, but subtract for geometric series. You divide by 2 for arithmetic series, but by $r - 1$ for geometric series.

15. $t_5 = 15\,552$

16. 13 terms

17. $x^{15} - 1 = (x - 1)(x^2 + x + 1)(x^4 + x^3 + x^2 + x + 1)(x^8 - x^7 + x^5 - x^4 + x^3 - x + 1)$

18. a)



b) The formula for the sum of geometric series gives the sum of the first n terms.
c) Yes. Consecutive terms of this series are getting smaller and smaller, so the sum is getting closer and closer to 1.

Lesson 7.7, p. 466

1. 1, 13, 78, and 286

2. a) $(x + 2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$
b) $(x - 1)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$
c) $(2x - 3)^3 = 8x^3 - 36x^2 + 54x - 27$

3. a) $(x + 5)^{10} = x^{10} + 50x^9 + 1125x^8 + \ldots$
b) $(x - 2)^8 = x^8 - 16x^7 + 112x^6 - \ldots$
c) $(2x - 7)^9 = 512x^9 - 16\,128x^8 + 225\,792x^7 - \ldots$

4. a) $(k + 3)^4 = k^4 + 12k^3 + 54k^2 + 108k + 81$
b) $(y - 5)^6 = y^6 - 30y^5 + 375y^4 - 2500y^3 + 9375y^2 - 18\,750y + 15\,625$
c) $(3q - 4)^4 = 81q^4 - 432q^3 + 864q^2 - 768q + 256$
d) $(2x + 7y)^3 = 8x^3 + 84x^2y + 294xy^2 + 343y^3$
e) $(\sqrt{2x} + \sqrt{3})^6 = 8x^6 + 24\sqrt{6}x^5 + 180x^4 + 120\sqrt{6}x^3 + \sqrt{270}x^2 + 54\sqrt{6}x + 27$
f) $(2z^3 - 3y^2)^5 = 32z^{15} - 240z^{12}y^2 + 720z^9y^4 - 1080z^6y^6 + 810z^3y^8 - 243y^{10}$

5. a) $(x - 2)^{13} = x^{13} - 26x^{12} + 312x^{11} - \ldots$
b) $(3y + 5)^9 = 19\,683y^9 + 295\,245y^8 + 1\,968\,300y^7 + \ldots$
c) $(z^5 - z^3)^{11} = z^{55} - 11z^{53} + 55z^{51} - \ldots$
d) $(\sqrt{a} + \sqrt{5})^{10} = a^5 + 10\sqrt{5a}\,a^4 + 225a^4 + \ldots$

e) $\left(3b^2 - \frac{2}{b}\right)^{14} = 4\,782\,969b^{28} - 44\,641\,044b^{25} + 193\,444\,524b^{22} + ...$

f) $(5x^3 + 3y^2)^8 = 390\,625x^{24} + 1\,875\,000x^{21}y^2 + 3\,937\,500x^{18}y^4 + ...$

6. **a)** The sum of all the numbers in a row of Pascal's triangle is equal to a power of 2.

b) If you alternately subtract and add the numbers in a row of Pascal's triangle, the result is always zero.

7. 1, 1, 2, 3; These are terms in the Fibonacci sequence.

8. 252 ways

9. Write $(x + y + z)^{10} = [x + (y + z)]^{10}$ and use the pattern for expanding a binomial twice.

10. $(3x - 5y)^6 = 729x^6 - 7290x^5y + 30\,375x^4y^2 - 67\,500x^3y^3 + 84\,375x^2y^4 - 56\,250xy^5 + 15\,625y^6$

11. To expand $(a + b)^n$, where $n \in \mathbf{N}$, write the numbers from the nth row of Pascal's triangle. Each term in the expansion is the product of a number from Pascal's triangle, a power of a, and a power of b. The exponents of a go down, term by term, to zero. The exponents of b start at zero and go up, term by term, to n.

12. The 1st differences of a cubic correspond to the differences between the $(x + 1)$th value and the xth value. The 1st differences of a cubic are quadratic. The 2nd differences of a cubic correspond to the 1st differences of a quadratic, and the 3rd differences of a cubic correspond to the 2nd differences of a quadratic. Since the 2nd differences of a quadratic are constant, the 3rd differences of a cubic are constant.

13. $\left(\frac{1}{2} + \frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{10} + 10\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right) + 45\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + ...$

The first three terms represent the probablility of getting heads 10, 9, and 8 times, respectively.

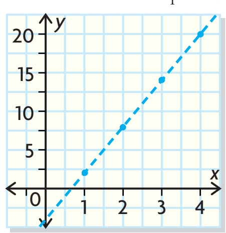
Chapter Review, pp. 468–469

1. **a)** Arithmetic sequence with first term 2, and each term afterward increases by 6.

b) General term: $t_n = 6n - 4$

Recursive formula: $t_1 = 2$, $t_n = t_{n-1} + 6$, where $n > 1$

c)



2. Check if the difference between consecutive terms is constant.

3.

	i)	ii)
a)	$t_n = 15n + 43$	$t_1 = 58$, $t_n = t_{n-1} + 15$, where $n > 1$
b)	$t_n = 9n - 58$	$t_1 = -49$, $t_n = t_{n-1} + 9$, where $n > 1$
c)	$t_n = 87 - 6n$	$t_1 = 81$, $t_n = t_{n-1} - 6$, where $n > 1$

4. $t_{100} = -3348$

5. 9 weeks

6. Check if the ratio of consecutive terms is constant.

7.

	i)	ii)
a)	geometric	$t_6 = 1215$
b)	neither	—
c)	geometric	$t_6 = 0.000\,09$
d)	arithmetic	$t_6 = 210$
e)	arithmetic	$t_6 = -26$
f)	geometric	$t_6 = 121.5$

8.

	i)	ii)	iii)
a)	$t_1 = 7$, $t_n = -3t_{n-1}$, where $n > 1$	$t_n = 7 \times (-3)^{n-1}$	7, −21, 63, −189, 567
b)	$t_1 = 12$, $t_n = \left(\frac{1}{2}\right)t_{n-1}$, where $n > 1$	$t_n = 12 \times \left(\frac{1}{2}\right)^{n-1}$	12, 6, 3, $\frac{3}{2}$, $\frac{3}{4}$
c)	$t_1 = 9$, $t_n = 4t_{n-1}$, where $n > 1$	$t_n = 9 \times 4^{n-1}$	9, 36, 144, 576, 2304

9.

	i)	ii)
a)	arithmetic	9, 13, 17, 21, 25
b)	neither	$\frac{1}{4}, \frac{1}{11}, \frac{1}{18}, \frac{1}{25}, \frac{1}{32}$
c)	neither	0, 3, 8, 15, 24
d)	neither	−17, −16, −14, −11, −7

10. **a)** 47 104 000 bacteria

b) No. The bacteria would eventually run out of food and space in the culture.

11. \$1933.52

12. $t_n = n^2 + 3n$

13. $t_{100} = \frac{100}{299}$

14. **a)** $S_{50} = 9850$ **b)** $S_{50} = -3850$ **c)** $S_{50} = 27\,275$ **d)** $S_{50} = -6575$ **e)** $S_{50} = 2590$ **f)** $S_{50} = 11\,750$

15. **a)** $S_{25} = 3900$ **b)** $S_{25} = 5812.5$ **c)** $S_{25} = -6000$ **d)** $S_{25} = -1700$ **e)** $S_{25} = 375$ **f)** $S_{25} = 950$

16. **a)** $S_{13} = 949$ **b)** $S_{124} = 252\,774$ **c)** $S_{50} = 25$

17. 325 m

18. **a)** $t_6 = 2673$, $S_6 = 4004$

b) $t_6 = 11\,111.1$, $S_6 \doteq 12\,345.654$

c) $t_6 = -192$, $S_6 = -126$

d) $t_6 = 4320$, $S_6 = 89\,775$

e) $t_6 = -\frac{4131}{32}$ or $-129.093\,75$, $S_6 = -\frac{2261}{32}$

f) $t_6 = \frac{243}{6250}$, $S_6 = \frac{3724}{3125}$

19. **a)** $S_8 = -131\,070$ **b)** $S_8 \doteq 3276.087\,34$ **c)** $S_8 \doteq 426.660$ **d)** $S_8 = 136\,718.4$

20. 61 425 orders

21. $S_{10} = 12\,276$

22. **a)** 7161

b) 1533

c) $\frac{25\,999}{64}$

d) 64 125

e) 18 882.14

23. **a)** $(a + 6)^4 = a^4 + 24a^3 + 216a^2 + 864a + 1296$

b) $(b - 3)^5 = b^5 - 15b^4 + 90b^3 - 270b^2 + 405b - 243$

c) $(2c + 5)^3 = 8c^3 + 60c^2 + 150c + 125$

d) $(4 - 3d)^6 = 4096 - 18\,432d + 34\,560d^2 - 34\,560d^3 + 19\,440d^4 - 5832d^5 + 729d^6$

e) $(5e - 2f)^4 = 625e^4 - 1000e^3f + 600e^2f^2 - 160ef^3 + 16f^4$

f) $\left(3f^2 - \frac{2}{f}\right)^4 = 81f^8 - 216f^5 + 216f^2 - \frac{96}{f} + \frac{16}{f^4}$

Chapter Self-Test, p. 470

1. i) / ii)

a) 45, 135, 405, 1215, 3645 — geometric

b) $\frac{5}{3}, \frac{8}{5}, \frac{11}{7}, \frac{14}{9}, \frac{17}{11}$ — neither

c) 5, 10 , 15, 20, 25 — arithmetic

d) 5, 35, 245, 1715, 12 005 — geometric

e) 19, −18, 19, −18, 19 — neither

f) 7, 13, 19, 25, 31 — arithmetic

2. i) / ii)

a) $t_n = (-9) \times (-11)^{n-1}$ — $t_1 = -9,\ t_n = -11t_{n-1}$, where $n > 1$

b) $t_n = 1281 - 579n$ — $t_1 = 702,\ t_n = t_{n-1} - 579$, where $n > 1$

3. a) 26 terms
b) 8 terms

4. a) $(x - 5)^4 = x^4 - 20x^3 + 150x^2 - 500x + 625$
b) $(2x + 3y)^3 = 8x^3 + 36x^2y + 54xy^2 + 27y^3$

5. a) $S_{31} = 7099$
b) $S_{10} = 259\ 586.8211$

6. $t_{123} = \frac{3}{2}$

7. $17 850

8. a) 61, 99, 160
b) $p^6 + 6p, p^7 + 7p, p^8 + 8p$
c) $-\frac{5}{18}, -\frac{11}{21}, -\frac{17}{24}$

Chapter 8

Getting Started, p. 474

1. i) / ii) / iii)

a) 23, 27 — $t_n = 3 + 4n$ — $t_1 = 7,\ t_n = t_{n-1} + 4$, where $n > 1$

b) −50, −77 — $t_n = 85 - 27n$ — $t_1 = 58,\ t_n = t_{n-1} - 27$, where $n > 1$

c) 1280, 5120 — $t_n = 5 \times 4^{n-1}$ — $t_1 = 5,\ t_n = 4t_{n-1}$, where $n > 1$

d) −125, 62.5 — $t_n = 1000 \times \left(\frac{-1}{2}\right)^{n-1}$ — $t_1 = 1000$, $t_n = \left(\frac{-1}{2}\right)t_{n-1}$, where $n > 1$

2. a) $t_5 = 147$ b) $d = 101$ c) $a = -257$ d) $t_{100} = 9742$

3. a) geometric—There is a constant rate between the terms.
b) $t_1 = 8000,\ t_n = (1.05)t_{n-1}$, where $n > 1$
c) $t_n = 8000 \times (1.05)^{n-1}$
d) $t_{10} \doteq 12\ 410.6257$

4. a) $S_{10} = 120$ c) $S_{10} = -285$
b) $S_{10} = 0$ d) $S_{10} = 5\ 456\ 000$

5. a) (1st year) 210 000, (2nd year) 220 500, (3rd year) 231 525
b) about 325 779

6. $x = 12$

7. a) $x \doteq 19.93$ c) $x \doteq 11.26$
b) $x \doteq 3.48$ d) $x \doteq 8.72$

8.

Example: $f(x) = 5 \times 2^x$	**Visual representation:** The graph looks like one of these shapes: 
Definition in your own words: A function of the form $f(x) = a \times b^x$, where a and b are constants. When x increases or decreases by a constant, y is multiplied by a constant.	**Personal association:** half-life geometric sequences

Lesson 8.1, pp. 481–482

1. a) i) (1st year) $532, (2nd year) $564, (3rd year) $596
ii) $980
b) i) (1st year) $1301.25, (2nd year) $1352.50, (3rd year) $1403.75
ii) $2018.75
c) i) (1st year) $26 250, (2nd year) $27 500, (3rd year) $28 750
ii) $43 750
d) i) (1st year) $1739.10, (2nd year) $1778.20, (3rd year) $1817.30
ii) $2286.50

2. a) $P = \$2000$ c) $r = 6\%/a$
b) $I = \$600$ d) $A(t) = 2000 + 120t$

3. 3 years and 132 days

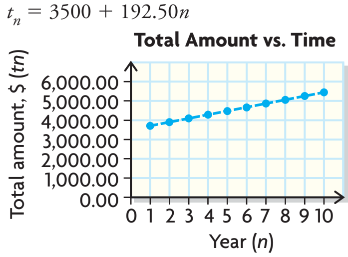
4. about 28%/a

5. a) $I = \$192, A = \$692$ d) $I = \$9.60, A = \$137.60$
b) $I = \$3763.20, A = \$6963.20$ e) $I = \$3923.08, A = \$53\ 923.08$
c) $I = \$260, A = \$5260$ f) $I = \$147.95, A = \$4647.95$

6. about 7.84%/a

7. $47 619.05

8. a) $192.50
b) (1st year) $3692.50, (2nd year) $3885.00, (3rd year) $4077.50, (4th year) $4270.00, (5th year) $4462.50
c) $t_n = 3500 + 192.50n$
d)



9. a) $3740 b) 27.2%/a

10. a) $1850
b) $t_n = 1850 + 231.25n$
c) 24 years and 157 days

11. 66 years and 8 months

12. $P = \$750$, $r = 3.7\%/a$ $A(t) = P + Prt$; $P = 750$; $Prt = 27.75t$; $750rt = 27.75t$; $750r = 27.75$; $r = 0.037$

13. $D = \frac{1}{r}$

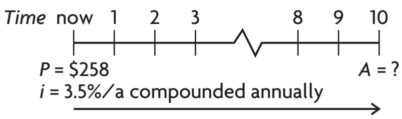
14. $23 400

Lesson 8.2, pp. 490–492

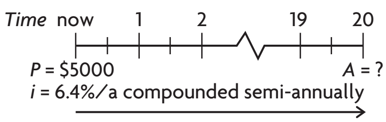
1.

	Interest Rate per Compounding Period, *i*	Number of Compounding Periods, *n*
a)	0.027	10
b)	0.003	36
c)	0.007 25	28
d)	0.000 5	43.3

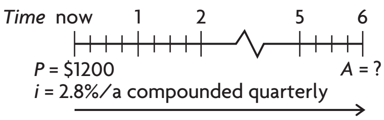
2. **a)** **i)** (1st year) \$10 720.00, (2nd year) \$11491.84, (3rd year) \$12 319.25, (4th year) \$13 206.24, (5th year) \$14 157.09
ii) $A(n) = 10\,000(1.072)^n$
b) **i)** (1st half-year) \$10 190.00, (2nd half-year) \$10 383.61, (3rd half-year) \$10 580.90, (4th half-year) \$10 781.94, (5th half-year) \$10 986.79
ii) $A(n) = 10\,000(1.019)^n$
c) **i)** (1st quarter) \$10 170.00, (2nd quarter) \$10 342.89, (3rd quarter) \$10 518.72, (4th quarter) \$10 697.54, (5th quarter) \$10 879.40
ii) $A(n) = 10\,000(1.017)^n$
d) **i)** (1st month) \$10 090.00, (2nd month) \$10 180.81, (3rd month) \$10 272.44, (4th month) \$10 364.89, (5th month) \$10 458.17
ii) $A(n) = 10\,000(1.009)^n$

3. **a)**


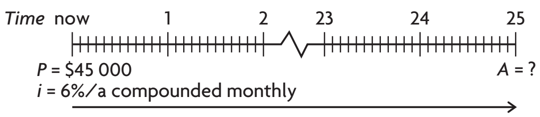
\$363.93

b)


\$17 626.17

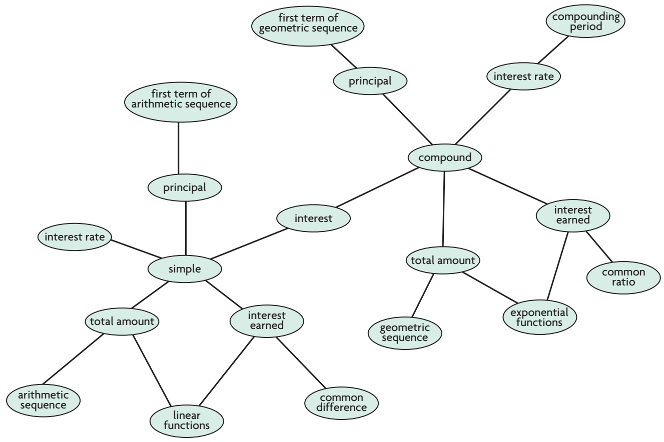
c)


\$1418.69

d)


\$200 923.64

4. **a)** $A = \$4502.04$, $I = \$502.04$
b) $A = \$10\,740.33$, $I = \$3240.33$
c) $A = \$16\,906.39$, $I = \$1906.39$
d) $A = \$48\,516.08$, $I = \$20\,316.08$
e) $A = \$881.60$, $I = \$31.60$
f) $A = \$2332.02$, $I = \$107.02$

5. **a)** 6% **b)** \$4000
6. about 13 years
7. \$3787.41
8. $P = \$5000$, $i = 9\%$/a compounded monthly
9. Plan B; Plan A = \$1399.99; Plan B = \$1049.25
10. \$1407.10
11. \$14 434.24
12. If he invests for 26 years or less, the first option is better.
13. Answers may vary. For example, How long will it take for both investments to be worth the same amount?
a) \$5000 at 5%/a compounded annually
b) \$3000 at 7%/a compounded annually
Answer: about 27 years
14. 6.5%/a compounded quarterly, 6.55%/a compounded semi-annually, 6.45%/a compounded monthly, 6.6%/a compounded annually
15. \$4514.38
16. \$4543.12
17. \$3427.08
18.


19. \$13 995.44
20. **a)** about 6.40% **b)** about 4.28% **c)** about 3.24%

Lesson 8.3, pp. 498–499

1. **a)** \$6755.64 **c)** \$10 596.47
b) \$73 690.81 **d)** \$3.46
2. Lui
3. **a)** $PV = \$7920.94$, $I = \$2079.06$
b) $PV = \$4871.78$, $I = \$1328.22$
c) $PV = \$8684.66$, $I = \$11\,315.34$
d) $PV = \$8776.74$, $I = \$4023.26$
4. \$8500.00
5. \$1900.00
6. \$5586.46
7. \$10 006.67
8. 8.85%
9. Franco, \$204.20
10. \$7200.00
11. **a)** 8-year guarantee **b)** \$8324.17
12. With present value, you are looking back in time. Present value is an exponential function with ratio $(1 + i)^{-1}$, so the amount is decreasing the further you go into the past, just like the amount of radioactive material decreases as time goes on.
13. about $16\frac{1}{2}$ years
14. 11.14%
15. about \$3047.98
16. $PV = \dfrac{A}{1 + in}$

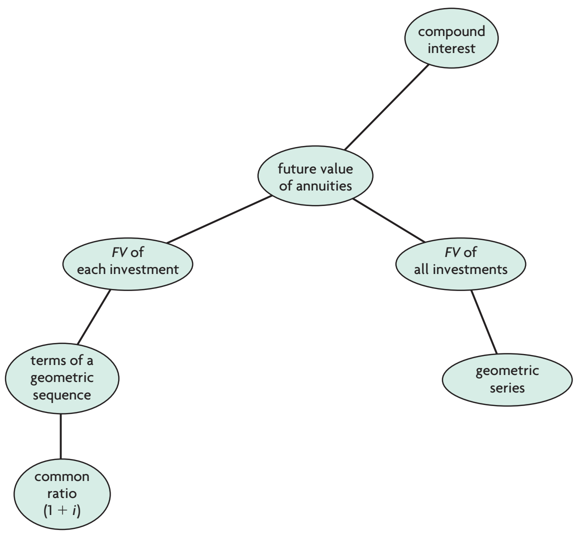
Answers

Mid-Chapter Review, p. 503

1. a) $I = \$5427.00$, $A = \$10\,827.00$
   b) $I = \$51.20$, $A = \$451.20$
   c) $I = \$3300.00$, $A = \$18\,300.00$
   d) $I = \$278.42$, $A = \$2778.42$
2. 3 years and 53 days
3. a) \$64.60 b) \$950.00 c) 6.8%/a
4. a) $A = \$8805.80$, $I = \$2505.80$
   b) $A = \$34\,581.08$, $I = \$20\,581.08$
   c) $A = \$822\,971.19$, $I = \$702\,971.19$
   d) $A = \$418.17$, $I = \$120.17$
5. 11 years and 5 months
6. 18.85%/a
7. \$2572.63
8. \$350.00
9. a) 9.40%/a b) \$8324.65

Lesson 8.4, pp. 511–512

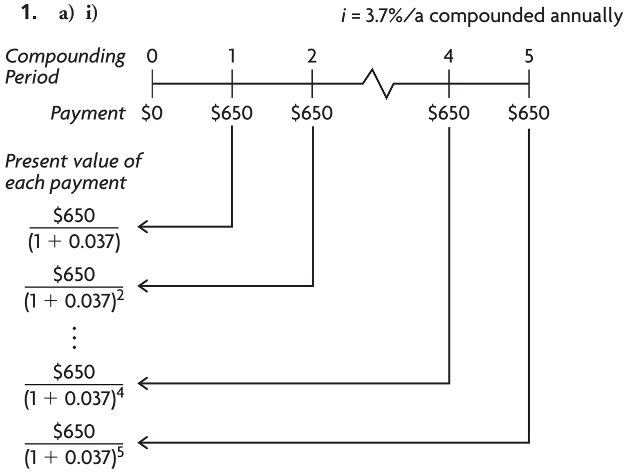
1. a) (1st investment) \$16 572.74, (2nd investment) \$15 316.76, (3rd investment) \$14 155.97, (4th investment) \$13 083.15
   b) geometric
   c) \$188 191.50
2. a) \$167 778.93 c) \$9920.91
   b) \$146 757.35 d) \$49 152.84
3. \$59 837.37
4. \$4889.90
5. a) \$20 051.96 c) \$79 308.62
   b) \$1569.14 d) \$57 347.07
6. a) \$148.77 b) \$638.38
7. Investment (a)—more compounding periods
8. 5 years and 9 months
9. a) \$198.25 b) \$926 980.31
10. 6.31%
11.



12. a) 3 years and 8 months b) \$918.30
13. \$924.32
14. 76 payments

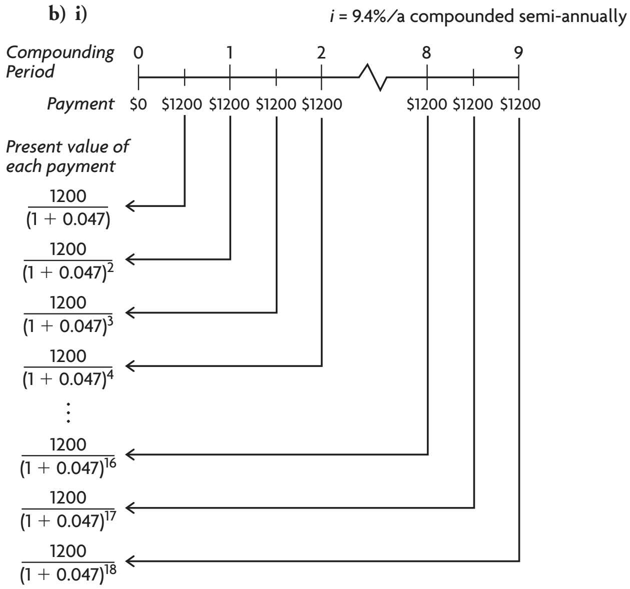
Lesson 8.5, pp. 520–522

1. a) i)



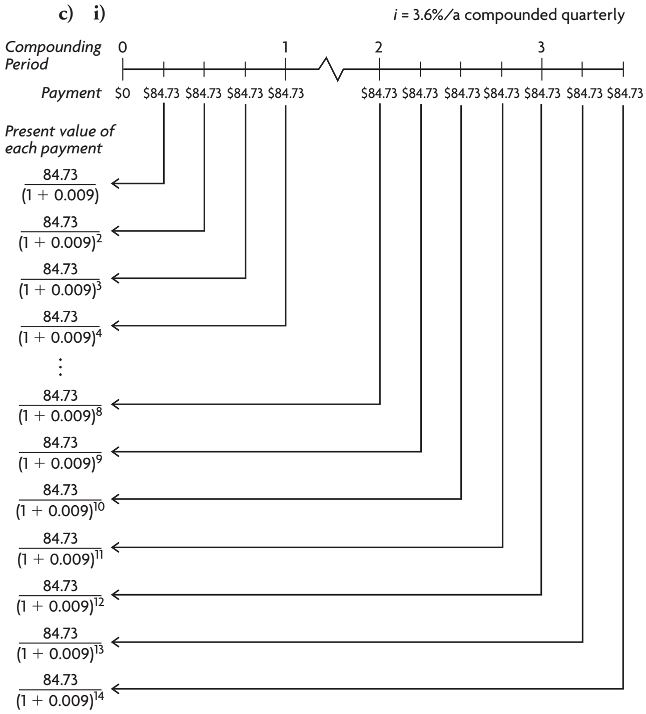
   ii) $PV = 650(1.037)^{-1} + 650(1.037)^{-2} + 650(1.037)^{-3} + \ldots + 650(1.037)^{-5}$
   iii) \$2918.23
   iv) \$331.77

   b) i)



   ii) $PV = 1200(1.047)^{-1} + 1200(1.047)^{-2} + 1200(1.047)^{-3} + \ldots + 1200(1.047)^{-18}$
   iii) \$14 362.17
   iv) \$7237.83

c) i)

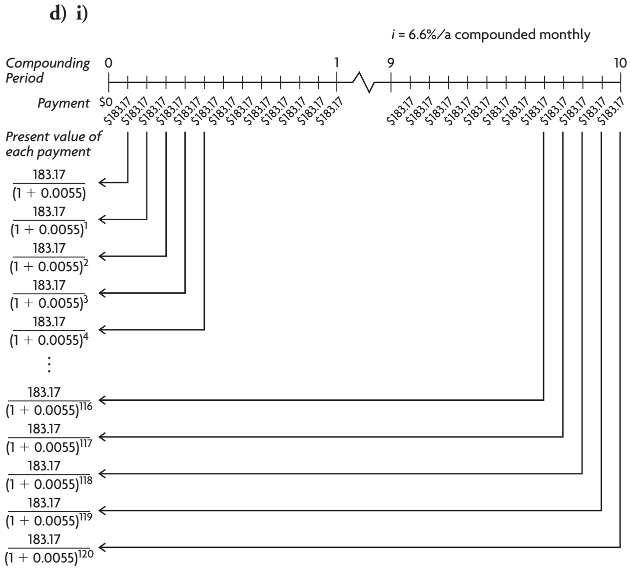


ii) $PV = 84.73(1.009)^{-1} + 84.73(1.009)^{-2} + 84.73(1.009)^{-3} + \ldots + 84.73(1.009)^{-14}$

iii) \$1109.85

iv) \$76.37

d) i)



ii) $PV = 183.17(1.0055)^{-1} + 183.17(1.0055)^{-2} + 183.17(1.0055)^{-3} + \ldots + 183.17(1.0055)^{-120}$

iii) \$16 059.45

iv) \$5920.95

2. a) i) $PV_1 = \$7339.45$, $PV_2 = \$6733.44$, $PV_3 = \$6177.47$, $PV_4 = \$5667.40$, $PV_5 = \$5199.45$, $PV_6 = \$4770.14$, $PV_7 = \$4376.27$

ii) $PV = 8000(1.09)^{-1} + 8000(1.09)^{-2} + 8000(1.09)^{-3} + \ldots + 8000(1.09)^{-7}$

iii) \$40 263.62

b) i) $PV_1 = \$288.46$, $PV_2 = \$277.37$, $PV_3 = \$266.70$, $PV_4 = \$256.44$, $PV_5 = \$246.58$, $PV_6 = \$237.09$, $PV_7 = \$227.98$

ii) $PV = 300(1.04)^{-1} + 300(1.04)^{-2} + 300(1.04)^{-3} + \ldots + 300(1.04)^{-7}$

iii) \$1800.62

c) i) $PV_1 = \$735.29$, $PV_2 = \$720.88$, $PV_3 = \$706.74$, $PV_4 = \$692.88$, $PV_5 = \$679.30$, $PV_6 = \$665.98$, $PV_7 = \$652.92$, $PV_8 = \$640.12$

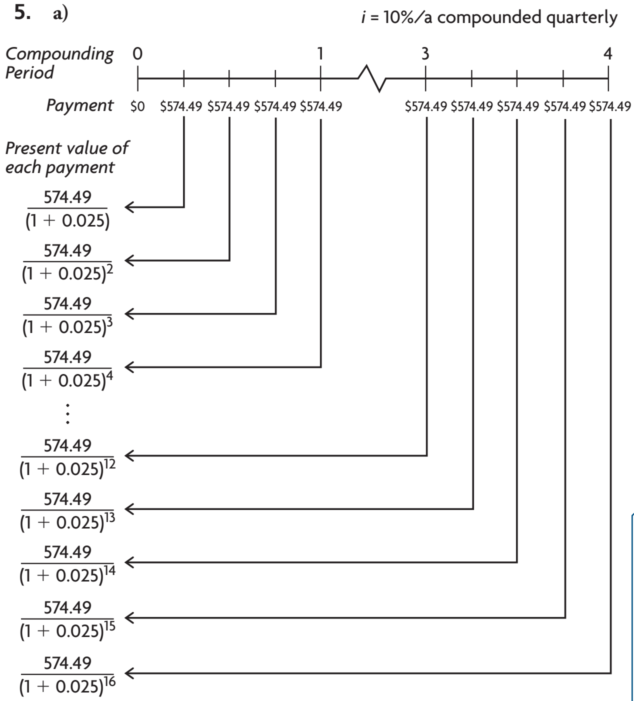
ii) $PV = 750(1.02)^{-1} + 750(1.02)^{-2} + 750(1.02)^{-3} + \ldots + 750(1.02)^{-8}$

iii) \$5494.11

3. **a)** \$20 391.67 **c)** \$2425.49
b) \$4521.04 **d)** \$1093.73

4. \$64.90

5. a)



b) $PV = 574.49(1.025)^{-1} + 574.49(1.025)^{-2} + 574.49(1.025)^{-3} + \ldots + 574.49(1.025)^{-16}$

c) \$574.49

6. **a)** \$418.89 **b)** \$31.11

7. \$971.03

8. **a)** (7-year term) \$1029.70, (10-year term) \$810.72
b) \$10 791.60
c) Answers may vary. For example, how much they can afford per month, interest rates possibly dropping

9. Choose bank financing. TL: \$33 990.60 B: \$33 156.00

10. **a)** (5-year term) \$716.39, (10-year term) \$432.08, (15-year term) \$342.61
b) (5-year term) \$7983.40, (10-year term) \$16 849.60, (15-year term) \$26 669.80

11. **a)** \$316.84 **b)** \$28.16

12. 19.00%

13. $1572.66

14. $219.85

15. **a)** Option B

b) If the person lives 29 or more years and/or the interest rate is 6.2% or less.

16. **a)** Answers may vary. For example, a lump sum is a one-time payment unlike an annuity, which has multiple payments. If a contest prize can be collected either as a lump sum or an annuity, the annuity will earn interest until the last payment is made. The lump sum will not earn any interest unless the contest winner deposits that amount into an account that earns interest.

b) Answers may vary. For example, future value is the value of an investment some time in the future unlike present value, which is the value of the investment now. If the parents of a child are planning for their child's education, they could deposit an amount (present value) into an account that earns interest. Once the child reaches the appropriate age, the balance in the account (future value) will be used for tuition and other fees.

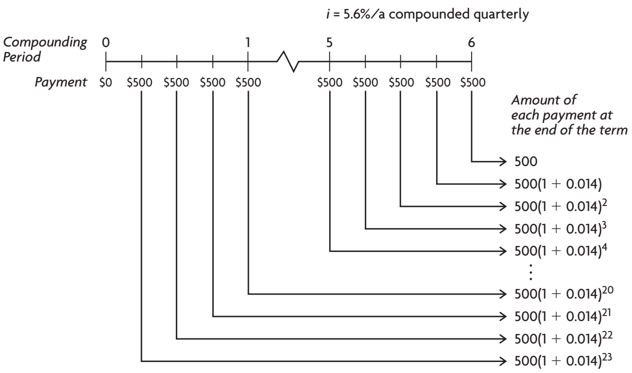
17. **a)**

	A	B	C
1	Year	Investment	Value of Each Investment at the End of 10 Years
2	1	$500.00	$1 024.77
3	2	$500.00	$946.23
4	3	$500.00	$873.71
5	4	$500.00	$806.75
6	5	$500.00	$744.92
7	6	$500.00	$687.83
8	7	$500.00	$635.12
9	8	$500.00	$586.44
10	9	$500.00	$541.50
11	10	$500.00	$500.00
12			$7 347.29

$$PV = \frac{7347.29}{(1.083)^{10}}$$
$$= \$3310.11$$

b) Answers may vary. For example, How much would you need to invest now at 5.6%/a compounded quarterly to provide $500 for the next six years?

Answer:



	A	B	C
1	Year	Investment	Value of Each Investment at the End of 10 Years
2	1	$500.00	$688.40
3	2	$500.00	$678.90
4	3	$500.00	$669.53
5	4	$500.00	$660.29
6	5	$500.00	$651.17
7	6	$500.00	$642.17
8	7	$500.00	$633.31
9	8	$500.00	$624.56
10	9	$500.00	$615.94
11	10	$500.00	$607.44
12	11	$500.00	$599.05
13	12	$500.00	$590.78
14	13	$500.00	$582.62
15	14	$500.00	$574.58
16	15	$500.00	$566.65
17	16	$500.00	$558.82
18	17	$500.00	$551.11
19	18	$500.00	$543.50
20	19	$500.00	$535.99
21	20	$500.00	$528.59
22	21	$500.00	$521.30
23	22	$500.00	$514.10
24	23	$500.00	$507.00
25	24	$500.00	$500.00
26			$14 145.78

$$PV = \frac{14\,145.78}{(1.014)^{24}}$$
$$= \$10\,132.49$$

c) $PV = \dfrac{FV}{(1+i)^n}$

18. 5 years and 3 months

19. $R = W \times \dfrac{1-(1+i)^{-n}}{(1+i)^m - 1}$

Lesson 8.6, pp. 530–531

1. **a)** 12 years **b)** 7 years **c)** 19 years **d)** 8 years

2. **a)** 7.10% **b)** 5.80% **c)** 16.30% **d)** 22.19%

3. $99.86

4. **a)** $817.76 **b)** 5 years and 2 months **c)** $36 368.74

5. $3651.03

6. 8 years

7. **a)** $1143.52 **b)** 9 years

8. 12.36%

9. **a)** Answers may vary. For example, for a loan of $3500 at 6.6%/a compounded monthly and amortized over 2 years, the monthly payment will be $156.07. If the interest rate became 13.2%/a compounded monthly, to keep the same amortization period, *R* would have to be $166.73, which is not double the original *R*-value.

b) Answers may vary. For example, for a loan of $3500 at 6.6%/a compounded monthly and amortized over 2 years, the monthly payment will be $156.07. If the loan became $7000, to keep the same amortization period, *R* would have to be $312.14, which is double the original *R*-value.

10. 3 years and 3 months

11. E.g.,

Type of Technology	Advantages	Disadvantages
spreadsheet	• can set up the spreadsheet so that you only need to type equations once and just input the values in a question • can see how changing one or more values affects the rest of the calculation	• need a computer
graphing calculator	• may be more easily available	• have to make too many keystrokes • could mistype a number or equation if there are several operations to be performed • can't see how changing one value affects the rest of the values

12. $4697.58 or less
13. **a)** (amortization period) 19.58 years
b) (amortization period) 13.64 years
c) (amortization period) 13.57 years
14. 5 years and 2 months

Chapter Review, pp. 534–535

1. **a)** $I = \$2100.00$, $A = \$5600.00$
b) $I = \$4950.00$, $A = \$19\,950.00$
c) $I = \$25.39$, $A = \$305.39$
d) $I = \$474.04$, $A = \$1324.04$
e) $I = \$176.40$, $A = \$21\,176.40$
2. **a)** 5.40% **b)** $3445.00 **c)** 18 years and 6 months
3. **a)** $5000 **b)** 10% **c)** 30 years
4. about 11 years
5. **a)** $A = \$8631.11$, $I = \$4331.11$
b) $A = \$1604.47$, $I = \$1104.47$
c) $A = \$30\,245.76$, $I = \$5245.76$
d) $A = \$607.31$, $I = \$300.31$
6. **a)** 8.00%/a **b)** $5000
7. 1.73%
8. **a)** $5784.53 **c)** $64 089.29
b) $1032.07 **d)** $589.91
9. $667.33
10. 11.10%/a
11. **a)** $FV = \$46\,332.35$, $I = \$16\,332.35$
b) $FV = \$13\,306.97$, $I = \$3806.97$
c) $FV = \$31\,838.87$, $I = \$1838.87$
12. 12 years and 7 months
13. $263.14
14. **a)** $PV = \$2276.78$, $I = \$423.22$
b) $PV = \$17\,185.88$, $I = \$4189.12$
c) $PV = \$2069.70$, $I = \$530.52$
d) $PV = \$1635.15$, $I = \$259.71$
15. **a)** $1022.00 **b)** $109 280.00
16. 20.40%
17. $29.12
18. $182.34
19. 4 years
20. $1979.06

Chapter Self-Test, p. 536

1. **a)** $A = \$1309.00$, $I = \$459.00$
b) $A = \$15\,913.05$, $I = \$10\,453.05$
c) $A = \$21\,005.02$, $I = \$3065.02$
2. **a)** (Loan #1) simple interest − common difference, (Loan #2) compound interest − common ratio
b) (Loan #1) 4.00%, (Loan #2) 6.00%
c) (Loan #1) $3650.00, (Loan #2) $870.00
d) (Loan #1) $5110.00, (Loan #2) $1558.04
3. $12 075.91
4. $22 831.55
5. 5.88%/a compounded monthly
6. 5.98%
7. $205.30

Cumulative Review Chapters 7–8, pp. 538–539

1. (a)	**5.** (c)	**9.** (a)	**13.** (c)
2. (c)	**6.** (a)	**10.** (a)	**14.** (d)
3. (c)	**7.** (a)	**11.** (b)	**15.** (b)
4. (a) and (c)	**8.** (b)	**12.** (c)	**16.** (b)

17. **a)** $t_1 = 350$, $t_n = 0.32(t_{n-1}) + 350$
b) 514.7 mg
c) 54 h
18. **a)** $t_n = 25(1.005)^{n-1}$
b) 168 payments, $6557.62
c)

A	B	C	C
1	Payment number	Payment	Future Value of each payment
2	1	$25.00	=C2*(1.005)^167
3	=B2 + 1	$25.00	=C2*(1.005)^166
4			
168	=B167 + 1	$25.00	=C2*(1.005)^1
169	=B168 + 1	$25.00	=C2
			=SUM(C2:C169)

d) $13 115.24

Appendix A

A–1 Operations with Integers, p. 542

1. **a)** 3 **c)** −24 **e)** −6
b) 25 **d)** −10 **f)** 6
2. **a)** $<$ **c)** $>$
b) $>$ **d)** $=$
3. **a)** 55 **c)** −7 **e)** $\frac{15}{7}$
b) 60 **d)** 8 **f)** $\frac{1}{49}$
4. **a)** 5 **c)** −9 **e)** −12
b) 20 **d)** 76 **f)** −1

5. a) 3 c) −2
   b) −1 d) 1

A–2 Operations with Rational Numbers, p. 544

1. a) $-\frac{1}{2}$ c) $-\frac{19}{12}$ e) $-\frac{41}{20}$
   b) $\frac{7}{6}$ d) $-8\frac{7}{12}$ f) 1
2. a) $-\frac{16}{25}$ c) $\frac{2}{15}$ e) $-3\frac{2}{5}$
   b) $-\frac{9}{5}$ d) $\frac{3}{2}$ f) $32\frac{7}{24}$
3. a) 2 c) $\frac{16}{9}$ e) $\frac{15}{2}$
   b) $-4\frac{3}{4}$ d) $-\frac{9}{2}$ f) $\frac{2}{3}$
4. a) $\frac{1}{5}$ c) $\frac{1}{15}$ e) $\frac{36}{5}$
   b) $\frac{3}{10}$ d) $-\frac{1}{18}$ f) $-\frac{3}{8}$

A–3 Exponent Laws, p. 546

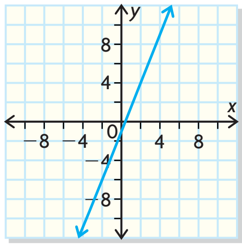
1. a) 16 c) 9 e) −125
   b) 1 d) −9 f) $\frac{1}{8}$
2. a) 2 c) 9 e) −16
   b) 31 d) $\frac{1}{18}$ f) $\frac{13}{36}$
3. a) 9 c) 4 194 304
   b) 50 d) $\frac{1}{27}$
4. a) x^8 c) y^7 e) x^6
   b) m^9 d) a^{bc} f) $\frac{x^{12}}{y^9}$
5. a) x^5y^6 c) $25x^4$
   b) $108m^{12}$ d) $\frac{4u^2}{v^2}$

A–4 The Pythagorean Theorem, pp. 547–548

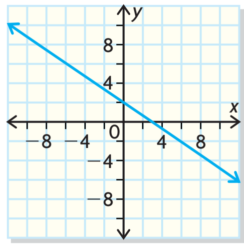
1. a) $x^2 = 6^2 + 8^2$ c) $9^2 = y^2 + 5^2$
   b) $c^2 = 13^2 + 6^2$ d) $8.5^2 = a^2 + 3.2^2$
2. a) 10 cm c) 7.5 cm
   b) 14.3 cm d) 7.9 cm
3. a) 13.93 c) 23.07
   b) 6 d) 5.23
4. a) 11.2 m c) 7.4 cm
   b) 6.7 cm d) 4.9 m
5. 10.6 cm
6. 69.4 m

A–5 Graphing Linear Relationships, p. 550

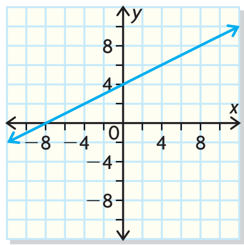
1. a) $y = 2x + 3$ c) $y = -\frac{1}{2}x + 2$
   b) $y = \frac{1}{2}x - 2$ d) $y = 5x + 9$
2. a)



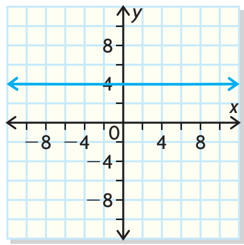
c)



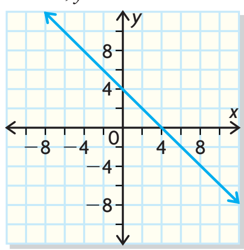
b)



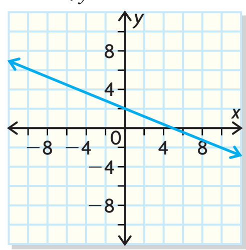
d)



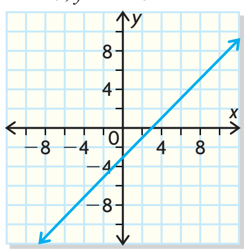
3. a) x-int: 10, y-int: 10 c) x-int: 5, y-int: 50
   b) x-int: 8, y-int: 4 d) x-int: 2, y-int: 4
4. a) x-int: 4, y-int: 4



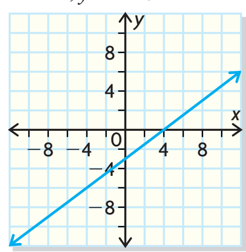
c) x-int: 5, y-int: 2



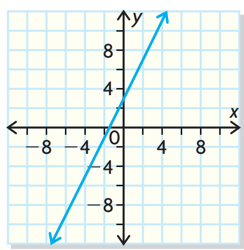
b) x-int: 3, y-int: −3



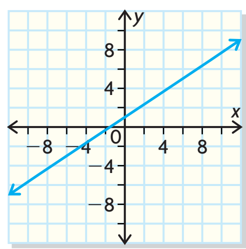
d) x-int: 4, y-int: −3



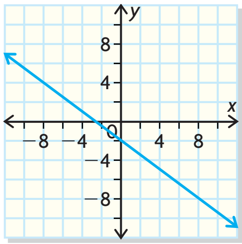
5. a)



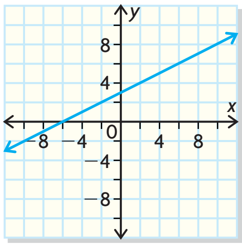
b)



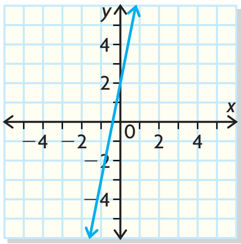
c)



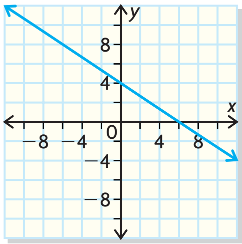
d)



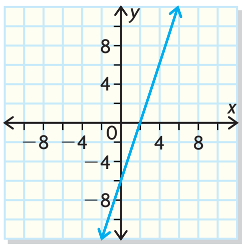
6. a)



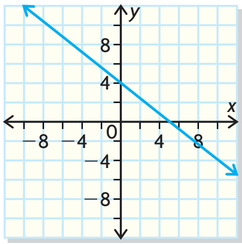
c)



b)



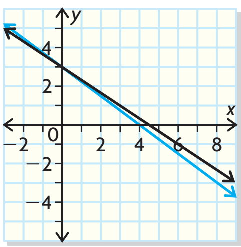
d)

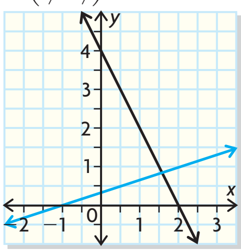


A–6 Solving Linear Systems, p. 553

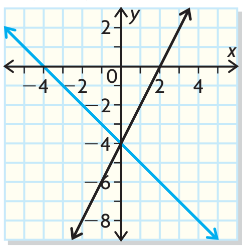
1. a) $(3, 2)$ b) $(5, -10)$

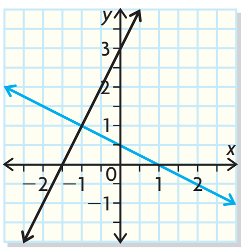
2. a) $(0, 3)$ c) $\left(\frac{11}{7}, \frac{6}{7}\right)$





b) $(0, -4)$ d) $(-1, 1)$





3. a) $(1, -1)$ d) $(2, 1)$

b) $(6, 9)$ e) $\left(\frac{9}{11}, \frac{28}{11}\right)$

c) $(6, -6)$ f) $\left(\frac{11}{7}, \frac{6}{7}\right)$

A–7 Evaluating Algebraic Expressions and Formulas, p. 554

1. a) 28 b) -17 c) 1 d) $\frac{9}{20}$

2. a) $\frac{1}{6}$ b) $\frac{5}{6}$ c) $\frac{-17}{6}$ d) $\frac{-7}{12}$

3. a) $82.35\ \text{cm}^2$ b) $58.09\ \text{m}^2$ c) $10\ \text{m}$ d) $4849.05\ \text{cm}^3$

A–8 Expanding and Simplifying Algebraic Expressions, p. 555

1. a) $-2x - 5y$ c) $-9x - 10y$
b) $11x^2 - 4x^3$ d) $4m^2n - p$

2. a) $6x + 15y - 6$ c) $3m^4 - 2m^2n$
b) $5x^3 - 5x^2 + 5xy$ d) $4x^7y^7 - 2x^6y^8$

3. a) $8x^2 - 4x$ c) $-13m^5n - 22m^2n^2$
b) $-34h^2 - 23h$ d) $-x^2y^3 - 12xy^4 - 7xy^3$

4. a) $12x^2 + 7x - 10$ c) $20x^2 - 23xy - 7y^2$
b) $14 + 22y - 12y^2$ d) $15x^6 - 14x^3y^2 - 8y^4$

A–9 Factoring Algebraic Expressions, p. 556

1. a) $4(1 - 2x)$ c) $3m^2n^3(1 - 3mn)$
b) $x(6x - 5)$ d) $14x(2x - y)$

2. a) $(x + 2)(x - 3)$ c) $(x - 5)(x - 4)$
b) $(x + 2)(x + 5)$ d) $3(y + 4)(y + 2)$

3. a) $(3y - 2)(2y + 1)$ c) $(5ax - 3)(a + 2)$
b) $(3x + 1)(4x - 1)$ d) $6(2x + 1)(x - 2)$

A–10 Solving Quadratic Equations Algebraically, p. 558

1. a) $x = 3, 2$ d) $x = \frac{3}{2}, \frac{4}{3}$

b) $x = \frac{5}{2}, \frac{1}{3}$ e) $y = -\frac{5}{2}, \frac{7}{3}$

c) $m = 4, 3$ f) $n = \frac{3}{5}, \frac{4}{3}$

2. a) $2, -1$ d) $-\frac{1}{2}, \frac{2}{3}$

b) $4, -5$ e) $-\frac{4}{3}, \frac{1}{2}$

c) $3, -5$ f) $3, -5$

3. a) $1 + \frac{\sqrt{3}}{2}, 1 - \frac{\sqrt{3}}{2}$ d) $-2, \frac{3}{5}$

b) $\frac{3}{2}, -\frac{3}{2}$ e) $3, -2$

c) $\frac{1}{2}, -\frac{1}{3}$ f) $7, 2$

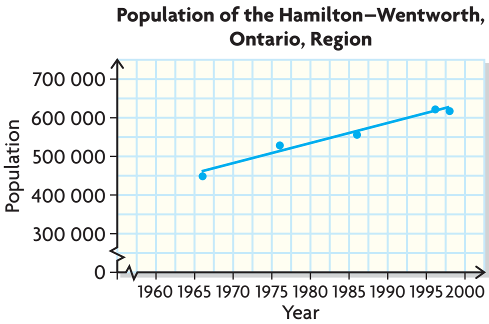
4. at 6 s

5. in the year 2010

Answers

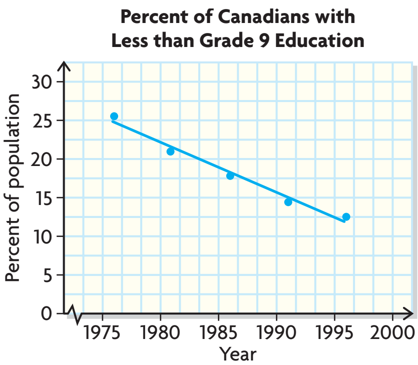
A–11 Creating Scatter Plots and Lines or Curves of Good Fit, p. 561

1. **a)** **i)**



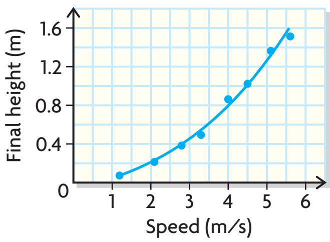
ii) The data displays a strong positive correlation.

b) **i)**



ii) The data displays a strong negative correlation.

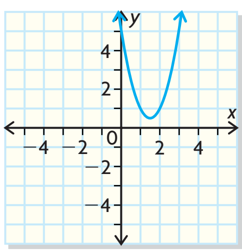
2. **a)**



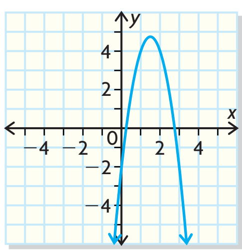
b) The motion sensor's measurements are consistent since the curve goes through several of the points.

A–12 Using Properties of Quadratic Relations to Sketch Their Graphs, p. 563

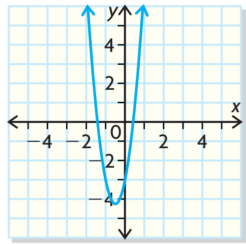
1. **a)**



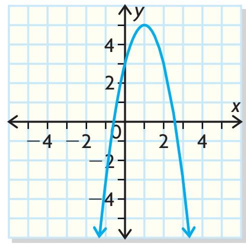
b)



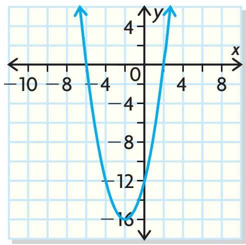
c)



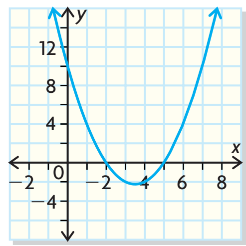
d)



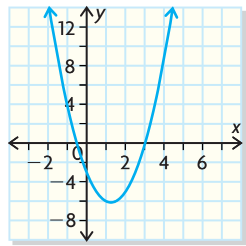
2. **a)**



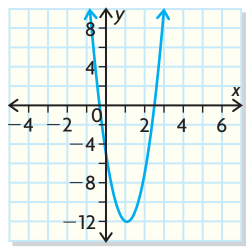
b)



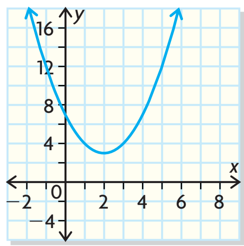
c)



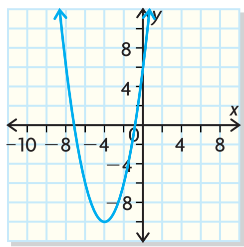
d)



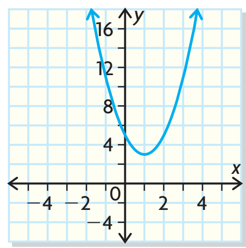
3. **a)**



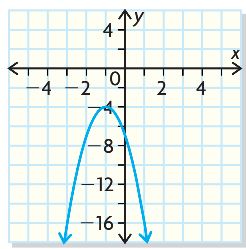
b)



c)



d)



A–13 Completing the Square to Convert to the Vertex Form of a Parabola, p. 564

1. **a)** $y = (x + 1)^2$ **c)** $y = (x + 3)^2$
b) $y = (x + 2)^2$ **d)** $y = (x + 5)^2$

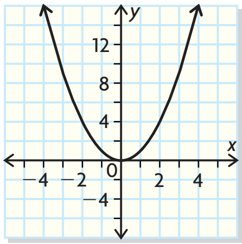
2. **a)** $y = (x + 1)^2 + 1$ **c)** $y = (x - 6)^2 + 4$
b) $y = (x + 2)^2 + 2$ **d)** $y = (x - 9)^2 - 1$

3. **a)** $y = 2(x - 1)^2 + 5; x = 1; (1, 5)$
b) $y = 5(x + 1)^2 + 1; x = -1; (-1, 1)$
c) $y = -3(x + 2)^2 + 14; x = -2; (-2, 14)$
d) $y = -2(x - 1.5)^2 + 6.5; x = 1.5; (1.5, 6.5)$

4. **a)** 6 m **b)** 1 s

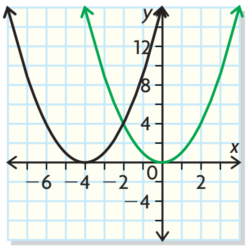
A–14 Transformations of Quadratic Relations, p.566

1. a) (2, 7) d) (7, 1)
   b) (−2, −1) e) (3, −3)
   c) (−5, 1) f) (2, 8)
2. a) (−4, 9) c) (−6, 7)
   b) (3, −3) d) (−5, −4)
3. a) $y = x^2 + 2$ c) $y = (x - 4)^2$
   b) $y = 3x^2$ d) $y = x^2 - 2$
4. a)

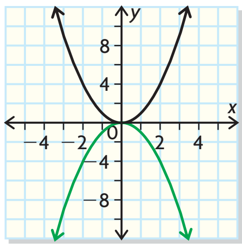


Graphs are identical.

d)

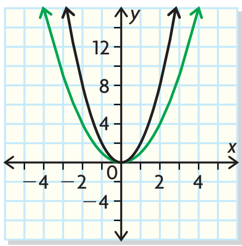


b)

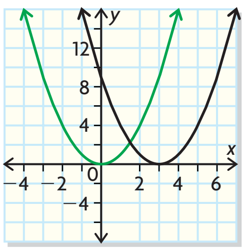


Graphs are identical.

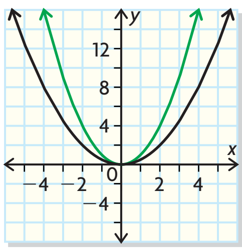
e)



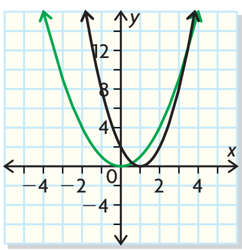
c)



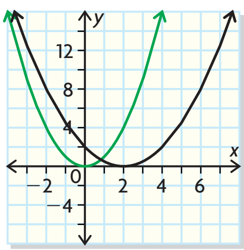
f)



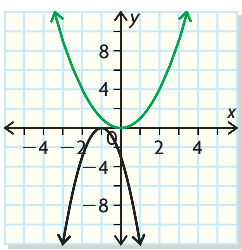
5. a)



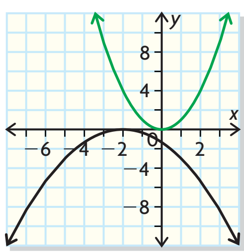
c)



b)

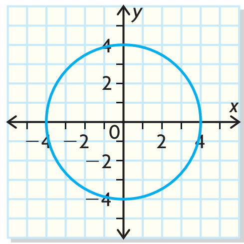


d)

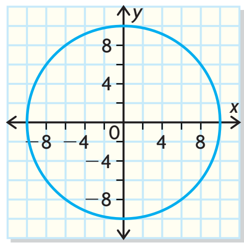


A–15 Equations of Circles Centred at the Origin, p. 568

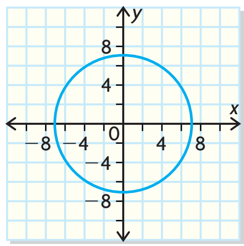
1. a) $x^2 + y^2 = 9$ c) $x^2 + y^2 = 64$
   b) $x^2 + y^2 = 49$ d) $x^2 + y^2 = 1$
2. a) 3 b) 9 c) 3.87 d) 5.20 e) 2.50 f) 4.20
3. 5, −5
4. a)



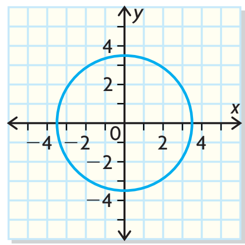
c)



b)



d)



A–16 Trigonometry of Right Triangles, p. 572

1. 27 m
2. a) $\sin A = \frac{12}{13}$, $\cos A = \frac{5}{13}$, $\tan A = \frac{12}{5}$
   b) $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$, $\tan A = \frac{4}{3}$
   c) $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$
   d) $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$, $\tan A = \frac{3}{4}$
3. a) 4.4 b) 6.8 c) 5.9 d) 26.9
4. a) 39° b) 54° c) 49° d) 25°
5. a) 12.5 cm b) 20.3 cm c) 19.7 cm d) 24°
6. a) 12.4 cm b) 5.7 cm c) 27° d) 46°
7. 8.7 m
8. 84.2 m
9. 195 m

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1. a) 10.3 b) 36.2° c) 85.1° d) 12.4 e) 47.3° f) 5.8
2. a) 16° b) 42.3° c) 23.4 d) 13.2 e) 33.1° f) 30.3
3. a) $t = 6.1$ cm, $\angle A = 72.8°$, $\angle C = 48.2°$
   b) $\angle A = 33.8°$, $\angle B = 42.1°$, $\angle C = 104.1°$
   c) $\angle F = 31.5°$, $\angle E = 109.5°$, $e = 25.8$ cm
4. 46.6
5. 1068.3 m
6. 12.2 m

Answers

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